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Lightlike hypersurfaces in Lorentz-Minkowski space

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1 Introduction

The study of the extrinsic differential geometry of submanifolds in Lorentz-Minkowski space is of special interest in Relativity Theory. In particular, lightlike hypersurfaces, which can be constructed as ruled hypersurfaces over spacelike submanifolds of codimension 2, provide good models for the study of different horizon types ([2, 3, 13]). Singularity theory tools have proven to be useful in the description, of geometrical properties of submanifolds immersed in different ambient spaces, from both the local and global viewpoint [11, 4, 5, 6, 7, 8, 9]. The natural connection between geometry and singularities relies on the basic fact that the contacts of a submanifold with the models (invariant under the action of a suitable transformation group) of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Legendrian maps ([1, 14]). When working in Lorentz-Minkowski space, the properties associated to the contacts of a given submanifold with lightlcones have a special relevance. In [7] it has been constructed the Lorentz invariant of spacelike submanifolds of codimension two in Lorentz-Minkowski space concerning their contacts with lightlike hyperplanes. Here we give a brief survey on these results in §2 and §3. In the case for Lorentz-Minkowski 4-space, which is the most important case from the view point of Relativity theory, it has been already investigated the lightlike hypersurfaces along a spacelike surface by using the Cartan's framework[7]. The framework constructed in [9] is slightly different from the framework in [7] but it is much more geometric than the framework in [7]. In this paper, we investigate singularities of lightlike hypersurfaces along spacelike submanifolds of codimension two in Lorentz-Minkowski space with general dimension by using the framework in [9]. The main techniques are given by the theory of Legendrian singularities.

We include in §2 the basic notions in Lorentz-Minkowski space that shall be used throughout the paper. In §3 we review the lightcone Gauss-Kronecker curvature which plays a principal role in this theory. The §4 is devoted to the study of the lightlike hypersurface. We review in §6 the classification result of singularities of generic lightlike hypersurfaces in Lorentz-Minkowski 4-space in [7]. Finally, we include the appendix containing the basic definitions and results on Legendrian singularities that shall be used in the paper.

2 Basic facts and notations on Lorentz-Minkowski space

We introduce in this section some basic notions on Lorentz-Minkowski $n+1$-space and spacelike submanifolds of codimension two. For basic concepts and properties, see [15].

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \ldots, n)\ \}$ be an $n+1$-dimensional cartesian space. For any $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ *Lorentz-Minkowski* $n+1$*-space*. We write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We have the canonical projection $\pi : \mathbb{R}^{n+1}_1 \to \mathbb{R}^n$ defined by $\pi(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Here we identify $\{\boldsymbol{0}\} \times \mathbb{R}^n$ with $\mathbb{R}^n$ and it is considered as Euclidean n-space whose scalar product is induced from the pseudo scalar product $\langle , \rangle$. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *Hyperbolic n-space* by

$$H^n_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}$$

and *de Sitter n-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}.$$

We define

$$LC^* = \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}_1\ | x_0 \neq 0,\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}$$

and we call it *the* (*open*) *lightcone* at the origin. Then we call the *future lightcone* to the subset

$$LC^*_+ = \{\boldsymbol{x} \in LC^*\ | x_0 > 0\}.$$

If $\boldsymbol{x} = (x_0, x_1, \ldots, x_2)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in S^{n-1}_+ = \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\}.$$

We call S^{n-1}_+ the *lightcone* (or, *spacelike*) *unit* $n-1$*-sphere*.

For any $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ by

$$\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}^i = (x^i_0, x^i_1, \ldots, x^i_n)$. We can easily check that

$$\langle \boldsymbol{x}, \boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n \rangle = \det(\boldsymbol{x}, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^n),$$

so that $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ is pseudo orthogonal to any $\boldsymbol{x}^i$ $(i = 1, \ldots, n)$.

3 Local differential geometry on spacelike submanifolds of codimension two

We review the results in [9] which constructed the basic geometrical tools for the study of spacelike submanifolds of codimension two in Lorentz-Minkowski $(n+1)$-space. Consider the orientation of $\mathbb{R}^{n+1}_1$ provided by the volume form $\boldsymbol{l}_0 \wedge \cdots \wedge \boldsymbol{l}_n$, where $\{\boldsymbol{l}_i\}_{i=0}^n$ is the dual basis of the canonical basis $\{\boldsymbol{e}_i\}_{i=0}^n$. We also give $\mathbb{R}^{n+1}_1$ a timelike orientation by choosing $\boldsymbol{e}_0 = (1, 0, \ldots, 0)$ as future timelike vector field. We consider a spacelilke embedding $\boldsymbol{X} : U \longrightarrow \mathbb{R}^{n+1}_1$ from an open subset $U \subset \mathbb{R}^{n-1}$. We write $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. We say that $\boldsymbol{X}$ is *spacelike* if $\boldsymbol{X}_{u_i}$ $i = 1, \ldots, n-1$ are always spacelike vectors. Therefore, the tangent space T_pM of M at p is a spacelike subspace (i.e., consists of spacelike vectors) for any point $p \in M$. In this case, the pseudo-normal space N_pM is a timelike plane (i.e., Lorentz plane) (cf.,[15]). We denote by $N(M)$ the pseudo-normal bundle over M. Since this is a trivial bundle, we can arbitrarily choose a future directed unit timelike normal section $\boldsymbol{n}^T(u) \in N_p(M)$, where $p = \boldsymbol{X}(u)$. Here, we say that $\boldsymbol{n}^T$ is *future directed* if $\langle \boldsymbol{n}^T, \boldsymbol{e}_0 \rangle < 0$. Therefore we can construct a spacelike unit normal section $\boldsymbol{n}^S(u) \in N_p(M)$ by

$$\boldsymbol{n}^S(u) = \frac{\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)}{\|\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|},$$

and we have $\langle \boldsymbol{n}^T, \boldsymbol{n}^T \rangle = -1$, $\langle \boldsymbol{n}^T, \boldsymbol{n}^S \rangle = 0$, $\langle \boldsymbol{n}^S, \boldsymbol{n}^S \rangle = 1$. Although we could also choose $-\boldsymbol{n}^S(u)$ as a spacelike unit normal section with the above properties, we fix the direction $\boldsymbol{n}^S(u)$ throughout this paper. We call $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ a *future directed normal frame* along $M = \boldsymbol{X}(U)$. Clearly, the vectors $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$ are lightlike. Here we choose $\boldsymbol{n}^T + \boldsymbol{n}^S$ as a lightlike normal vector field along M. Since $\{\boldsymbol{X}_{u_1}(u), \ldots, \boldsymbol{X}_{u_{n-1}}(u)\}$ is a basis of T_pM, the system $\{\boldsymbol{n}^T(u), \boldsymbol{n}^S(u), \boldsymbol{X}_{u_1}(u), \ldots, \boldsymbol{X}_{u_{n-1}}(u)\}$ provides a basis for $T_p\mathbb{R}^{n+1}_1$. In [9] we have shown the following basic fact:

Lemma 3.1 *Given two future directed unit timelike normal sections $\boldsymbol{n}^T(u), \bar{\boldsymbol{n}}^T(u) \in N_p(M)$, the corresponding lightlike normal sections $\boldsymbol{n}^T(u) + \boldsymbol{n}^S(u), \bar{\boldsymbol{n}}^T(u) + \bar{\boldsymbol{n}}^S(u)$ are parallel.*

Under the identification of M and U through $\boldsymbol{X}$, we have the linear mapping provided by the derivative of the lightcone normal vector field $\boldsymbol{n}^T + \boldsymbol{n}^S$ at each point $p \in M$,

$$d_p(\boldsymbol{n}^T + \boldsymbol{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_p(M).$$

Consider the orthogonal projections $\pi^t : T_pM \oplus N_p(M) \longrightarrow T_p(M)$ and $\pi^n : T_p(M) \oplus N_p(M) \longrightarrow N_p(M)$. We define

$$d_p(\boldsymbol{n}^T + \boldsymbol{n}^S)^t = \pi^t \circ d_p(\boldsymbol{n}^T + \boldsymbol{n}^S)$$

and

$$d_p(\boldsymbol{n}^T+\boldsymbol{n}^S)^n = \pi^n \circ d_p(\boldsymbol{n}^T+\boldsymbol{n}^S).$$

We respectively call the linear transformations $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S) = -d_p(\boldsymbol{n}^T+\boldsymbol{n}^S)^t$ and $d_p(\boldsymbol{n}^T+\boldsymbol{n}^S)^n$ of T_pM, the *$(\boldsymbol{n}^T,\boldsymbol{n}^S)$-shape operator* of $M=\boldsymbol{X}(U)$ at $p=\boldsymbol{X}(u)$ and the *normal connection with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ of M at p.

The eigenvalues of $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S)$, denoted by $\{\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)\}_{i=1}^{n-1}$, are called the *lightcone principal curvatures* with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at p. Then the *lightcone Gauss-Kronecker curvature* with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at p is defined as

$$K_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(p) = \det S_p(\boldsymbol{n}^T,\boldsymbol{n}^S).$$

We say that a point p is a *$(\boldsymbol{n}^T,\boldsymbol{n}^S)$-umbilic point* if all the principal curvatures coincide at p and thus $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S) = \kappa(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)1_{T_pM}$, for some function κ. We say that M is *totally $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-umbilic* if all points on M are $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-umbilic.

We deduce now the lightcone Weingarten formula. Since $\boldsymbol{X}_{u_i}$ $(i=1,\dots n-1)$ are spacelike vectors, we have a Riemannian metric (the *hyperbolic first fundamental form*) on M defined by $ds^2=\sum_{i=1}^{n-1} g_{ij}du_idu_j$, where $g_{ij}(u)=\langle \boldsymbol{X}_{u_i}(u),\boldsymbol{X}_{u_j}(u)\rangle$ for any $u\in U$. We also have a *lightcone second fundamental invariant with respect to the normal vector field* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ defined by $h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i}(u),\boldsymbol{X}_{u_j}(u)\rangle$ for any $u\in U$. In [9] we have shown the following lightcone Weingarten formula with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$:

(a) $(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i} = \langle \boldsymbol{n}^S,\boldsymbol{n}^T_{u_i}\rangle(\boldsymbol{n}^T+\boldsymbol{n}^S) - \sum_{j=1}^{n-1} h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$

(b) $\pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{n-1} h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

Here $\left(h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\right) = \left(h_{ik}(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)\left(g^{kj}\right)$ and $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

These formulae induce an explicit expression of the lightcone curvature in terms of the Riemannian metric and the lightcone second fundamental invariant as follows:

$$K_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S) = \frac{\det\left(h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Since $\langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)(u),\boldsymbol{X}_{u_j}(u)\rangle = 0$, we have

$$h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \langle \boldsymbol{n}^T(u)+\boldsymbol{n}^S(u),\boldsymbol{X}_{u_iu_j}(u)\rangle.$$

Therefore the lightcone second fundamental invariant at a point $p_0=\boldsymbol{X}(u_0)$ depends only on the values, $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$, respectively assumed by the vector fields $\boldsymbol{n}^T+\boldsymbol{n}^S$ and $\boldsymbol{X}_{u_iu_j}$ at the point p_0. And thus, the lightcone curvature depends only on $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$, $\boldsymbol{X}_{u_i}(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$ too, independently of the choice of the normal vector fields $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$. We write $K_\ell(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)(u_0)$ as the lightcone curvature at p_0 with respect to $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S) = (\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))$. We might also say that a point p_0 is *$(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$-umbilic* because the lightcone $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-shape operator at p_0 only depends on the normal vectors $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$. Analogously, we say that the point p_0 is a *$(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$-parabolic point* of M if $K_\ell(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)(u_0)=0$. And we say that p_0 is a *$(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$-flat point* if it is $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$-umbilic and $K_\ell(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)(u_0)=0$.

For any spacelike embedding $\boldsymbol{X}: U\to\mathbb{R}_1^{n+1}$ from an open subset $U\subset\mathbb{R}^{n-1}$, we consider a future directed unit timelike normal section $\boldsymbol{n}^T(u)\in N_p(M)$ and the corresponding spacelike unit normal section $\boldsymbol{n}^S(u) \in N_p(M)$ constructed in the previous section, where $p = \boldsymbol{X}(u)$. By Lemma 3.1, if we choose another future directed unit timelike normal section $\bar{\boldsymbol{n}}^T(u)$, then we have $(\widetilde{\boldsymbol{n}^T + \boldsymbol{n}^S})(u) = (\widetilde{\bar{\boldsymbol{n}}^T + \bar{\boldsymbol{n}}^S})(u) \in S^{n-1}_+$. Therefore we define the *lightcone Gauss map* of $M = \boldsymbol{X}(U)$ as

$$\begin{array}{rccc} \widetilde{\mathbb{L}}: & U & \longrightarrow & S^{n-1}_+ \\ & (u) & \longmapsto & (\widetilde{\boldsymbol{n}^T + \boldsymbol{n}^S})(u). \end{array}$$

This induces a linear mapping $d\widetilde{\mathbb{L}}_p : T_pM \to T_p\mathbb{R}^{n+1}_1$ under the identification of U and M, where $p = \boldsymbol{X}(u)$. We have the following normalized lightcone Weingarten formula:

$$\pi^t \circ \widetilde{\mathbb{L}}_{u_i} = -\sum_{j=1}^{n-1} \frac{1}{\ell_0(u)} h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j},$$

where $\mathbb{L}(u) = (\ell_0(u), \ell_1(u), \dots, \ell_n(u))$.

We call the linear transformation $\widetilde{S}_p = -\pi^t \circ d\widetilde{\mathbb{L}}_p$ the *normalized lightcone shape operator* of M at p. The eigenvalues $\{\widetilde{\kappa}_i(p)\}_{i=1}^{n-1}$ of $\widetilde{S}_p$ are called *normalized lightcone principal curvatures.* By the above proposition, we have $\widetilde{\kappa}_i(p) = (1/\ell_0)\kappa_i(\boldsymbol{n}^T, \boldsymbol{n}^S)(p)$. The *normalized lightcone Gauss-Kronecker curvature* of M is defined to be $\widetilde{K}_\ell(u) = \det \widetilde{S}_p$. Then we have the following relation between the normalized lightcone Gauss-Kronecker curvature and the lightcone Gauss-Kronecker curvature:

$$\widetilde{K}_\ell(u) = \left(\frac{1}{\ell_0(u)}\right)^{n-1} K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)(u).$$

It is clear from the corresponding definitions that the lightcone Gauss map, the normalized lightcone principal curvatures and the normalized lightcone Gauss-Kronecker curvatures are independent on the choice of of the normal frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$.

We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is a *lightlike umbilical point* if $\widetilde{S}_p = \widetilde{\kappa}(p)1_{T_pM}$. By the above proposition, p is a lightlike umbilic point if and only if it is a $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic point for any $(\boldsymbol{n}^T, \boldsymbol{n}^S)$. We say that M is *totally lightlike umbilic* if all points on M are lightlike umbilic, as usual. We also say that p is a *lightlike parabolic point* if $\widetilde{K}_\ell(u) = 0$. Moreover, p is called a *lightlike flat point* if p is both lightlike umbilic and parabolic. The spacelike submanifold M is called *totally lightlike flat* provided every point of M is lightlike flat.

4 Lightlike hypersurfaces

We define a hypersurface

$$LH_M : U \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1}_1$$

by

$$LH_M(p,t) = LH_{\boldsymbol{X}}(u,t) = \boldsymbol{X}(u) + t(\widetilde{\boldsymbol{n}^T + \boldsymbol{n}^S})(u),$$

where $p = \boldsymbol{X}(u)$. We call LH_M the *lightlike hypersurface* along M. We remark that we can also define $LH^-_M(p,t) = \boldsymbol{X}(u) + t(\widetilde{\boldsymbol{n}^T - \boldsymbol{n}^S})(u)$ as another lightlike hypersurface.

However, the properties of LH_M^- is the same as the properties of LH_M, so that we only consider LH_M.

In general, a hypersurface $H \subset \mathbb{R}_1^{n+1}$ is called a *lightlike hypersurface* if it is tangent to a lightcone at any point. It is known that any lightlike hypersurface is given by the construction above at least locally (cf. [10]).

We introduce the notion of Lorentzian distance-squared functions on spacelike submanifold of codimention two, which is useful for the study of singularities of lightlike hypersurfaces.

First we define a family of functions $G : M \times \mathbb{R}_1^{n+1} \longrightarrow \mathbb{R}$ on a spacelike submanifold $M = \boldsymbol{X}(U)$ of codimension two by

$$G(p, \boldsymbol{\lambda}) = G(u, \boldsymbol{\lambda}) = \langle \boldsymbol{X}(u) - \boldsymbol{\lambda}, \boldsymbol{X}(u) - \boldsymbol{\lambda} \rangle,$$

where $p = \boldsymbol{X}(u)$. We call G the *Lorentzian distance-squared function* on the spacelike submanifold M. For any fixed $\boldsymbol{\lambda}_0 \in \mathbb{R}_1^{n+1}$, we write $g(p) = G_{\lambda_0}(p) = G(p, \boldsymbol{\lambda}_0)$ and have the following proposition.

Proposition 4.1 *Let M be a spacelike submanifold of codimension two and $G : M \times \mathbb{R}_1^{n+1} \to \mathbb{R}$ the Lorentzian distance-squared function on M. Suppose that $p_0 \neq \boldsymbol{\lambda}_0$. Then we have the following*:

(1) $g(p_0) = \partial g/\partial u_i(p_0) = 0$ $(i = 1, \dots, n-1)$ *if and only if* $p_0 - \boldsymbol{\lambda}_0 = \mu\widetilde{(\boldsymbol{n}^T \pm \boldsymbol{n}^S)}(p_0)$ *for some* $\mu \in \mathbb{R} \setminus \{0\}$.

(2) $g(p_0) = \partial g/\partial u_i(p_0) = \det\mathcal{H}(g)(p_0) = 0$ $(i = 1, \dots, n-1)$ (*where* $\det\mathcal{H}(g)(p_0)$ *is the determinant of the Hessian matrix*) *if and only if*

$$p_0 - \boldsymbol{\lambda}_0 = \mu\widetilde{(\boldsymbol{n}^T \pm \boldsymbol{n}^S)}(p_0)$$

for some $\mu \in \mathbb{R} \setminus \{0\}$ *such that* $1/\mu$ *is one of the non-zero normalized lightcone principal curvatures* $\widetilde{\kappa}_i^{\mp}(p_0), (i = 1, \dots, n-1)$.

Proof. (1) The condition $g(p) = \langle \boldsymbol{X}(u) - \boldsymbol{\lambda}_0, \boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \rangle = 0$ means that $\boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \in LC_0$. We can observe that $dg(p) = \langle d\boldsymbol{X}(u), \boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \rangle = 0$ if and only if $\boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \in N_pM$. Hence $g(p_0) = dg(p_0) = 0$ if and only if $p_0 - \boldsymbol{\lambda}_0 \in N_pM \cap LC_0$. This is equivalent to the condition that $p_0 - \boldsymbol{\lambda}_0 = \mu\widetilde{(\boldsymbol{n}^T \pm \boldsymbol{n}^S)}(p_0)$ for some $\mu \in \mathbb{R} \setminus \{0\}$.

(2) We can calculate that

$$\frac{\partial g}{\partial u_i} = 2\langle \boldsymbol{X}_{u_i}, \boldsymbol{X} - \boldsymbol{\lambda}_0 \rangle$$

and

$$\frac{\partial^2 g}{\partial u_i \partial u_j} = 2\left\{ \boldsymbol{X}_{u_i u_j}, \boldsymbol{X} - \boldsymbol{\lambda}_0 \rangle + \langle \boldsymbol{X}_{u_i}, \boldsymbol{X}_{u_j} \rangle \right\}.$$

Under the condition $p_0 - \boldsymbol{\lambda}_0 = \mu\widetilde{(\boldsymbol{n}^T \pm \boldsymbol{n}^S)}(p_0)$ that we have

$$\frac{\partial^2 g}{\partial u_i \partial u_j} = 2\left\{ \boldsymbol{X}_{u_i u_j}, \mu\widetilde{(\boldsymbol{n}^T \pm \boldsymbol{n}^S)}(p_0) \rangle + g_{ij}(u) \right\}.$$

Therefore, we have

$$\left(\frac{\partial^2 g}{\partial u_i \partial u_j}\right)(g^{k\ell}) = \left(2\left\{-\mu \widetilde{h}^i_j + \delta^i_j\right\}\right).$$

It follows that $\det\mathcal{H}(g)(p_0) = 0$ if and only if $1/\mu$ is an eigenvalue of $(\widetilde{h}^i_j(p))$. □

Thus Proposition 4.1 means that the discriminant set of the Lorentzian distance-squared function G is given by

$$\mathcal{D}_G = \left\{\boldsymbol{\lambda} \;\middle|\; \boldsymbol{\lambda} = \boldsymbol{X}(p) + u(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(p),\ p \in M, u \in \mathbb{R}\ \right\},$$

which is the image of the lightlike hypersurface along M. Therefore a singular point of the lightlike hypersurface is a point $\boldsymbol{\lambda}_0 = \boldsymbol{X}(p_0) + u_0(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(p_0)$ at which $u_0 = -1/\kappa^{\mp}_i(p_0)$, $i = 1, 2$.

We now explain the reason why such a correspondence exists from the point of view of contact geometry. Let $\pi : PT^*(\mathbb{R}^{n+1}_1) \longrightarrow \mathbb{R}^{n+1}_1$ be the projective cotangent bundle with its canonical contact structure. We next review the geometric properties of this bundle. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^{n+}_1) \to PT^*(\mathbb{R}^{n+1}_1)$ and the differential map $d\pi : TPT^*(\mathbb{R}^{n+1}_1) \to T\mathbb{R}^{n+1}_1$ of π. For any $X \in TPT^*(\mathbb{R}^{n+1}_1)$, there exists an element $\alpha \in T^*(\mathbb{R}^{n+1}_1$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(\mathbb{R}^{n+1}_1)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^{n+1}_1)$ by

$$K = \{X \in TPT^*(\mathbb{R}^{n+1}_1) \mid \tau(X)(d\pi(X)) = 0\}.$$

Via the coordinates $(v_0, v_1, \ldots, v_n)$, we have the trivialization $PT^*(\mathbb{R}^{n+1}_1) \cong \mathbb{R}^{n+1}_1 \times P^n(\mathbb{R})^*$, and call

$$((v_0, v_1, \ldots, v_n), [\xi_0 : \xi_1 : \cdots : \xi_n])$$

homogeneous coordinates of $PT^*(\mathbb{R}^{n+1}_1)$, where $[\xi_0 : \xi_1 : \cdots : \xi_n]$ are the homogeneous coordinates of the dual projective space $P^n(\mathbb{R})^*$.

It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=0}^n \mu_i \xi_i = 0$, where $d\tilde{\pi}(X) = \sum_{i=0}^n \mu_i \partial/\partial v_i$. An immersion $i : L \to PT^*(\mathbb{R}^{n+1}_1)$ is said to be *a Legendrian immersion* if $\dim L = n$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called *the Legendrian map* and the set $W(i) = \mathrm{image}\,\pi \circ i$, the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$. In the appendix, we give a quick survey of the theory of Legendrian singularities . For additional definitions and basic results on generating families, we refer to ([1], Chapter 21). By the preceding arguments, the lightlike hypersurface $LH^{\pm}_M$ is the discriminant set of the Lorentzian distance-squared function G. We have the following proposition (See the appendix for the definition of a Morse family).

Proposition 4.2 *Let G be the Lorentzian distance-squared function on M. For any point $(u, \boldsymbol{\lambda}) \in G^{-1}(0)$, G is a Morse family around $(u, \boldsymbol{\lambda})$.*

Proof. We denote that

$$\boldsymbol{X}(u) = (X_0(u), X_1(u), \ldots, X_n(u)) \text{ and } \boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n).$$

By definition, we have

$$G(u,\boldsymbol{\lambda}) = -(X_0(u)-\lambda_0)^2 + (X_1(u)-\lambda_1)^2 + \cdots + (X_n(u)-\lambda_n)^2.$$

We now prove that the mapping

$$\Delta^* G = \left(G, \frac{\partial G}{\partial u_1}, \ldots, \frac{\partial G}{\partial u_{n-1}}\right)$$

is non-singular at $(u,\boldsymbol{\lambda}) \in G^{-1}(0)$. Indeed, the Jacobian matrix of $\Delta^* G$ is given by

$$\begin{pmatrix} \mathbf{A} & \begin{matrix} 2(X_0-\lambda_0) & -2(X_1-\lambda_1) & \cdots & -2(X_n-\lambda_n) \\ 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_{n-1}} & -2X_{1u_{n-1}} & \cdots & -2X_{nu_{n-1}} \end{matrix} \end{pmatrix},$$

where $\mathbf{A}$ is the following matrix:

$$\begin{pmatrix} 2\langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_1}\rangle & \cdots & 2\langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_{n-1}}\rangle \\ 2(\langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_1}\rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_1u_1}\rangle) & \cdots & 2(\langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_{n-1}}\rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_1u_{n-1}}\rangle) \\ \vdots & \ddots & \vdots \\ 2(\langle \boldsymbol{X}_{u_{n-1}}, \boldsymbol{X}_{u_1}\rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_{n-1}u_1}\rangle) & \cdots & 2(\langle \boldsymbol{X}_{u_{n-1}}, \boldsymbol{X}_{u_{n-1}}\rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_{n-1}u_{n-1}}\rangle) \end{pmatrix}.$$

Since $\boldsymbol{X}$ is an immersion, the rank of the matrix

$$\begin{pmatrix} 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_{n-1}} & -2X_{1u_{n-1}} & \cdots & -2X_{nu_{n-1}} \end{pmatrix}.$$

is equal to $n-1$. Moreover, $\boldsymbol{X}-\boldsymbol{\lambda}$ is lightlike, so that it is linearly independent of tangent vectors $\boldsymbol{X}_{u_1}, \ldots, \boldsymbol{X}_{u_{n-1}}$. This means that the rank of the matrix

$$\begin{pmatrix} 2(X_0-\lambda_0) & -2(X_1-\lambda_1) & \cdots & -2(X_n-\lambda_n) \\ 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_{n-1}} & -2X_{1u_{n-1}} & \cdots & -2X_{nu_{n-1}} \end{pmatrix}$$

is equal to n. Therefore the Jacobi matrix of $\Delta^* G$ is non-singular at $(u,\boldsymbol{\lambda}) \in G^{-1}(0)$. □

Since G is a Morse family, we have the Legendrian immersion

$$L_G^{\pm} : \Sigma_*(G) \longrightarrow PT^*(\mathbb{R}_1^{n-1})$$

by

$$L_G^{\pm}(u,\boldsymbol{\lambda}) = (\boldsymbol{\lambda}, [(X_0(u)-\lambda_0) : (\lambda_1 - X_1(u)) : \cdots : (\lambda_n - X_n(u))]),$$

where

$$\Sigma_*(G) = (\Delta^* G)^{-1}(0) = \{(u,\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} = LH_M^{\pm}(u,t) \text{ for some } t \in \mathbb{R}\}.$$

We observe that G is a generating family of the Legendrian immersion $L_G^{\pm}$ whose wave front is $LH_M^{\pm}$ (cf. the appendix). Therefore we might say that the Lorentzian distance-squared function G on M gives a Lorentz-Minkowski-canonical generating family for the Legendrian lift of $LH_M^{\pm}$.

5 Contact with lightcones

In this section we describe Montaldi's characterization of submanifolds contact in terms of $\mathcal{K}$-equivalence. It is then adapted to lightlike hypersurfaces and their indicatrices. We begin with the following basic observations.

Proposition 5.1 *Let $\boldsymbol{\lambda}_0 \in \mathbb{R}^{n+1}_1$ and M a spacelike submanifold of codimension two without umbilic points satisfying $\widetilde{K}_\ell \neq 0$. Then $M \subset LC_{\lambda_0}$ if and only if $\boldsymbol{\lambda}_0$ is an isolated singular value of the lightlike hypersurface $LH^{\pm}_M$ and $LH^{\pm}_M(U \times \mathbb{R}) \subset LC_{\lambda_0}$.*

Proof. In the first place, we remark that $\widetilde{K}_\ell(u) \neq 0$ if and only if

$$\{(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}), (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_1}, \dots, (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_{n-1}}\}$$

is linearly independent at $u \in U$. By definition, $M \subset LC_{\lambda_0}$ if and only if $g_{\lambda_0}(u) \equiv 0$ for any $u \in U$, where $g_{\lambda_0}(u) = G(u, \boldsymbol{\lambda}_0)$ is the Lorentzian distance-squared function on M. It follows from Proposition 4.1 that there exists a smooth function $\mu : U \longrightarrow \mathbb{R}$ such that

$$\boldsymbol{X}(u) = \boldsymbol{\lambda}_0 + \mu(u)(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(u).$$

Therefore we have

$$LH^{\pm}_M(u,t) = \boldsymbol{\lambda}_0 + (t + \mu(u))(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(u).$$

Hence we have $LH^{\pm}_M(U \times \mathbb{R}) \subset LC_{\lambda_0}$. Moreover, it follows that

$$\begin{aligned} \frac{\partial LH^{\pm}_M}{\partial t} &= (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(u), \\ \frac{\partial LH^{\pm}_M}{\partial u_i} &= \mu_{u_i}(u)(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(u) + (t + \mu(u))(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_i}(u), \end{aligned}$$

from which we obtain

$$\begin{aligned} &\left(\frac{\partial LH^{\pm}_M}{\partial t} \wedge \frac{\partial LH^{\pm}_M}{\partial u_1} \wedge \dots \wedge \frac{\partial LH^{\pm}_M}{\partial u_{n-1}}\right) \\ &= (t + \mu(u))^{n-1}(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}) \wedge (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_1} \wedge \dots \wedge (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_{n-1}}. \end{aligned}$$

By the assumption, we have

$$\boldsymbol{X} - \boldsymbol{\lambda}_0 = \mu(u)(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(u).$$

Since $\boldsymbol{X} - \boldsymbol{\lambda}_0$ is lightlike and $\boldsymbol{X}_{u_i}$,$(i = 1, \dots, n-1)$ are spacelike, $\boldsymbol{X} - \boldsymbol{\lambda}_0, \boldsymbol{X}_{u_1}, \dots, \boldsymbol{X}_{u_{n-1}}$ are linearly independent. Therefore we have

$$\boldsymbol{0} \neq (\boldsymbol{X} - \boldsymbol{\lambda}_0) \wedge \boldsymbol{X}_{u_1} \wedge \dots \wedge \boldsymbol{X}_{u_{n-1}} = \mu(u)^n(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}) \wedge (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_1} \wedge \dots \wedge (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})_{u_{n-1}},$$

so that

$$\left(\frac{\partial LH^{\pm}_M}{\partial t} \wedge \frac{\partial LH^{\pm}_M}{\partial u_1} \wedge \dots \wedge \frac{\partial LH^{\pm}_M}{\partial u_{n-1}}\right) = 0$$

if and only if $t + \mu(u) = 0$ under the assumption that $\widetilde{K}_\ell \neq 0$. This means that $\boldsymbol{\lambda}_0$ is an isolated singularity of $LH^{\pm}_M$. The converse assertion is trivial. $\square$

Motivated by the proposition above, we now consider the contact of spacelike submanifolds of codimension two with lightcones in view of Montaldi's theorem [14]. Let X_i and Y_i, $i = 1, 2$, be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact of* X_1 and Y_1 at y_1 is same type as *the contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Since this definition of contact is local, we can replace $\mathbb{R}^n$ by arbitrary n-manifold. Montaldi gives in [14] the following characterization of contact by using $\mathcal{K}$-equivalence.

Theorem 5.2 *Let* X_i *and* Y_i, $i = 1, 2$, *be submanifolds of* $\mathbb{R}^n$ *with* $\dim X_1 = \dim X_2$ *and* $\dim Y_1 = \dim Y_2$. *Let* $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ *be immersion germs and* $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ *be submersion germs with* $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. *Then*

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

if and only if $f_1 \circ g_1$ *and* $f_2 \circ g_2$ *are* $\mathcal{K}$*-equivalent.*

Turning to lightlike hypersurfaces, we now consider the function

$$\mathcal{G} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}$$

defined by $\mathcal{G}(\boldsymbol{x}, \boldsymbol{\lambda}) = \langle \boldsymbol{x} - \boldsymbol{\lambda}, \boldsymbol{x} - \boldsymbol{\lambda} \rangle$. Given $\boldsymbol{\lambda}_0 \in \mathbb{R}^{n+1}_1$, we denote $\mathfrak{g}_{\lambda_0}(\boldsymbol{x}) = \mathcal{G}(\boldsymbol{x}, \boldsymbol{\lambda}_0)$, so that we have $\mathfrak{g}^{-1}_{\lambda_0}(0) = LC_{\lambda_0}$. For any $u_0 \in U$, we take the point $\boldsymbol{\lambda}^{\pm}_0 = \boldsymbol{X}(u_0) + t_0(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(u_0)$ and have

$$\mathfrak{g}_{\lambda^{\pm}_0} \circ \boldsymbol{X}(u_0)) = \mathcal{G} \circ (\boldsymbol{X} \times id_{\mathbb{R}^{n+1}_1})(u_0, \boldsymbol{\lambda}^{\pm}_0) = G(p_0, \boldsymbol{\lambda}^{\pm}_0) = 0,$$

where $p_0 = \boldsymbol{X}(u_0)$ and $t_0 = -1/\widetilde{\kappa}^{\mp}_i(u_0)$, $i = 1, \ldots, n-1$. We also have relations

$$\frac{\partial \mathfrak{g}_{\lambda^{\pm}_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial G}{\partial u_i}((p_0), \boldsymbol{\lambda}^{\pm}_0) = 0, \; i = 1, \ldots, n-1.$$

These imply that the lightcone $\mathfrak{g}^{-1}_{\lambda^{\pm}_0}(0) = LC_{\lambda^{\pm}_0}$ is tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$. In this case, we call each $LC_{\lambda^{\pm}_0}$ the *tangent lightcone* of $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$.

We now describe the contacts of spacelike surfaces with lightcones. Let $LH^{\sigma}_{M,i} : (U, u_i) \longrightarrow (LC^*_+, \boldsymbol{v}^{\sigma}_i)$, $i = 1, 2$, be two lightlike hypersurface germs of spacelike submanifold germs of codimension two $\boldsymbol{X}_i : (U, u_i) \longrightarrow (\mathbb{R}^{n+1}_1, p_i)$, where $\sigma = \pm$. We say that $LH^{\sigma}_{M,1}$ and $LH^{\sigma}_{M,2}$ are $\mathcal{A}$*-equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2))$ and $\Phi : (\mathbb{R}^{n+1}_1, \boldsymbol{\lambda}^{\sigma}_1) \longrightarrow (\mathbb{R}^{n+1}_1, \boldsymbol{\lambda}^{\sigma}_2)$ such that $\Phi \circ LH^{\sigma}_{M,1} = LM^{\sigma}_{M,2} \circ \phi$. If both of the regular sets of $LM^{\sigma}_{M,i}$ are dense in (U, u_i), it follows from Proposition A.2 of the appendix that $LH^{\sigma}_{M,1}$ and $LH^{\sigma}_{M,2}$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian lift germs are Legendrian equivalent. This condition is also equivalent to that two generating families G_1 and G_2 are P-$\mathcal{K}$-equivalent by Theorem A.3, where $G_i : (U \times \mathbb{R}^4_1, ((x_i, y_i), \boldsymbol{\lambda}^{\sigma}_i)) \longrightarrow \mathbb{R}$ denotes the Lorentzian distance-squared function germ of $\boldsymbol{X}_i$.

On the other hand, if we denote $g_{i,\lambda_i^\sigma}(u) = G_i(u, \boldsymbol{\lambda}_i^\sigma)$, then we have $g_{i,\lambda_i^\pm}(u) = \mathfrak{g}_{\lambda_i^\pm} \circ \boldsymbol{x}_i(u)$. By Theorem 5.2, $K(\boldsymbol{X}_1(U), LC_{\lambda_1^\sigma}, \boldsymbol{\lambda}_1^\sigma) = K(\boldsymbol{x}_2(U), LC\lambda_2^\sigma, \boldsymbol{\lambda}_2^\sigma)$ if and only if $\widetilde{g}_{1,\lambda_1}$ and $\widetilde{g}_{2,\lambda_2}$ are $\mathcal{K}$-equivalent. Therefore, we can apply Proposition A.4 to our situation. We denote by $Q^\sigma(\boldsymbol{X}, u_0)$ the local ring of the function germ $\widetilde{g}_{\lambda_0^\sigma} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{\lambda}_0^\sigma = LC_M^\sigma(u_0, t_0)$. We remark that we can explicitly write the local ring as follows:

$$Q^\pm(\boldsymbol{X}, u_0) = \frac{C_{u_0}^\infty(U)}{\langle \widetilde{\langle \boldsymbol{X}(u), \boldsymbol{n}^T \pm \boldsymbol{n}^S(u_0)\rangle} - 1 \rangle_{C_{u_0}^\infty(U)}},$$

where $C_{u_0}^\infty(U)$ is the local ring of function germs at u_0.

Theorem 5.3 *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (\mathbb{R}_1^{n+1}, \boldsymbol{X}_i(u_i))$, $i = 1, 2$, be spacelike surface germs such that the corresponding Legendrian lift germs are Legendrian stable. For $\sigma = +$ or $-$, the following conditions are equivalent:*

(1) *The lightlike hypersurface germs $LH_{M_1}^\sigma$ and $LH_{M_2}^\sigma$ are $\mathcal{A}$-equivalent.*
(2) *G_1 and G_2 are P-$\mathcal{K}$-equivalent.*
(3) *g_{1,λ_1} and g_{2,λ_2} are $\mathcal{K}$-equivalent.*
(4) *$K(\boldsymbol{X}_1(U), LC_{\lambda_1^\sigma}, \boldsymbol{\lambda}_1^\sigma) = K(\boldsymbol{X}_2(U), LC\lambda_2^\sigma, \boldsymbol{\lambda}_2^\sigma)$.*
(5) *$Q^\sigma(\boldsymbol{X}_1, u_1)$ and $Q^\sigma(\boldsymbol{X}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.*

Proof. The preceding arguments shows that (3) and (4) are equivalent. The other assertions follow from Proposition A.4. □

Given a spacelike submanifold germ of codimension two $\boldsymbol{X} : (U, u_0) \longrightarrow (\mathbb{R}_1^{n+1}, \boldsymbol{X}(u_0))$, we call $(\boldsymbol{X}^{-1}(\widetilde{LC}_{\lambda^\pm}), (u_0))$ the *tangent lightcone indicatrix germ* of $\boldsymbol{X}$, where $\boldsymbol{\lambda}^\pm = \boldsymbol{X}(u_0) + t_0(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(u_0)$ and $t_0 = -1/\widetilde{\kappa}_i^\mp(u_0)$, $i = 1, 2$. As a corollary of Theorem 5.3, we have

Corollary 5.4 *Under the assumptions of Theorem 5.3, if the lightlike hypersurface germs $LH_{M_1}^\sigma$ and $LH_{M_2}^\sigma$ are $\mathcal{A}$-equivalent, then tangent lightcone indicatrix germs*

$$(\boldsymbol{X}_1^{-1}(LC_{\lambda_1^\pm}), (u_1)) \quad \text{and} \quad (\boldsymbol{X}_2^{-1}(LC_{\lambda_2^\pm}), (u_2))$$

are diffeomorphic as set germs.

Proof. Notice that the tangent lightcone indicatrix germ of $\boldsymbol{X}_i$ is the zero level set of g_{i,λ_i}. Since $\mathcal{K}$-equivalence among function germs preserves the zero-level sets of function germs, the assertion follows from Theorem 5.3. □

6 Lightlike hypersurfaces in Lorentz-Minkowski four-space

In [7] generic singularities of lightlike hypersurfaces in $\mathbb{R}_1^4$ have been classified. We consider the space of spacelike embeddings $\mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}_1^4)$ with the Whitney C^∞-topology. We have shown the following theorem.

Theorem 6.1 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the Legendrian lift of the corresponding lightlike hypersurface $LH^{\pm}_M$ at each point is Legendrian stable.*

By the classification results on stable Legendrian mappings, we have the following

Corollary 6.2 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding lightlike hypersurfaces $LH^{\pm}_M$ at any point $(x, y, u) \in U \times \mathbb{R}$ is $\mathcal{A}$-equivalent to one of the map germs A_k $(1 \leq k \leq 4)$ or $D^{\pm}_4$: where, A_k, $D^{\pm}_4$-map germ $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^4, 0)$ are given by*

(A_1) $f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0)$,

(A_2) $f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3)$,

(A_3) $f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3)$,

(A_4) $f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_1, u_2)$,

(D^+_4) $f(u_1, u_2, u_3) = (2(u_1^2 + u_2^2) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3)$,

(D^-_4) $f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3)$.

By using the generic normal forms of generating families (i.e. Lorentzian distance squared functions) and Corollary 4.4, we have the following

Corollary 6.3 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding tangent lightcone indicatrix at any point $(x_0, y_0) \in U$ is diffeomorphic to one of the germs in the following list:*

(1) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 + y^2 = 0\}$ *(ordinary cusp)*

(2) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^4 \pm y^2 = 0\}$ *(tachnode or point)*

(3) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^5 + y^2 = 0\}$ *(rhamphoid cusp)*

(4) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 - xy^2 = 0\}$ *(three lines)*

(5) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 + y^3 = 0\}$ *(line)*

A Generating families

Here we give a quick survey on the theory of Legendrian singularities mainly developed by Arnol'd-Zakalyukin [1, 16]. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is *a Morse family* if the map germ

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is submersive, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $(n-1)$-dimensional submanifold

$$\Sigma_*(F) = \left\{(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \;\middle|\; F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \cdots : \frac{\partial F}{\partial x_n}(q, x)\right]\right)$$

is a Legendrian immersion. Then we have the following fundamental theorem in the theory of Legendrian singularities ([1] §20.7 [16], Page 27).

Proposition A.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F *a generating family* of Φ_F, and the corresponding wave front is $W(\Phi_F) = \pi_n(\Sigma_*(F))$, where $\pi_n : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the canonical projection.

We now introduce an equivalence relation among Legendrian immersion germs. Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L',p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibers of π and that $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined by the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

Proposition A.2 *Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L',p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i$ and $\pi \circ i'$ are dense respectively. Then i, i' are Legendrian equivalent if and only if wave front sets $W(i), W(i')$ are diffeomorphic as set germs. Here $\pi : PT^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the canonical projection of the projective cotangent bundle.*

This result has been firstly pointed out by Zakalyukin ([17], Assertion 1.1). In his original assertion, he assume that the representatives of $\pi \circ i$ and $\pi \circ i'$ are proper. However, we remark that we can get rid of such an assumption. The assumption in the above proposition is a generic condition for i, i'. In particular, if i and i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x,u) = (\psi_1(q,x), \psi_2(x))$ for $(q,x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *$\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{0\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{0\}, \ldots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{0\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [12].) The main result in the theory ([1], §20.8 and [16], THEOREM 2) is the following:

Theorem A.3 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Then*

(1) *Φ_F and Φ_G are Legendrian equivalent if and only if F, G are P-$\mathcal{K}$-equivalent, and*

(2) *Φ_F is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $F \mid \mathbb{R}^k \times \{0\}$.*

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, 0)$, we do not need the notion of stably P-$\mathcal{K}$-equivalences under this situation (cf. [16], Page 27). By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, we have the following classification result of Legendrian stable germs (cf. [7]). For any map germ $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, we define *the local ring of f* by $Q(f) = \mathcal{E}_n / f^*(\mathfrak{M}_p)\mathcal{E}_n$.

Proposition A.4 *Let F and $G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Suppose that Φ_F and Φ_G are Legendrian stable. The the following conditions are equivalent.*

(1) *$(W(\Phi_F), 0)$ and $(W(\Phi_G), 0)$ are diffeomorphic as germs.*

(2) *Φ_F and Φ_G are Legendrian equivalent.*

(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras,*

where $f = F|\mathbb{R}^k \times \{0\}, g = G|\mathbb{R}^k \times \{0\}$.

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