

PONTRYAGIN-THOM-SZÜCS TYPE CONSTRUCTION FOR NON-POSITIVE CODIMENSIONAL SINGULAR MAPS WITH PRESCRIBED SINGULAR FIBERS

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ABSTRACT. We give a Pontryagin-Thom-Szücs type construction for non-positive codimensional singular maps, and obtain results about cobordism and bordism groups of -1 codimensional stable maps with prescribed singular fibers.

1. INTRODUCTION

The purpose of this paper is to show many results about (co)bordisms of non-positive codimensional singular maps (if we have a map $f: M^m \rightarrow P^p$ of an m -dimensional manifold into a p -dimensional manifold, then the *codimension* of the map f is the integer $p-m$) with prescribed singular fibers by giving a Pontryagin-Thom-Szücs type construction in an analogous way to the case of positive codimensional singular maps and by finding analogous theorems and arguments.

The classical Pontryagin-Thom construction is an elementary method to study homotopy groups (or stable homotopy groups) of Thom-spaces of vector bundles via cobordisms of embeddings (resp. immersions), or on the contrary, study cobordisms via homotopy groups. Szücs extended the classical Pontryagin-Thom construction for cobordisms of immersions with restricted self-intersections [33], for cobordisms of singular maps with various types of singularities [32, 35] and in general for cobordisms of positive codimensional singular maps together with Rimányi [21]. Later it turned out that Pontryagin-Thom-Szücs type construction can be used effectively to study Thom polynomials [20] and eliminations of singularities by cobordism [37, 38].

In the case of codimension -1 , we obtained already some results and applications, e.g., computing the cobordism groups of fold maps with prescribed singular fibers in low dimensions and eliminating singular fibers by cobordism [13, 14, 15, 16].

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Recently several techniques have been developed in order to describe cobordisms of singular maps. Ando and Sadykov constructed spectra [1, 4, 23] by using the h-principle of Ando [2, 3], and Szűcs constructed spectra by using a compression theorem [38]. The main advantage of the Pontryagin-Thom-Szűcs type construction is that it also gives the possibility to handle in an elementary way cobordisms of singular maps with *global* restrictions like restrictions about the multiplicities or symmetries of singularities or singular or regular fibers.

In Section 2 we give basic definitions, in Section 3 we state our main results, in Section 4 we give the Pontryagin-Thom-Szűcs type construction for non-positive codimensional singular maps, in Section 5 we reduce the structure groups of -1 codimensional singularities to compact subgroups, in Section 6 we compute some cobordism groups of cusp maps.

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2. PRELIMINARIES

Notations. In this paper the symbol “ \amalg ” denotes the disjoint union. The symbol $T\xi^k$ denote the the Thom space of the bundle ξ^k . The symbol $\pi_n^s(X)$ (or π_n^s) denotes the n th stable homotopy group of the space X (resp. spheres). The symbol “ id_A ” denotes the identity map of the space A . The symbol ε denotes a small positive number, and Ω_n denotes the oriented cobordism group of closed oriented n -dimensional manifolds. All manifolds and maps are smooth of class C^∞ .

2.1. Smooth maps. In this paper, all smooth maps are supposed to be non-positive codimensional, proper and stable. We also suppose that a smooth map can be stratified by the singular strata, and the strata are smooth submanifolds of the target.

2.2. Bundle structure on a family of a map. Let $h: X^l \rightarrow \mathbb{R}^k$ be a smooth map of an l -dimensional manifold, where $l \geq k \geq 1$. Let \mathcal{G} be a subgroup of the *automorphism group* $\text{AUT}(h)$ of h , i.e., the group of pairs (α, β) , where $\alpha: X^l \rightarrow X^l$ and $\beta: \mathbb{R}^k \rightarrow \mathbb{R}^k$ are diffeomorphisms (suppose that β is also a linear transformation), and $h \circ \alpha = \beta \circ h$ holds. By the Milnor construction, we can construct the universal “ h bundle”

$$E\mathcal{G} \times_{\mathcal{G}} (h: X^l \rightarrow \mathbb{R}^k) \longrightarrow B\mathcal{G},$$

i.e., whose “total space” is the fiberwise map

$$E\mathcal{G} \times_{\mathcal{G}} X^l \xrightarrow{E\mathcal{G} \times_{\mathcal{G}} h} E\mathcal{G} \times_{\mathcal{G}} \mathbb{R}^k$$

denoted by $\chi: \xi \rightarrow \eta$ (the restriction of χ to any fiber is a map equivalent to h). Now, if we have a map $\varphi: W \rightarrow B\mathcal{G}$, then by pull-back, we obtain a family

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$$\begin{array}{ccc}
 \varphi^*\xi & \xrightarrow{\varphi^*\chi} & \varphi^*\eta \\
 & \searrow & \swarrow \\
 & W &
 \end{array}$$

of the map h parametrized by the space W . Moreover, because of the properties of the Milnor construction, this family is locally trivial and has structure group \mathcal{G} .

Conversely, if we have a family

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\tilde{\chi}} & \tilde{Y} \\
 & \searrow & \swarrow \\
 & W &
 \end{array}$$

of the map h parametrized by the space W , and this family is locally trivial and has structure group \mathcal{G} , then it can be induced from the universal h bundle $\chi \rightarrow B\mathcal{G}$.

Since in the case of Pontryagin-Thom-Szűcs type construction for non-positive codimension we want to induce families of maps from universal bundles obtained by the Milnor construction similarly to above, our families in hand should be locally trivial. Moreover, since we want to use transversality in the total space of the bundle $E\mathcal{G} \times_{\mathcal{G}} \mathbb{R}^k \rightarrow B\mathcal{G}$, the topological group \mathcal{G} should be compact or finite dimensional Lie group. In the case of positive codimension, see [10, 21, 34, 39].

2.3. Fiber-germs, singular fibers and τ -maps. Let $f: Q^q \rightarrow N^n$ be a smooth map between smooth manifolds of dimensions q and n respectively, $q \geq n \geq 1$, and $p \in Q^q$. The *fiber-germ* \mathfrak{F} over $f(p)$ is the map germ

$$f: (f^{-1}(U_{f(p)}), f^{-1}(f(p))) \rightarrow (U_{f(p)}, f(p)),$$

where $U_{f(p)}$ denotes a small neighbourhood of the point $f(p)$ (see “singular and regular fibers” in [26, Chapter 1]).

In this paper, we do not make difference between a fiber-germ \mathfrak{F} and its suspensions

$$\begin{aligned}
 f \times \text{id}_{\mathbb{R}^s}: & ((f \times \text{id}_{\mathbb{R}^s})^{-1}(U_{f(p)} \times \mathbb{R}^s), (f \times \text{id}_{\mathbb{R}^s})^{-1}(f(p) \times \{0\})) \rightarrow \\
 & (U_{f(p)} \times \mathbb{R}^s, f(p) \times \{0\}),
 \end{aligned}$$

$s > 0$, although they are not right-left equivalent. E.g., we will always suppose that a fiber-germ \mathfrak{F} is not a suspension.

In this paper, instead of a fiber-germ \mathfrak{F} (which is never a suspension by the previous paragraph), often we consider a representative $\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_{\varepsilon}^m$, where D_{ε}^m is a small m -dimensional disk around $f(p)$, $s_{\mathfrak{F}}$ denotes $f^{-1}(D_{\varepsilon}^m)$ and $\sigma_{\mathfrak{F}}$ denotes $f|_{f^{-1}(D_{\varepsilon}^m)}$ (and $\sigma_{\mathfrak{F}}$ is not right-left equivalent to a map of the form $\tilde{\sigma}_{\mathfrak{F}} \times \text{id}_{\mathbb{R}^s}$, $s > 0$). Let $\kappa(\mathfrak{F})$ denote m . We will refer to the representative $\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_{\varepsilon}^m$ as the fiber-germ itself.

If $p \in Q^q$ is a singular point of the map f , then we call the fiber-germ over $f(p)$ a *singular fiber*. If the fiber-germ has only regular points, then we call the fiber-germ over $f(p)$ a *regular fiber*, and the representative $\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_{\varepsilon}^m$ is chosen to be the map $f|_{f^{-1}(f(p))}: f^{-1}(f(p)) \rightarrow f(p)$ (i.e., $\kappa(\mathfrak{F}) = m = 0$).

Definition 2.1. Let $f: Q^q \rightarrow N^n$ be a smooth map. Let \mathfrak{F} be a fiber-germ of f and let $S_{\mathfrak{F}}$ denote the $\kappa(\mathfrak{F})$ codimensional submanifold of N^n over which the fiber-germs are equivalent to \mathfrak{F} . We say that the *local triviality condition* holds for the family of $\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_{\varepsilon}^{\kappa(\mathfrak{F})}$ parametrized by $S_{\mathfrak{F}}$ if it is a locally trivial bundle with fiber $\sigma_{\mathfrak{F}}$ and structure group $\text{AUT}(\sigma_{\mathfrak{F}})$. Furthermore, we say that the *compact structure group condition* holds for the family of $\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_{\varepsilon}^{\kappa(\mathfrak{F})}$ parametrized by $S_{\mathfrak{F}}$ if its structure group can be reduced to a compact subgroup of $\text{AUT}(\sigma_{\mathfrak{F}})$.

For example, a family of a regular fiber $\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D^0$ always satisfies the local triviality condition, and if the self-diffeomorphism group of the manifold $s_{\mathfrak{F}}$ can be reduced to a compact group (e.g. when $s_{\mathfrak{F}}$ is one dimensional), then it satisfies the compact structure group condition as well.

Definition 2.2. We say that a smooth map f is *locally trivial* if all of its fiber-germ families satisfy the local triviality condition. We say that a locally trivial smooth map f is *compact* if all of its fiber-germ families satisfy the compact structure group condition.

We introduce some notion which are analogues of “ τ -maps” of the positive codimensional case [21].

Definition 2.3. Let $\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_{\varepsilon}^{\kappa(\mathfrak{F})}$ be a fiber-germ. Let g_1 be an auto-diffeomorphism of $s_{\mathfrak{F}}$ and g_2 be an auto-diffeomorphism of $D_{\varepsilon}^{\kappa(\mathfrak{F})}$ such that $g_2 \circ \sigma_{\mathfrak{F}} = \sigma_{\mathfrak{F}} \circ g_1$. We call the pair (g_1, g_2) an *automorphism* of the fiber-germ $\sigma_{\mathfrak{F}}$. The *automorphism group* of $\sigma_{\mathfrak{F}}$ consists of this kind of pairs (g_1, g_2) . Let $\text{AUT}(\sigma_{\mathfrak{F}})$ denote the automorphism group of $\sigma_{\mathfrak{F}}$.

Hence $\text{AUT}(\sigma_{\mathfrak{F}})$ is a subgroup of the topological group $\text{DIFF}(s_{\mathfrak{F}}) \times \text{DIFF}(D_{\varepsilon}^{\kappa(\mathfrak{F})})$.

Definition 2.4. Let $\sigma_{\mathfrak{F}}$ be a fiber-germ and let \mathcal{G} be a subgroup of the automorphism group of $\sigma_{\mathfrak{F}}$. We call the pair $(\sigma_{\mathfrak{F}}, \mathcal{G})$ a *global fiber-germ*.

Definition 2.5. Let τ_0 and τ_1 be two sets of global fiber-germs. $\tau_0 \prec \tau_1$ if for every $(\sigma_{\mathfrak{F}}, \mathcal{G}_{\mathfrak{F}}) \in \tau_0$ there exists an element $(\sigma'_{\mathfrak{F}}, \mathcal{G}'_{\mathfrak{F}}) \in \tau_1$ such that $\sigma_{\mathfrak{F}} = \sigma'_{\mathfrak{F}}$ and $\mathcal{G}_{\mathfrak{F}}$ is a subgroup of $\mathcal{G}'_{\mathfrak{F}}$.

Definition 2.6. For a locally trivial map f let τ_f be the set of the global fiber-germs $(\sigma_{\mathfrak{F}}, \mathcal{G}_{\mathfrak{F}})$ where each $\sigma_{\mathfrak{F}}$ is a fiber-germ of f and the corresponding group $\mathcal{G}_{\mathfrak{F}}$ is the structure group of the bundle of this fiber-germ. We call the set τ_f the *global fiber-germ set of f* .

Definition 2.7. Let τ be a set of global fiber-germs. We say that the locally trivial map f is a τ -map if $\tau_f \prec \tau$.

2.4. Some -1 codimensional singular maps. Let Q^{n+1} and N^n be smooth manifolds of dimensions $n+1$ and n respectively. Let $p \in Q^{n+1}$ be a singular point of a smooth map $f: Q^{n+1} \rightarrow N^n$. The smooth map f has a *fold singularity* at the singular point p , if we can write f in some local coordinates at p and $f(p)$ in the form

$$f(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n-1}, x_n^2 \pm x_{n+1}^2).$$

A smooth map $f: Q^{n+1} \rightarrow N^n$ is called a *fold map*, if f has only fold singularities.

The smooth map f has a *cuspl singularity* at the singular point p , if we can write f in some local coordinates at p and $f(p)$ in the form

$$f(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n-1}, x_n^3 + x_1 x_n - x_{n+1}^2).$$

A smooth map $f: Q^{n+1} \rightarrow N^n$ is called a *cuspl map*, if f has only fold and cuspl singularities.

The possible regular and singular fibers of stable fold and cuspl maps for $n \leq 4$ are classified in [19, 26, 28, 41].

We will prove that every -1 codimensional stable map is locally trivial and compact.

2.5. Bordisms and cobordisms of fold maps with prescribed fiber-germs. In the following, we suppose that the singular maps are locally trivial.

Definition 2.8 (Cobordism). Let N^n be an n -dimensional manifold. Let τ and τ' be two sets of global fiber-germs. Two τ -maps f_0 and f_1 of closed q -dimensional manifolds Q_0^q and Q_1^q into N^n are τ' -cobordant, if

- (1) there exists a τ' -map $F: X^{q+1} \rightarrow N^n \times [0, 1]$ from a compact $(q+1)$ -dimensional manifold X^{q+1} ,
- (2) $\partial X^{q+1} = Q_0^q \amalg Q_1^q$,
- (3) $F|_{Q_0^q \times [0, \varepsilon]} = f_0 \times \text{id}_{[0, \varepsilon]}$ and $F|_{Q_1^q \times (1-\varepsilon, 1]} = f_1 \times \text{id}_{(1-\varepsilon, 1]}$, where $Q_0^q \times [0, \varepsilon)$ and $Q_1^q \times (1-\varepsilon, 1]$ are small collar neighbourhoods of ∂X^{q+1} with the identifications $Q_0^q = Q_0^q \times \{0\}$, $Q_1^q = Q_1^q \times \{1\}$.

We call the map F a τ' -cobordism between f_0 and f_1 .

This clearly defines an equivalence relation on the set of τ -maps of closed q -dimensional manifolds into N^n .

For two sets τ and τ' of global fiber-germs let us denote the cobordism classes of τ -maps under τ' -cobordisms by

$$\text{Cob}_{N, \tau, \tau'}(q, n - q)$$

($\text{Cob}_{N, \tau}(q, n - q)$ for $\tau = \tau'$).

When the target manifold N^n has the form $\mathbb{R}^1 \times M^{n-1}$, we can define a commutative semigroup operation on $\text{Cob}_{N, \tau, \tau'}(q, n - q)$ by the far away

disjoint union. In the case of $N^n = \mathbb{R}^n$, we have an abelian group, which we denote by $\text{Cob}_{\tau, \tau'}(q, n - q)$ ($\text{Cob}_{\tau}(q, n - q)$ for $\tau = \tau'$).

Definition 2.9 (Bordism). Let N_i^n ($i = 0, 1$) be two closed oriented n -dimensional manifolds. Let τ and τ' be two sets of global fiber-germs. Two τ -maps f_i ($i = 0, 1$) of closed q -dimensional manifolds Q_0^q and Q_1^q into N_0^n and N_1^n , respectively, are τ' -bordant, if

- (1) there exists a τ' -map $F: X^{q+1} \rightarrow Y^{n+1}$ of a compact $(q + 1)$ -dimensional manifold X^{q+1} to a compact oriented $(n + 1)$ -dimensional manifold Y^{n+1} ,
- (2) $\partial X^{q+1} = Q_0^q \amalg Q_1^q$ and $\partial Y^{n+1} = N_0^n \amalg -N_1^n$,
- (3) $F|_{Q_0^q \times [0, \varepsilon]} = f_0 \times \text{id}_{[0, \varepsilon]}$ and $F|_{Q_1^q \times (1 - \varepsilon, 1]} = f_1 \times \text{id}_{(1 - \varepsilon, 1]}$, where $Q_0^q \times [0, \varepsilon]$ and $Q_1^q \times (1 - \varepsilon, 1]$ are small collar neighbourhoods of ∂X^{q+1} with the identifications $Q_0^q = Q_0^q \times \{0\}$, $Q_1^q = Q_1^q \times \{1\}$.

We call the map F a τ' -bordism between f_0 and f_1 .

This clearly defines an equivalence relation on the set of τ -maps of closed q -dimensional manifolds into closed oriented n -dimensional manifolds.

For two sets τ and τ' of global fiber-germs let us denote the bordism classes of τ -maps under τ' -bordisms by $\mathcal{B}or_{\tau, \tau'}(q, n - q)$ ($\mathcal{B}or_{\tau}(q, n - q)$ for $\tau = \tau'$).

We define a commutative group operation on $\mathcal{B}or_{\tau, \tau'}(q, n - q)$ in the usual way by the disjoint union.

3. MAIN THEOREMS

The following theorem is the non-positive codimensional analogue of [21].

Theorem 3.1. *For every integers $q \geq n \geq 1$ and set of global fiber-germs τ , there is a Pontryagin-Thom-Szűcs type construction for locally trivial τ -maps of q -dimensional manifolds into n -dimensional manifolds, i.e.,*

- (1) *there exists a universal τ -map $\xi_{\tau}: U_{\tau} \rightarrow \Gamma_{\tau}$ such that¹ for every locally trivial τ -map $g: Q^q \rightarrow N^n$ there exists a commutative diagram*

$$\begin{array}{ccc} Q^q & \longrightarrow & U_{\tau} \\ g \downarrow & & \xi_{\tau} \downarrow \\ N^n & \longrightarrow & \Gamma_{\tau} \end{array}$$

moreover the arising map $N^n \rightarrow \Gamma_{\tau}$ is unique up to homotopy. It will be denoted by χ_g . The space Γ_{τ} is constructed by gluing together total spaces of vector bundles corresponding to the possible singular fibers in τ and their automorphisms.

¹The spaces U_{τ} and Γ_{τ} are not (finite dimensional) manifolds and so ξ_{τ} is not a smooth map.

(2) For every n -dimensional manifold N^n there is a natural map

$$\chi_*^N: \text{Cob}_{N^n, \tau}(q, n - q) \rightarrow [\dot{N}^n, \Gamma_\tau]$$

between the set of τ -cobordism classes $\text{Cob}_{N^n, \tau}(n + 1, -1)$ and the set of homotopy classes $[\dot{N}^n, \Gamma_{-1}]$. The map χ_*^N maps a τ -cobordism class $[g]$ of a locally trivial map g into the homotopy class of the inducing map $\chi_g: \dot{N}^n \rightarrow \Gamma_\tau$.

(3) If n, q, τ are such that the τ -maps and their cobordisms are not only locally trivial but also compact, then the natural map χ_*^N is a bijection.

Similarly, there is a natural map $\text{Bor}_\tau(q, n - q) \rightarrow \Omega_n(\Gamma_\tau)$, which is a bijection if the maps in hand are compact.

Definition 3.2. Let $\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_\varepsilon^{\kappa(\mathfrak{F})}$ be a singular fiber. We say that the singular fiber $\sigma_{\mathfrak{F}}$ is *weakly coorientable* if for every automorphism (g_1, g_2) of $\sigma_{\mathfrak{F}}$ the diffeomorphism g_2 preserves the orientation. (See similar notions for example in [26, 28]).

The following theorem can be obtained by using an analogue of the spectral sequence argument for example in [36] and Proposition 5.3. Let $q \geq n \geq 1$, $q - n = 1$, and τ be a set of global fiber-germs with maximal symmetry groups such that

- (1) every singular fiber has only fold singularities,
- (2) τ contains all the regular fibers and definite fold singular fibers, and
- (3) if \mathfrak{F} is a singular fiber in τ , then any singular fiber of the form

$$\mathfrak{F} \cup \text{some regular fibers} \cup \text{some definite fold singular fibers}$$

is in τ .

Theorem 3.3. *The rank of the homology group $H_n(\Gamma_\tau; \mathbb{Z})$ is equal to the rank of the n th homology group of the universal complex (see [26, 27]) for weakly coorientable singular fibers in τ . If p is an odd prime and $p > n$, then the homology group $H_n(\Gamma_\tau; \mathbb{Z})$ does not have any p -torsion. \square*

Since -1 codimensional fold maps are compact, we obtain the following.

Corollary 3.4. *If p is an odd prime and $p > n$, then the cobordism group $\text{Cob}_\tau(n + 1, -1)$ does not have any p -torsion.*

Corollary 3.5. *The rational bordism group $\text{Bor}_\tau(n + 1, -1) \otimes \mathbb{Q}$ is isomorphic to $\bigoplus_{k+l=n} H_k(\Gamma_\tau; \mathbb{Q}) \otimes \Omega_l$.*

Analogously to [14], we can apply the results of [31, 38] to the Pontryagin-Thom-Szűcs type construction for -1 codimensional stable maps with prescribed singular fibers as follows.

For a global singular fiber $(\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_\varepsilon^{\kappa(\mathfrak{F})}, \mathcal{G})$, let $\text{Imm}_N^{\mathcal{G}}(n - \kappa(\mathfrak{F}), \kappa(\mathfrak{F}))$ denote the cobordism group of immersions of $(n - \kappa(\mathfrak{F}))$ -dimensional manifolds into N^n with normal bundle induced from the universal bundle

$$E\mathcal{G} \times_{\mathcal{G}} D_\varepsilon^{\kappa(\mathfrak{F})}.$$

Let $q \geq n \geq 1$, $q - n = 1$, and τ be a set of global fiber-germs such that

- (1) every singular fiber has only stable singularities,
- (2) τ contains all the regular fibers and definite fold singular fibers, and
- (3) if \mathfrak{F}_1 and \mathfrak{F}_2 are fiber-germs in τ , then the fiber-germ $\mathfrak{F}_1 \cup \mathfrak{F}_2$ (with the appropriate symmetry group) is also in the set τ .

Let $\sigma_{\mathfrak{F}}$ be a connected singular fiber in τ , such that there is no connected singular fiber in τ whose boundary has the singular fiber \mathfrak{F} (i.e., $\sigma_{\mathfrak{F}}$ is a "top" singular fiber). Let τ_0 denote the set obtained from τ by leaving out the singular fiber $\sigma_{\mathfrak{F}}$ and all of its multi singular fibers.

Theorem 3.6. *There is a fibration*

$$\Gamma_{\tau_0} \hookrightarrow \Gamma_{\tau} \rightarrow \Gamma(T(EG \times_G D_{\varepsilon}^{\kappa(\mathfrak{F})})),$$

where $T(EG \times_G D_{\varepsilon}^{\kappa(\mathfrak{F})})$ denotes the Thom-space of the bundle $EG \times_G D_{\varepsilon}^{\kappa(\mathfrak{F})}$ and $\Gamma(T(EG \times_G D_{\varepsilon}^{\kappa(\mathfrak{F})}))$ denotes the classifying space for the immersions of $\kappa(\mathfrak{F})$ codimensional immersions with normal bundle induced from the bundle $EG \times_G D_{\varepsilon}^{\kappa(\mathfrak{F})}$. \square

Corollary 3.7. *Let $f: Q^{n+1} \rightarrow N^n$ be a τ -map. Then f is τ -cobordant to a τ_0 -map if and only if the immersion into N^n corresponding to the \mathfrak{F} -family of f is zero in the cobordism group $\text{Imm}^{\mathcal{G}}(n - \kappa(\mathfrak{F}), \kappa(\mathfrak{F}))$. \square*

4. THE CLASSIFYING SPACE FOR τ -MAPS

Proof of Theorem 3.1. Given a topological group G let $EG \rightarrow BG$ denote the Milnor construction for the universal G -bundle. For a global fiber-germ $(\sigma_{\mathfrak{F}}, \mathcal{G})$, let $\xi_{\mathfrak{F}}^{\mathcal{G}}$ denote the "total space" of the "bundle"

$$EG \times_G (\sigma_{\mathfrak{F}}: s_{\mathfrak{F}} \rightarrow D_{\varepsilon}^{\kappa(\mathfrak{F})}) \rightarrow BG,$$

i.e., the fiberwise map

$$(4.1) \quad EG \times_G s_{\mathfrak{F}} \rightarrow EG \times_G D_{\varepsilon}^{\kappa(\mathfrak{F})}.$$

Let us define the boundary $\partial \xi_{\mathfrak{F}}^{\mathcal{G}}$ of $\xi_{\mathfrak{F}}^{\mathcal{G}}$ as the map

$$EG \times_G (\partial \sigma_{\mathfrak{F}}: \partial s_{\mathfrak{F}} \rightarrow \partial D_{\varepsilon}^{\kappa(\mathfrak{F})})$$

of $\xi_{\mathfrak{F}}^{\mathcal{G}}$. This is a smooth map, the fiber-germs of which have codimension less than $\kappa(\mathfrak{F})$.

If $\sigma_{\mathfrak{F}}$ is a regular fiber, then the boundary $\partial \xi_{\mathfrak{F}}^{\mathcal{G}}$ is the empty set.

We can define a partial ordering on the set of global fiber-germs.

Definition 4.1. We say that the global fiber-germ $\mathfrak{a} = (\sigma_{\mathfrak{F}_0}, \mathcal{G}_0)$ is *less than* the global fiber-germ $\mathfrak{g} = (\sigma_{\mathfrak{F}_1}, \mathcal{G}_1)$ (denoted by $\mathfrak{a} < \mathfrak{g}$) if $\kappa(\mathfrak{F}_0) < \kappa(\mathfrak{F}_1)$, $\partial \xi_{\mathfrak{F}_1}^{\mathcal{G}_1}$ has a fiber-germ of type $\sigma_{\mathfrak{F}_0}$ and the structure group of the fiber-germ $\sigma_{\mathfrak{F}_0}$ in $\partial \xi_{\mathfrak{F}_1}^{\mathcal{G}_1}$ is a subgroup of the structure group \mathcal{G}_0 .

Note that for any group \mathcal{G}_1 the relation $(\sigma_{\mathfrak{F}_0}, \text{AUT}(\sigma_{\mathfrak{F}_0})) < (\sigma_{\mathfrak{F}_1}, \mathcal{G}_1)$ holds if and only if the boundary $\partial\sigma_{\mathfrak{F}_1}$ of the fiber-germ $\sigma_{\mathfrak{F}_1}$ has a fiber-germ of type $\sigma_{\mathfrak{F}_0}$.

Definition 4.2. We say that a global fiber-germ set τ is *closed* if for any global fiber-germ $(\sigma_{\mathfrak{F}}, \mathcal{G})$ in τ and for every fiber-germ $\sigma'_{\mathfrak{F}}$ of the map $\partial\xi_{\mathfrak{F}}^{\mathcal{G}}$ there exists a global fiber-germ $(\sigma'_{\mathfrak{F}}, \mathcal{G}')$ in τ such that $(\sigma'_{\mathfrak{F}}, \mathcal{G}') < (\sigma_{\mathfrak{F}}, \mathcal{G})$.

Note that for a map f the global fiber-germ set τ_f is closed.

Conversely, for a closed global fiber-germ set τ , we can construct by induction a map ξ_{τ} such that $\tau_{\xi_{\tau}}$ is equal to τ as follows.

The following construction of the map ξ_{τ} is very similar to the constructions of the classifying spaces of Szűcs [33, 35] and the universal map of Rimányi and Szűcs [21]. We suggest to study and understand the construction in [21] before reading the following definition. In our case the transversality and the uniqueness up to homotopy are included without mentioning.

Definition 4.3. Let τ be a closed global fiber-germ set. We construct the map ξ_{τ} by induction. Let ξ_{τ}^0 denote the disjoint union $\amalg_{\sigma_{\mathfrak{F}}} EG \times_{\mathcal{G}} \sigma_{\mathfrak{F}}$ where $(\sigma_{\mathfrak{F}}, \mathcal{G})$ runs over the global regular fibers in τ (i.e., $\sigma_{\mathfrak{F}}$ with $\kappa(\mathfrak{F}) = 0$).

Suppose that we have constructed the map ξ_{τ}^k , ($k \geq 0$). Then for every global fiber-germ $(\sigma_{\mathfrak{F}}, \mathcal{G})$ in τ with $\kappa(\mathfrak{F}) = k + 1$, the map $\partial\xi_{\mathfrak{F}}^{\mathcal{G}}$ has only global fiber-germs in $\tau_{\xi_{\tau}^k}$ because τ is closed. Because of the induction hypothesis, we have a glueing “map” $\varphi_{\mathfrak{F}}^{\mathcal{G}}: \partial\xi_{\mathfrak{F}}^{\mathcal{G}} \rightarrow \xi_{\tau}^k$ between the sources and targets of the maps $\partial\xi_{\mathfrak{F}}^{\mathcal{G}}$ and ξ_{τ}^k , which – restricted to each stratum of the map $\partial\xi_{\mathfrak{F}}^{\mathcal{G}}$ – is a bundle-map into the “total spaces” $\xi_{\mathfrak{F}'}^{\mathcal{G}'}$ (where $\xi_{\mathfrak{F}'}^{\mathcal{G}'} \subseteq \xi_{\tau}^k$) for every global fiber-germ $(\sigma_{\mathfrak{F}'}, \mathcal{G}')$ with $(\sigma_{\mathfrak{F}'}, \mathcal{G}') < (\sigma_{\mathfrak{F}}, \mathcal{G})$. Now, by using these glueing maps, we obtain the desired map $\xi_{\tau}^{k+1} = \xi_{\tau}^k \cup_{\varphi_{\mathfrak{F}}^{\mathcal{G}}} \xi_{\mathfrak{F}}^{\mathcal{G}}$, where $(\sigma_{\mathfrak{F}}, \mathcal{G})$ runs over the set of global fiber-germs in τ with $\kappa(\mathfrak{F}) = k + 1$. Let ξ_{τ} denote the limit $\xi_{\tau}^1 \subseteq \xi_{\tau}^2 \subseteq \dots \subseteq \xi_{\tau}^k \subseteq \dots$.

The map ξ_{τ} defined in Definition 4.3 has a target space Γ_{τ} , i.e., the target spaces

$$\Gamma_{\mathfrak{F}}^{\mathcal{G}} := EG \times_{\mathcal{G}} D_{\varepsilon}^{\kappa(\mathfrak{F})}$$

of the maps $\xi_{\mathfrak{F}}^{\mathcal{G}}$, see (4.1), glued together.

Proposition 4.4. *The map $\xi_{\tau}: U_{\tau} \rightarrow \Gamma_{\tau}$ is the Pontryagin-Thom-Szűcs type construction for Theorem 3.1.*

Proof. The homotopy groups of Γ_{τ} provide us the corresponding cobordism groups according to the Pontryagin-Thom-Szűcs type construction as follows.

Let $f: Q^q \rightarrow N^n$ be a τ -map. Then, we obtain a map $\mathfrak{S}: \dot{N}^n \rightarrow \Gamma_{\tau}$ as follows. Let $U_{\mathfrak{F}(f)}$ denote the small tubular neighbourhood of the f -image²

²In the case of a regular fiber let $U_{\mathfrak{F}(f)}$ denote only the f -image.

of the fiber-germs of type $\sigma_{\mathfrak{F}}$ in N^n . By a map $\mathfrak{S}_{\mathfrak{F}}: U_{\mathfrak{F}(f)} \rightarrow \Gamma_{\mathfrak{F}}^{\mathcal{G}}$, we induce the $\sigma_{\mathfrak{F}}$ bundle of f from the block $\Gamma_{\mathfrak{F}}^{\mathcal{G}}$ of Γ_{τ} for every global fiber-germ $(\sigma_{\mathfrak{F}}, \mathcal{G}')$ in τ_f , such that the regular neighbourhood $U_{\mathfrak{F}(f)}$ is induced from the target $EG \times_{\mathcal{G}} D_{\varepsilon}^{\kappa(\mathfrak{F})}$ of the map $\xi_{\mathfrak{F}}^{\mathcal{G}}$, and the preimage $f^{-1}(U_{\mathfrak{F}(f)})$ is induced from the source $EG \times_{\mathcal{G}} s_{\mathfrak{F}}$ of the map $\xi_{\mathfrak{F}}^{\mathcal{G}}$. Then, we glue together these maps $\mathfrak{S}_{\mathfrak{F}}: U_{\mathfrak{F}(f)} \rightarrow \Gamma_{\tau}$ into a map $\mathfrak{S}: N^n \rightarrow \Gamma_{\tau}$ by the generalized Pontryagin-Thom construction [21].

Conversely, let $\mathfrak{S}: N^n \rightarrow \Gamma_{\tau}$ be an element of $[N^n, \Gamma_{\tau}]$. If the structure groups of the global fiber-germs in τ are compact, then we can construct a τ -map of a closed q -dimensional manifold into N^n by glueing together the $\sigma_{\mathfrak{F}}$ bundles $\mathfrak{S}^*(\xi_{\mathfrak{F}}^{\mathcal{G}}) \rightarrow \mathfrak{S}^{-1}(\text{zero section of } \Gamma_{\mathfrak{F}}^{\mathcal{G}})$ induced from the $\sigma_{\mathfrak{F}}$ bundles $\xi_{\mathfrak{F}}^{\mathcal{G}} \rightarrow B\mathcal{G}$.

This can be applied for homotopies and cobordisms as well. Details are left to the reader. □

This completes the proof of Theorem 3.1. □

5. BUNDLE STRUCTURE ON -1 CODIMENSIONAL STABLE MAPS

Let $f: Q^{n+1} \rightarrow N^n$ be a stable map in general position. The map f can be considered as “locally trivial bundles of singularities” glued together. More precisely, we have the following theorems. The first is an analogue of [34] while the second is an analogue of the positive codimensional structure group reduction [10, 21, 39].

For each singular fiber $\sigma_{\mathfrak{F}}$ of the map f let $S_{\mathfrak{F}}$ denote the submanifold in N^n which is the f -image of singular fibers of type $\sigma_{\mathfrak{F}}$. Note that $S_{\mathfrak{F}}$ is an $(n - \kappa(\mathfrak{F}))$ -dimensional submanifold.

Let $P_{\mathfrak{F}}$ denote the total space of the disk bundle associated with the normal bundle of the submanifold $S_{\mathfrak{F}}$. The manifold $P_{\mathfrak{F}}$ is embedded into N^n in a natural way, onto a regular neighbourhood of $S_{\mathfrak{F}}$. Hence, we have the projection $\pi_{\mathfrak{F}}: P_{\mathfrak{F}} \rightarrow S_{\mathfrak{F}}$, and we also have the projection $f^{-1}(P_{\mathfrak{F}}) \rightarrow S_{\mathfrak{F}}$ defined by the map $\pi_{\mathfrak{F}} \circ f'$, where $f' = f|_{f^{-1}(P_{\mathfrak{F}})}$. Therefore, we have the commutative diagram

$$\begin{array}{ccc}
 f^{-1}(P_{\mathfrak{F}}) & \xrightarrow{f'} & P_{\mathfrak{F}} \\
 \searrow \pi_{\mathfrak{F}} \circ f' & & \swarrow \pi_{\mathfrak{F}} \\
 & S_{\mathfrak{F}} &
 \end{array}$$

which gives us a $\sigma_{\mathfrak{F}}$ -family denoted by $\xi_{\mathfrak{F}}(f)$, i.e., the “total space” of $\xi_{\mathfrak{F}}(f)$ is the fiberwise map $f': f^{-1}(P_{\mathfrak{F}}) \rightarrow P_{\mathfrak{F}}$ between the total spaces of the bundles $\pi_{\mathfrak{F}} \circ f': f^{-1}(P_{\mathfrak{F}}) \rightarrow S_{\mathfrak{F}}$ with fiber $s_{\mathfrak{F}}$ and $\pi_{\mathfrak{F}}: P_{\mathfrak{F}} \rightarrow S_{\mathfrak{F}}$ with fiber $J^{\kappa(\mathfrak{F})}$, the “base space” of $\xi_{\mathfrak{F}}(f)$ is $S_{\mathfrak{F}}$, and the “fiber” of $\xi_{\mathfrak{F}}(f)$ is right-left equivalent to the fiber-singularity $\sigma_{\mathfrak{F}}$.

Then the following theorem can be proved by an argument similar to that in [34].

Theorem 5.1. *Let $\sigma_{\mathfrak{F}}$ be a singular fiber of the map f . Then the family $\xi_{\mathfrak{F}}(f)$ is a locally trivial bundle over $S_{\mathfrak{F}}$ with the singular fiber $\sigma_{\mathfrak{F}}$ as fiber, and with structure group $\text{AUT}(\sigma_{\mathfrak{F}})$. \square*

Proof. Since the singularities are stable, there is an open covering of $S_{\mathfrak{F}}$ such that over each open set the family is trivial. By doing the same process as in [34], we receive the result. \square

Theorem 5.2. *Let $\sigma_{\mathfrak{F}}$ be a singular fiber of the map f . Then the structure group of the bundle $\xi_{\mathfrak{F}}(f)$ can be reduced to a compact group.*

Proof. Let $N(S)$ denote the neighbourhood of the singular set in $f^{-1}(P_{\mathfrak{F}})$. The restriction

$$\begin{array}{ccc} N(S) & \xrightarrow{f'} & P_{\mathfrak{F}} \\ & \searrow \pi_{\mathfrak{F}} \circ f' & \swarrow \pi_{\mathfrak{F}} \\ & & S_{\mathfrak{F}} \end{array}$$

of the bundle $\xi_{\mathfrak{F}}(f)$ is a bundle $\nu_{\mathfrak{F}}(f)$ of a multi-germ around finite number of isolated points, hence its structure group can be reduced to a maximal compact subgroup by [10, 39].

The restriction of the map f to the complement $f^{-1}(P_{\mathfrak{F}}) - N(S)$ is a submersion with one dimensional manifolds as fibers. Hence, clearly we can put a Riemannian metric on $f^{-1}(P_{\mathfrak{F}}) - N(S)$, which is invariant under the compact structure group of the bundle $\nu_{\mathfrak{F}}(f)$ on $(f^{-1}(P_{\mathfrak{F}}) - N(S)) \cap \text{cl}(N(S))$. Therefore the structure group $\text{AUT}(\sigma_{\mathfrak{F}})$ can be reduced to a compact group. \square

Note that we showed a symmetry group, which keeps fixed a Riemannian metric on $s_{\mathfrak{F}}$. Let us denote this group by $\text{ISO}(\sigma_{\mathfrak{F}})$. In [14], we obtained the following.

Proposition 5.3. *For a singular fiber $\sigma_{\mathfrak{F}}$ with only indefinite singular points and no circle components, the group $\text{ISO}(\sigma_{\mathfrak{F}})$ is finite, and if p is an odd prime with $p > \kappa(\mathfrak{F})$, then its order $|\text{ISO}(\sigma_{\mathfrak{F}})|$ cannot be divided by p .*

6. CUSP COBORDISMS IN LOW DIMENSIONS

By Theorems 3.1 and 3.6 and the previous section, we can compute some cobordism groups of -1 codimensional cusp maps. Let us denote the set of all global regular fibers and singular fibers with only fold or cusp singularities and with maximal structure groups by “c”, hence $\text{Cob}_c(n+1, -1)$ denotes the cobordism group of cusp maps of closed $(n+1)$ -dimensional manifolds into \mathbb{R}^n . By introducing orientations on the source manifolds of maps and cobordisms in Definition 2.8, we obtain the notion of oriented cobordism group of cusp maps, denoted by $\text{Cob}_c^O(n+1, -1)$. Our theorems clearly work for oriented cobordisms as well.

Proposition 6.1. *The cobordism group $Cob_c^O(3, -1)$ is isomorphic to \mathbb{Z}_2 . The homomorphism $\gamma: Cob_c^O(3, -1) \rightarrow Imm(1, 1)$, which maps a cobordism class $[f]$ into the cobordism class of the immersion of the singular set³ of f , is an isomorphism.*

Proof. By Theorems 3.1 and 3.6, we have a homotopy exact sequence for cobordism groups, and the homomorphism γ clearly does not vanish, since it can be non-zero on fold maps [13]. \square

Proposition 6.2. *The rank of the cobordism group $Cob_c^O(4, -1)$ is 2. The two \mathbb{Z} summands are generated by the algebraic number of the singular fiber of type III^a and the algebraic number of the singular fiber of type III^b (for the notations of the singular fibers, see [26]).*

Proof. By [16] and the exact sequence from Theorem 3.6, we obtain the result. \square

Let us denote the set of all the global regular fibers and singular fibers with only cusp or fold singularities which have at most one singular point in each of their connected components and with maximal structure groups by “sc”, hence $Cob_{sc}^O(n+1, -1)$ denotes the oriented cobordism group of simple cusp maps of closed $(n+1)$ -dimensional manifolds into \mathbb{R}^n .

Proposition 6.3. *The simple cusp cobordism group $Cob_{sc}^O(n+1, -1)$ is isomorphic to $\pi_{n-1}^s(\mathbb{R}P^\infty)$.*

Proof. By [15] and easy geometric constructions the summand π_{n-1}^s of the oriented simple fold cobordism group goes to zero under the natural homomorphism into the group $Cob_{sc}^O(n+1, -1)$. But by the exact sequence from Theorem 3.6, the kernel of this natural homomorphism cannot be bigger. \square

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³Cusp maps restricted to their singular sets are not immersions, but one can consider them as immersions by “smoothing” the cusps.

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