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A REMARK ON $\bar{\partial}$-COHOMOLOGY WITH SUPPORTS IN THE COMPLEMENT OF A CONE SINGULARITY

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1. Introduction

Let

$$\Psi = \Sigma_{k=1}^{n} \psi_k(\mathbf{w}, z) d\bar{w}_k + \psi_{n+1} d\bar{z}$$

be a $\bar{\partial}$-closed form on $\mathbb{C}^n \times \mathbb{C}$. If it is assumed that Ψ has compact (specifically non-empty) support, then $\psi_{n+1}(\mathbf{w}, z) \neq 0$, for if the last component vanishes identically then this, together with the closedness of Ψ, implies that $\frac{\partial \psi_k}{\partial \bar{z}}$ vanishes identically, $1 \leq k \leq n$. Hence Ψ is the pullback of a form defined on $\mathbb{C}^n$, which contradicts the compactness of support. Now define

$$u(\mathbf{w}, z) = \frac{1}{2\pi \mathbf{i}} \int_{\mathbb{C}} \frac{\psi_{n+1}(\mathbf{w}, \zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

satisfying

$$\frac{\partial u}{\partial \bar{z}} = \psi_{n+1}(\mathbf{w}, z) \ ,$$

and note that

$$\frac{\partial u}{\partial \bar{w}_k} = \frac{1}{2\pi \mathbf{i}} \int_{\mathbb{C}} \frac{\frac{\partial \psi_{n+1}}{\partial \bar{w}_k} d\zeta \wedge d\bar{\zeta}}{\zeta - z} = \frac{1}{2\pi \mathbf{i}} \int_{\mathbb{C}} \frac{\frac{\partial \psi_k}{\partial \bar{\zeta}} d\zeta \wedge d\bar{\zeta}}{\zeta - z} = \psi_k(\mathbf{w}, z) \ .$$

Hence $\bar{\partial} u = \Psi$. If ψ_{n+1} moreover has rotational symmetry in the z-variable, about some fixed $z_0 \in \mathbb{C}$, then (as will be seen explicitly in the next section) u can easily be shown to have compact support, and is therefore the unique such solution of the above equation. In the following this simple phenomenon will be examined in a more global setting, namely that of the Cauchy-Riemann Equation for compactly supported forms of type (p, q) in the complement of an isolated singularity. A similar problem was addressed in [1] for the case of an isolated singular point for which the neighbouring smooth n-dimensional structure possesses a Kähler metric with L^n-curvature. In the present context a singular space $X \subset \mathbb{C}^N$ will be defined specifically by collapsing the zero section of a negative Hermitian-holomorphic line bundle $L \to M$, with M a compact, complex manifold (an application of the main theorem below will be presented in a forthcoming article). However, the Kähler metric naturally associated with this space will not be assumed to have any specific properties (other than smoothness) on the punctured smooth locus. This manuscript is based on a talk given at the second Japanese-Australian workshop on Real and Complex Singularities, held at the Research Institute of Mathematical Sciences, Kyoto University, in November 2007. The authors are grateful to Professors K. Miyajima and T. Akahori for helpful conversations at different stages of the preliminary research, and to Professor J. Slovak for his encouragement and support during the second author's visits to Masaryk University.

2. Cone Singularities and Cohomology with Supports

Let $\pi : L \to M$ be a holomorphic line bundle with Hermitian metric h over a compact complex n-dimensional manifold. Assume the curvature form of h to be strictly negative so that the dual bundle, L^* is very ample, i.e., there exist enough holomorphic sections $\sigma_i \in H^0(M, \mathcal{O}(L^*))$, $1 \leq i \leq N$, such that the map

$$M \hookrightarrow \mathbb{CP}_{N-1} \ ; \ \mapsto [\sigma_1(p) : \ldots\ldots : \sigma_N(p)]$$

is a holomorphic embedding. For any coordinate neighbourhood $\mathcal{U} \subset M$, consider the chart $\mathcal{U} \times \mathbb{C}$ with coordinates $(\mathbf{w}, z)$ on the total space of L, and note that the product $\sigma_i(\mathbf{w}) \cdot z$ locally defines a holomorphic function $f_i \in H^0(L, \mathcal{O})$, $1 \leq i \leq N$. The holomorphic map $\Phi : L \to \mathbb{C}^N$ such that $\Phi(\mathbf{w}, z) = (f_1(\mathbf{w}, z), \ldots, f_N(\mathbf{w}, z))$ is well-known to have the following properties:

- $\Phi^{-1}(\mathbf{0})$ corresponds to the zero section of L
- $\Phi \mid_{L \setminus \Phi^{-1}(\mathbf{0})}$ is a holomorphic embedding
- minimality of $N > n + 1$ implies that the image $X = \Phi(L)$ has an isolated singularity at the origin (we will denote $X_0 = X \setminus \{\mathbf{0}\}$).

Now the curvature $\mathbf{i}\bar{\partial}\partial \log(h)$ defines a Kähler form on M, and correspondingly the function $\rho(\mathbf{w}, z) = h(\mathbf{w})|z|^2$ determines a strongly plurisubharmonic function on the total space of L. In turn $\omega_{X_0} = \mathbf{i}\bar{\partial}\partial\rho$ is a Kähler form on $L \setminus \Phi^{-1}(\mathbf{0}) \cong X_0$. Given any $x \in X_0$, a choice of holomorphic normal frame for L at $\pi(x)$ allows us to assume

$$\rho(x) = h(\mathbf{w}(x))|z(x)|^2 = |z|^2 \ , \ \frac{\partial \rho}{\partial w_k}(x) = \frac{\partial h}{\partial w_k}(x) = 0 \ .$$

Moreover, a choice of Kähler normal coordinates on M at $\pi(x)$ further implies that

$$\omega_{X_0}(x) = \frac{\mathrm{i}}{2}(|z(x)|^2 dw_i \wedge d\bar{w}_i + dz \wedge d\bar{z}) \ .$$

Let $\varphi \in C_c^\infty(X_0, \Omega_{X_0}^{p,q})$ be a compactly supported form of type (p,q) on X_0. In the following $\Omega_{X_0}^{p,q}$ will be viewed alternatively as $\Omega_{X_0}^{0,q}(E)$, where E denotes the holomorphic vector bundle corresponding to $\bigwedge^p T^*X_0$. If $g_{i,\bar{j}}$ denotes the coefficients of the Kähler metric corresponding to ω_{X_0}, then $g^{i,\bar{j}}$ will similarly denote those of the dual metric on T^*X_0, while $\gamma^{\alpha,\bar{\beta}}$ will denote the metric induced on E. The volume form on TX_0 will then correspond to

$$\omega_{X_0}^{n+1} = (n+1)!(\frac{\mathrm{i}}{2})^{n+1} \det(g) d\mathbf{w} \wedge dz \wedge d\bar{\mathbf{w}} \wedge d\bar{z} \ (\dagger).$$

For notational simplicity it will be assumed that $q = 1$, though with appropriate modifications of the notation the general case can be handled similarly. In particular, the L^2-inner product may then be written

$$(\varphi, \bar{\partial}\psi) = \int_{X_0} \varphi_i^\alpha g^{k.\bar{i}} \gamma^{\alpha,\bar{\beta}} \frac{\partial \bar{\psi}^\beta}{\partial \bar{w}_k} \det(g) \ ,$$

(identifying $w_{n+1} \equiv z$), from which we obtain (cf. also [1])

$$\bar{\partial}^* \varphi = -\frac{\partial \varphi_i^\alpha}{\partial w_k} g^{k,\bar{i}} + \varphi_a^\alpha g^{b\bar{a}} (g^{k\bar{i}} \frac{\partial g_{b\bar{i}}}{\partial w_k}) - \varphi_i^\varepsilon g^{k\bar{i}} (\gamma^{\alpha\bar{\delta}} \frac{\partial \gamma_{\varepsilon\bar{\delta}}}{\partial w_k})$$

$$-\varphi_i^\alpha g^{k\bar{i}} \frac{\partial}{\partial w_k} (\log \det(g)) \ \ (*) \ .$$

Definition 1. *Let $\tau : M \to L$ denote a smooth section, and for any $\vartheta \in [0, 2\pi)$ define the fibre-preserving affine-linear map $\Gamma_{\tau,\vartheta} : L \to L$ such that*

$$\Gamma_{\tau,\vartheta}(\mathbf{w}, z) = (\mathbf{w}, e^{\mathrm{i}\vartheta}(z - \tau(\mathbf{w})) + \tau(\mathbf{w})) \ .$$

$\varphi \in C_c^{|\infty}(X_0, \Omega_{X_0}^{p,q})$ will be said to be relatively elliptic *if there exists a τ such that $\Gamma_{\tau,\vartheta}^* \varphi = \varphi$.*

Theorem 1. *Suppose $\psi \in C_c^\infty(X_0, \Omega_{X_0}^{p,q})$ is $\bar{\partial}$-closed and relatively elliptic, then there exists a unique $u \in C_c^\infty(X_0, \Omega_{X_0}^{p,q-1})$, which is also relatively elliptic, such that $\bar{\partial} u = \psi$.*

Proof. From (*), note that

$$\bar{\partial}^* \rho\varphi - \rho\bar{\partial}^* \varphi = -\frac{\partial \rho}{\partial w_k} \varphi_i^\alpha g^{k\bar{i}} \ ,$$

and hence

$$\bar\partial(\bar\partial^*\rho\varphi-\rho\bar\partial^*\varphi)_l=-\frac{\partial^2\rho}{\partial\bar w_l\partial w_k}\varphi_i^\alpha g^{k,\bar i}-\frac{\partial\rho}{\partial w_k}\frac{\partial\varphi_i^\alpha}{\partial\bar w_l}g^{k\bar i}-\frac{\partial\rho}{\partial w_k}\varphi_i^\alpha\frac{\partial g^{k\bar i}}{\partial\bar w_l}\ .$$

With respect to the normal form of ω_{X_0} represented by (†) at a given point $x\in X_0$, note that

$$\frac{\partial^2\rho}{\partial\bar w_l\partial w_k}\varphi_i^\alpha g^{k,\bar i}=\delta_{li}\varphi_i^\alpha=\varphi_l^\alpha\ .$$

If moreover φ is assumed to be $\bar\partial$-closed, then

$$-\frac{\partial\rho}{\partial w_k}\frac{\partial\varphi_i^\alpha}{\partial\bar w_l}g^{k\bar i}=-\frac{\partial\rho}{\partial w_k}\frac{\partial\varphi_l^\alpha}{\partial\bar w_i}g^{k\bar i}=-\xi^{01}(\varphi_l)\ ,$$

where $\xi^{01}=\bar z\frac{\partial}{\partial\bar z}$ is a globally defined vector field corresponding to the metric dual of $\bar\partial\rho$. Finally

$$-\frac{\partial\rho}{\partial w_k}\varphi_i^\alpha\frac{\partial g^{k\bar i}}{\partial\bar w_l}=\frac{\partial\rho}{\partial w_k}\varphi_i^\alpha g^{k\bar\mu}\frac{\partial g_{\beta\bar l}}{\partial\bar w_\mu}g^{\beta\bar i}\ ,$$

which gives φ_l^α if $l<n+1$, and zero if $l=n+1$. Hence

$$\begin{aligned}\bar\partial(\bar\partial^*\rho\varphi-\rho\bar\partial^*\varphi)_l&=-\bar z\frac{\partial\varphi_l^\alpha}{\partial\bar z}\ ,\ l<n+1\\&=-\varphi_{n+1}^\alpha-\bar z\frac{\partial\varphi_{n+1}^\alpha}{\partial\bar z}\ ,\ l=n+1\\&=-(\xi^{01}(\varphi_l^\alpha)-\varphi^\alpha([\xi^{01},\frac{\partial}{\partial\bar w_l}]))=-(L_{\xi^{01}}\varphi^\alpha)_l\ ,\end{aligned}$$

i.e., the Lie derivative of φ with respect to ξ^{01} (here $[*,*]$ denotes the standard Lie bracket on vector fields, and $\bar w_{n+1}$ is again identifed with $\bar z$). Note moreover that if J denotes the almost complex structure on the complexified real tangent bundle of X_0, then $L_{\xi^{01}}J=0$ (equivalently, for all vector fields σ , $[\xi^{01},J\sigma]=J[\xi^{01},\sigma]$) implies $\bar\partial$ commutes with $L_{\xi^{01}}$, i.e.,

$$L_{\xi^{01}}\bar\partial\varphi^\alpha=\bar\partial L_{\xi^{01}}\varphi^\alpha\ ,$$

(cf. [2]). In particular, whenever $L_{\xi^{01}}\varphi^\alpha$ is a closed form, it follows that $L_{\xi^{01}}\bar\partial\varphi^\alpha=0$, thus

$$\begin{aligned}0&=\bar z\frac{\partial}{\partial\bar z}(\frac{\partial\varphi_i^\alpha}{\partial\bar w_k}-\frac{\partial\varphi_k^\alpha}{\partial\bar w_i})\ ,\ i,k<n+1\ ,\ \text{or}\\&=\bar z\frac{\partial}{\partial\bar z}(\frac{\partial\varphi_i^\alpha}{\partial\bar w_k}-\frac{\partial\varphi_k^\alpha}{\partial\bar w_i})+(\frac{\partial\varphi_i^\alpha}{\partial\bar w_k}-\frac{\partial\varphi_k^\alpha}{\partial\bar w_i})\end{aligned}$$

when $i=n+1$, or $k=n+1$. Now φ_i^α is compactly supported for all i, hence the above equation implies $\frac{\partial\varphi_i^\alpha}{\partial\bar w_k}-\frac{\partial\varphi_k^\alpha}{\partial\bar w_i}\equiv0$, i.e., $\bar\partial\varphi=0$.

It follows that if ψ is a closed, compactly supported form, and φ is another compactly supported form satisfying $L_{\xi^{01}}\varphi = -\psi$, then

$$\bar\partial(\bar\partial^* \rho\varphi - \rho\bar\partial^*\varphi) = \psi \ .$$

The completion of the proof relies on the following

Lemma 1. *If* $\psi \in C_c^\infty(X_0, \Omega_{X_0}^{p,q})$ *is relatively elliptic, then* $u := \bar\partial^*\rho\varphi - \rho\bar\partial^*\varphi \in C_c^\infty(X_0, \Omega_{X_0}^{p,q-1})$ *and is also relatively elliptic.*

Proof. Given ψ, we must first solve the equation

$$L_{\xi^{01}}\varphi^\alpha = -\psi^\alpha \ .$$

The following explicit formulae imply smoothness of the solution in the local coordinates $(\mathbf{w}, z)$, namely

$$\varphi_l^\alpha(\mathbf{w}, z) = \frac{-1}{2\pi\mathbf{i}} \int_{\mathbf{C}} \frac{(\bar\zeta)^{-1}\psi_l^\alpha(\mathbf{w},\zeta)d\zeta \wedge d\bar\zeta}{\zeta - z} \ , \ l < n+1$$

and

$$\varphi_{n+1}^\alpha(\mathbf{w}, z) = \frac{-1}{2\pi\mathbf{i}\bar z} \int_{\mathbf{C}} \frac{\psi_{n+1}^\alpha(\mathbf{w},\zeta)d\zeta \wedge d\bar\zeta}{\zeta - z} \ .$$

Now

$$\bar\partial^*\rho\varphi^\alpha - \rho\bar\partial^*\varphi^\alpha = -\frac{\partial\rho}{\partial w_k}\varphi_l^\alpha g^{kl} = -\iota_{\xi^{01}}\varphi \ ,$$

which equals zero if $l < n+1$, and $-\bar z\varphi_{n+1}^\alpha$ when $l = n+1$. Hence

$$u^\alpha(\mathbf{w}, z) = \frac{1}{2\pi\mathbf{i}} \int_{\mathbf{C}} \frac{\psi_{n+1}^\alpha(\mathbf{w},\zeta)d\zeta \wedge d\bar\zeta}{\zeta - z} \ .$$

The condition that $\Gamma_{\tau,\vartheta}^*\psi = \psi$ means

$$\psi_l^\alpha(\Gamma_{\tau,\vartheta}(\mathbf{w}, z)) = \psi_l^\alpha(\mathbf{w}, z) \ , \ l < n+1$$

or

$$\psi_{n+1}^\alpha(\Gamma_{\tau,\vartheta}(\mathbf{w}, z)) = e^{\mathbf{i}\vartheta}\psi_{n+1}^\alpha(\mathbf{w}, z) \ , \ l = n+1 \ .$$

Hence

$$u^\alpha(\Gamma_{\tau,\vartheta}(\mathbf{w}, z)) = \frac{1}{2\pi\mathbf{i}} \int_{\mathbf{C}} \frac{\psi_{n+1}^\alpha(\mathbf{w},\zeta)d\zeta \wedge d\bar\zeta}{\zeta - e^{\mathbf{i}\vartheta}(z - \tau(\mathbf{w})) - \tau(\mathbf{w})}$$

$$= \frac{1}{2\pi\mathbf{i}} \int_{\mathbf{C}} \frac{\psi_{n+1}^\alpha(\mathbf{w},\zeta' - \tau(\mathbf{w})(e^{\mathbf{i}\vartheta} - 1))d\zeta' \wedge d\bar\zeta'}{\zeta' - e^{\mathbf{i}\vartheta}z} \ , \ (\zeta' = \zeta + \tau(\mathbf{w})(e^{\mathbf{i}\vartheta} - 1))$$

$$= \frac{1}{2\pi\mathbf{i}} \int_{\mathbf{C}} \frac{\psi_{n+1}^\alpha(\mathbf{w},e^{\mathbf{i}\vartheta}\zeta'' - \tau(\mathbf{w})(e^{\mathbf{i}\vartheta} - 1))d\zeta'' \wedge d\bar\zeta''}{e^{\mathbf{i}\vartheta}(\zeta'' - z)} \ , \ (\zeta'' = e^{-\mathbf{i}\vartheta}\zeta')$$

$$= \frac{1}{2\pi\mathbf{i}} \int_{\mathbf{C}} \frac{\psi_{n+1}^\alpha(\Gamma_{\tau,\vartheta}(\mathbf{w},\zeta''))d\zeta'' \wedge d\bar\zeta''}{e^{\mathbf{i}\vartheta}(\zeta'' - z)} = u^\alpha(\mathbf{w}, z) \ ,$$

hence $u^\alpha \in C^\infty(X_0, \Omega^{p,q}_{X_0})$ is relatively elliptic. But

$$\lim_{|z|\to\infty} |u^\alpha(\mathbf{w}, z)| \leq \lim_{|z|\to\infty} \frac{1}{\pi} \int_{\mathbb{C}} \frac{|\psi^\alpha_{n+1}| \rho d\rho \wedge d\theta}{|\zeta - z|} = 0$$

and $\frac{\partial u}{\partial \bar{z}} = \psi_{n+1}$ implies u^α is a holomorphic function of z in the complement of the support of ψ^α_{n+1}. Moreover u restricted to each fibre $L_\mathbf{w}$ is constant along circles centred at $\tau(\mathbf{w})$ by the ellipticity condition, hence $u^\alpha \mid_{\mathbb{C}\setminus supp(\psi_{n+1})}$ is constant, and therefore identically zero. □

The proof of the Theorem is now complete.

□

3. References

[1] Harris, A., and Tonegawa, Y., "L^p-Curvature and the Cauchy-Riemann Equation near an Isolated Singular point", *Nagoya Mathematical Journal* Vol. 164 (2001), 35-51

[2] Tanaka, N., "A Differential Geometric Study on Strongly Pseudo-Convex Manifolds", *Lectures in Mathematics, Kyoto University 9*, Kinokuniya Book-Store Co., Ltd. (1975)

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