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A method to investigate $T\mathcal{A}(f)$, $T\mathcal{L}(f)$ and its applications

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Abstract. In this note, we propose a method to investigate $T\mathcal{A}(f)$ and $T\mathcal{L}(f)$ and show its applications.
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Key words: $T\mathcal{A}(f)$, $T\mathcal{L}(f)$, $\mathcal{A}$-determinacy, $\mathcal{L}$-determinacy, $\mathcal{A}$-simple map-germ, $\mathcal{L}$-simple map-germ, number of branches, corank at most one.

1 How to investigate $T\mathcal{A}(f)$ and $T\mathcal{L}(f)$

In singularity theory of differentiable mappings, the $\mathcal{A}$-equivalence relation is the most important equivalence relation and the $\mathcal{L}$-equivalence relation is also important if we want to deal with the images of map-germs rather than map-germs themselves, but investigating them are not easy in general. The main reason why it is not easy is in the fact that we have to deal with mixed homomorphisms of finite type over f^* defined in [12], where $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is a given map-germ. For instance, the standard filtration $\mathcal{E}_n = m_n^0 \supset m_n^1 \supset m_n^2 \supset \cdots \supset m_n^\infty \supset \{0\}$ of $\mathcal{E}_n$ is not well compatible with mixed homomorphisms of finite type over f^*. However, if we replace the filtration $\mathcal{E}_n = m_n^0 \supset m_n^1 \supset m_n^2 \supset \cdots \supset m_n^\infty \supset \{0\}$ of $\mathcal{E}_n$ with the filtration $\mathcal{E}_n = f^*m_p^0\mathcal{E}_n \supset f^*m_p^1\mathcal{E}_n \supset f^*m_p^2\mathcal{E}_n \supset \cdots \supset f^*m_p^\infty\mathcal{E}_n \supset \{0\}$ of $\mathcal{E}_n$, then we can deal with everything easily by just one time using of Malgrange preparation theorem, which is the key of our method.

More precisely, for any i, j, k (resp. i, k) $\in \{0, 1, \cdots, \infty\}$ in general it is impossible to obtain the inclusion $m_n^i\theta_S(f) \subset tf(m_n^j\theta_S(n)) + \omega f(m_p^k\theta_{\{0\}}(p))$ (resp. $m_n^i\theta_S(f) \subset \omega f(m_p^k\theta_{\{0\}}(p))$) from the only one inclusion $m_n^i\theta_S(f) \subset$

$tf(m_n^j\theta_S(n))+\omega f(m_p^k\theta_{\{0\}}(p))+m_n^{i+1}\theta_S(f)$ (resp. $m_n^i\theta_S(f)\subset \omega f(m_p^k\theta_{\{0\}}(p))+m_n^{i+1}\theta_S(f)$). As the map-germ with one variable

$$f(x)=(x^i,x^{i+1},\cdots,x^{2i-1},0,\cdots,0)\quad (2\le i\le p),$$

which appears naturally as a singularity of pedal curve produced by a non-singular dual curve ([16]), suggests, in order to obtain the desired inclusion $m_n^i\theta_S(f)\subset tf(\theta_S(n))+\omega f(\theta_{\{0\}}(p))$ from the inclusion $m_n^i\theta_S(f)\subset tf(\theta_S(n))+\omega f(\theta_{\{0\}}(p))+m_n^{i+1}\theta_S(f)$, we need in general i-tuples successive inclusions $m_n^k\theta_S(f)\subset tf(m_n^r\theta_S(n))+\omega f(m_p^\ell\theta_{\{0\}}(p))+m_n^{k+1}\theta_S(f)$ $(k=i,i+1,\cdots,2i-1)$. (for details on this topic, for instance see [8], [22]). For the definition of $\theta_S(f),\theta_S(n),\theta_{\{0\}}(p)$, tf and ωf, see §3.

On the other hand, if we replace $m_n^i\theta_S(f)$ with $f^*m_p^i\theta_S(f)$, then we have the following proposition 1, which is a special case of Mather's lemma on mixed homomorphisms of finite type over $f^*:\mathcal{E}_p\to\mathcal{E}_n\times\cdots\times\mathcal{E}_n$ (see, (1.12) of [12]).

Throughout this note, let $S=\{s_1,\cdots,s_r\}$ be a finite subset of $\mathbf{R}^n$ with r elements, $f:(\mathbf{R}^n,S)\to(\mathbf{R}^p,0)$ be a germ of a C^∞ mapping at S such that $f(S)=0$ and for any i $(1\le i\le r)$ let f_i be the restriction of f to $(\mathbf{R}^n,s_i)$ (called a *branch* of f). The integer r is called the *number of branches of* f. Let $\mathcal{E}_n$ (resp. $\mathcal{E}_p$) be the $\mathbf{R}$-algebra of C^∞ function-germs at the origin in $(\mathbf{R}^n,0)$ (resp. $(\mathbf{R}^p,0)$) with usual operations, and let m_n (resp. m_p) be the unique maximal ideal of $\mathcal{E}_n$ (resp. $\mathcal{E}_p$). We define $m_q^0=\mathcal{E}_q$. For a given map-germ $f:(\mathbf{R}^n,S)\to(\mathbf{R}{,}0)$ and a non-negative integer i we put

$$\begin{aligned}{}_iQ(f)^n &= \frac{f^*m_p^i\theta_S(n)}{f^*m_p^{i+1}\theta_S(n)}\\ {}_iQ(f)^p &= \frac{f^*m_p^i\theta_S(f)}{f^*m_p^{i+1}\theta_S(f)}.\end{aligned}$$

Then, ${}_iQ(f)^q$ is an $\mathcal{E}_n$-module.

Proposition 1 *Let $f:(\mathbf{R}^n,S)\to(\mathbf{R}^p,0)$ be a C^∞ map-germ such that the dimension of $Q(f)$ is finite and let i be a non-negative integer. Suppose that*

$${}_iQ(f)^p=\{[g]\in{}_iQ(f)^p\mid g\in tf(m_n^j\theta_S(n))+\omega f(m_p^k\theta_{\{0\}}(p))\}$$

for some $j,k\in\{0,1,\cdots,\infty\}$. Then, we have

$$f^*m_p^i\theta_S(f)\subset tf(m_n^j\theta_S(n))+\omega f(m_p^k\theta_{\{0\}}(p)).$$

For the definition of $Q(f)$, see §3. We have the same inclusion as proposition 1 for $\omega f(m_p^k\theta_{\{0\}}(p))$ if we replace $tf(m_n^j\theta_S(n)) + \omega f(m_p^k\theta_{\{0\}}(p))$ with $\omega f(m_p^k\theta_{\{0\}}(p))$. Since the assumption of proposition 1 is equivalent to say the following:

$$f^*m_p^i\theta_S(f) \subset tf(m_n^j\theta_S(n)) + \omega f(m_p^k\theta_{\{0\}}(p)) + f^*m_p^{i+1}\theta_S(f),$$

we see that $f^*m_p^i\theta_S(f)$ is easier to deal with than $m_n^i\theta_S(f)$.

From these results, we see that the following method seems to be useful. For $\mathcal{G} = \mathcal{R}$ or $\mathcal{A}$ or $\mathcal{L}$, put

$${}_iQ(f,\mathcal{G})^p = \{[g] \in {}_iQ(f)^p \mid g \in T\mathcal{G}(f)\},$$

where $[g] = g + f^*m_p^{i+1}\theta_S(f)$. We see that ${}_iQ(f)^p$ is finite dimensional if $Q(f)$ is finite dimensional. Thus, in the case that $\dim_{\mathbf{R}} Q(f) < \infty$ ${}_iQ(f)^p$ can be decomposed in the following way:

$${}_iQ(f)^p = {}_iQ(f,\mathcal{G})^p + V,$$

where V is a finite dimensional vector subspace of ${}_iQ(f)^p$.
The method which we would like to propose in this note is to obtain the smallest i such that ${}_iQ(f)^p = {}_iQ(f,\mathcal{G})^p$. By our method, we can expect to improve the situations of calculations in many cases. For instance, consider the following fencing curves due to Arnold ([1]):

$$f_{a,b}(x) = (x^5, x^6 + ax^8 + bx^9, x^7) \quad (a, b \in \mathbf{R}).$$

We see easily that

$$(1) \qquad {}_2Q(f_{a,b})^3 = \{[g] \in {}_2Q(f_{a,b})^3 \mid g \in \omega f_{a,b}(m_p^2\theta_{\{0\}}(p))\}.$$

Thus, we have that ${}_2Q(f_{a,b})^3 = {}_2Q(f_{a,b},\mathcal{L})^3$. On the other hand, it is easily seen that ${}_1Q(f_{a,b})^3 \neq {}_1Q(f_{a,b},\mathcal{L})^3$. Threfore, 2 is the smallest i such that ${}_iQ(f_{a,b})^3 = {}_iQ(f_{a,b},\mathcal{L})^3$, and we see that $f_{a,b}$ is 9-$\mathcal{L}$-determined for any $a, b \in \mathbf{R}$ and it suffices to calculate only inside the 15 dimensional vector space generated by monomials $x^5, \cdots, x^9$ to calculate

$$\dim_{\mathbf{R}} \frac{T\mathcal{C}(f_{a,b})}{T\mathcal{L}(f_{a,b})} = 6, \quad \dim_{\mathbf{R}} \frac{T\mathcal{K}(f_{a,b})}{T\mathcal{A}(f_{a,b})} = 1$$

and to obtain the property that

$$(2) \quad (\alpha x^8 + \beta x^9)\frac{\partial}{\partial Y} \notin T\mathcal{A}(f_{a,b}) \quad \text{for any } a, b, \alpha, \beta \text{ such that } 2a\beta \neq 3b\alpha.$$

By (2) we see that $f_{a,b}$ is not 8-$\mathcal{A}$-determined if $a \neq 0$.

Our method works well also in complex holomorphic category. Hence, all results in this note hold also in complex holomorphic category.

This note is organized in the following way. In §2, we gather results obtained so far by using our method introduced in this section. In §3, we prepare several notions and notations. For theorem 1 stated in §2, a sketch of proof is given in §4. For theorem 2 and proposition 2 (resp. theorem 3 and corollary 1) stated in §2, a sketch of proofs is given in §5 (resp. §6). §7 is devoted to give a proof of proposition 3 stated in §2. Propositons 4 and 5 stated in §2 are proved sketchily in §8. Finally, in §9 we give a short proof of assertion 1 stated in §2.

2 Applications

In this section, we gather results obtained so far by using our method introduced in §1.

Theorem 1 ([17]) *Any a-$\mathcal{A}$-determined singular curve-germ $f : (\mathbf{R}, 0) \to (\mathbf{R}^p, 0)$ $(p \geq 2)$ is $(4a^2 + 2a - 1)$-$\mathcal{L}$-determined.*

For the definition of order of $\mathcal{G}$-determinacy ($\mathcal{G} = \mathcal{A}$ or $\mathcal{L}$), see §3.

Theorem 1 is a partial answer to the problem of Wall which asks to obtain a function $\ell(p, n, a)$ such that a-$\mathcal{A}$-determinacy implies ℓ-$\mathcal{L}$-determinacy for a map-germ f with $p \geq 2n$ (p. 512 of [22]). Note that our estimate in theorem 1 is not effective in general since for instance the example $f(x) = (x^i, x^{i+1}, \cdots, x^{2i-1}, 0, \cdots, 0)$ $(2 \leq i \leq p)$ given in §1 is $(2i-1)$-$\mathcal{G}$-determined but not $(2i-2)$-$\mathcal{G}$-determined for both of $\mathcal{G} = \mathcal{A}$ and $\mathcal{G} = \mathcal{L}$. On the other hand, note also that our function $\ell(p, 1, a)$ in theorem 1 does not depend on p. The author does not know whether or not this is a particular phenomenon which occurs only when $n = 1$.

Theorem 2 ([18]) *Let $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ $(n \leq p, np \neq 1)$ be an $\mathcal{A}$-simple map-germ with corank at most one. Then, the following inequality holds:*

$$\dim_{\mathbf{R}} Q(f) \leq \frac{p^2 + (n-1)r}{n(p-n) + (n-1)}.$$

For the definition of $Q(f)$, see §3.

Note that there are no upper bounds for $\dim_{\mathbf{R}} Q(f)$ in the case that $n = p = 1$ since for any positive integer δ the map-germ $f(x) = x^\delta$ is $\mathcal{A}$-simple and of corank at most one.

Theorem 3 ([18]) *Let $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$, $(n < p)$ be a C^∞ map-germ. Suppose that f is $\mathcal{A}$-simple. Then, the number of branches r is restricted in the following way:*

$$r < \frac{p^2}{n(p-n)}.$$

For the definitions of an $\mathcal{A}$-simple map-germ and a map-germ with corank at most one, see §3.

Note that there are no upper bounds for r in the case that $n = p$ since for any positive integer r a smooth finite covering with r fibers gives an example of $\mathcal{A}$-simple map-germ in the case that $n = p$. Note also that since $r \leq \dim_{\mathbf{R}} Q(f)$ the inequality $r \leq \frac{p^2}{n(p-n)}$ for an $\mathcal{A}$-simple map-germ with corank at most one can be obtained from theorem 2 as an immediate corollary . Thus, the point of theorem 3 is the sharpness of the inequality.

Since the left hand side of the inequality in theorem 3 is an integer while the right hand side is a rational number, the sharp inequality in theorem 3 suggests that there exists some special restrictions for the number of branches of an $\mathcal{A}$-simple map-germ when the right hand side is an integer. The rational number $\frac{p^2}{n(p-n)}$ can be an integer only when $p = 2n$ and in the case it attains its minimal value 4. Thus, we may guess that the classical cross ratio and the symplectic cross ratio ([19]) are the very invariants of special restrictions for the number of branches of an $\mathcal{A}$-simple map-germ .

It seems interesting also to compare theorem 2 with theorem 3 when the right hand side of the inequality in theorem 3 is an integer. The rational number $\frac{p^2+(n-1)r}{n(p-n)+(n-1)}$ for $p = 2n$, $r < 4$ can be an integer only when $n = 1$ and in this case it attains its maximal value 4. Although there are no $\mathcal{A}$-simple map-germs $f : (\mathbf{R}^n, S) \to (\mathbf{R}^{2n}, 0)$ with $r = 4$ by theorem 3, for instance map-germs $x \mapsto (x^4, x^5 + x^7)$ (taken from [3]), $\{x \mapsto (x, 0), x \mapsto (x^3, x^4)\}$ and $\{x \mapsto (x, 0), x \mapsto (0, x), x \mapsto (x^2, x^3)\}$ (these two are taken from [10]) give examples of $\mathcal{A}$-simple map-germs satisfying $\dim_{\mathbf{R}} Q(f) = 4$ in the case that $(n, p) = (1, 2)$. In particular, we can not expect the sharpness for the inequality of theorem 2.

Not only in the case above, the upper bound for $\dim_{\mathbf{R}} Q(f)$ given in theorem 1 is the best possible bound in the classification results of $\mathcal{A}$-simple map-germs listed here ([5], [6], [7], [9], [10], [11], [15], [20], [23]), and the upper bound for r is also the best possible bound in the classification results ([6], [10], [23]). However, if $n = r = 1$ and p is greater than 5, then the upper bound in theorem 1 is not the best estimate since the effect of $\mathcal{A}$-moduli sets in $\mathcal{K}$-simple orbits can not be disregarded as shown in [1].

For corresponding results on $\mathcal{L}$-simple singularities, we have the following which can be obtained easily.

Proposition 2 ([18]) *Let* $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ $(n \leq p, np \neq 1)$ *be an* $\mathcal{L}$*-simple map-germ with corank at most one. Then, the following inequality holds:*

$$\dim_{\mathbf{R}} Q(f) \leq \frac{p}{n}.$$

Corollary 1 ([18]) *Let* $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$, $(n \leq p)$ *be a* C^∞ *map-germ. Suppose that* f *is* $\mathcal{L}$*-simple. Then, the number of branches* r *is restricted in the following way:*

$$r \leq \frac{p}{n}.$$

Proposition 2 shows that if $n \leq p < 2n$ then any $\mathcal{L}$-simple map-germ with corank at most one must be an immersive mono-germ (i.e. an immersion germ with only one branches), and we can not expect to improve corollary 1 to hold the sharp inequality $r < \frac{p}{n}$. Furthermore, if $p = 2n$ then for instance the map-germ with three branches $\{f_1, f_2, f_3\}$ given by the following is not $\mathcal{L}$-simple though it is $\mathcal{A}$-simple (for the property that this map-germ is $\mathcal{A}$-simple, see proposition 5 below).

$$\begin{aligned} f_1(x_1, \cdots, x_n) &= (x_1, \cdots, x_n, 0, \cdots, 0), \\ f_2(x_1, \cdots, x_n) &= (0, \cdots, 0, x_1, \cdots, x_n), \\ f_3(x_1, \cdots, x_n) &= (x_1, \cdots, x_n, a_1x_1, \cdots, a_nx_n) \quad (a_1 \cdots a_n \neq 0). \end{aligned}$$

Thus we see that $\mathcal{L}$-simple singularities are quite restricted although $\mathcal{A}$-simple singularities are not so.

From the calcurations in §1 by using our method, for the Arnold's fencing curves $f_{a,b}$ we can see easily the following.

Proposition 3 *For any* $g : (\mathbf{R}, 0) \to (\mathbf{R}^p, 0)$ $(p \geq 3)$ *with* $\dim_{\mathbf{R}} Q(g) \geq 7$, *we see that* g *is adjacent to the set* $\cup_{(a,b)\neq(0,0)}\mathcal{A}(f_{a,b})$.

By proposition 3 we can say that the union $\cup_{(a,b)\neq(0,0)}\mathcal{A}(f_{a,b})$ dominates the set of singular curve-germs with multiplicities ≥ 7.

Next, we would like to investigate the exsistence of an $\mathcal{A}$-simple map-germ which is not $\mathcal{L}$-simple.

Proposition 4 ([18]) *Let* p *be an integer greater than* 1 *and let* $f : (\mathbf{R}, S) \to (\mathbf{R}^p, 0)$ *be an immersion such that* $\sum_{i=1}^{r} j^1 f_i(s_i)(\mathbf{R}) = \mathbf{R}^p$, *where the* 1*-jet* $j^1 f_i(s_i)$ *is regarded as a linear mapping. Then, we have the following:*

1. *Suppose that* $r = p$. *Then,* f *is* $\mathcal{L}$*-simple.*

2. *Suppose that* $r = p + 1$. *Then,* f *is* $\mathcal{A}$*-simple.*

3. *Suppose that $r \geq p+2$. Then, f is not $\mathcal{A}$-simple.*

Note that under the situation of proposition 4, f is not $\mathcal{L}$-simple if $r = p+1$ by corollary 1 and thus an f given in proposition 4 in the case that $r = p+1$ is an $\mathcal{A}$-simple map-germ which is not $\mathcal{L}$-simple.

Proposition 5 **([18])** *Let $f : (\mathbf{R}^n, S) \to (\mathbf{R}^{2n}, 0)$ be an immersion such that f_i is transverselly intersecting with f_j for any i, j $(1 \leq i, j \leq r,\ i \neq j)$.*

1. *Suppose that $r = 2$. Then, f is $\mathcal{L}$-simple.*

2. *Suppose that $r = 3$. Then, f is $\mathcal{A}$-simple.*

3. *Suppose that $r \geq 4$. Then, f is not $\mathcal{A}$-simple.*

Note that under the situation of proposition 5, f is not $\mathcal{L}$-simple if $r = 3$ by corollary 1 and thus an f given in proposition 5 in the case that $r = 3$ is an $\mathcal{A}$-simple map-germ which is not $\mathcal{L}$-simple.

さらに, このノートで説明している方法を使えば, 以下のように, 中山の補題の特別な場合もマルグランジュの予備定理を使って容易に示すことができる. 特別な場合といっても, 可微分写像の特異点論に登場する中山の補題は大抵は以下の形なので, 可微分写像の特異点論においては中山の補題は必要不可欠な道具とは言い難いようにも思える.

Assertion 1 M は, $M \subset m_n$ を満たす, $\mathcal{E}_n$ の有限生成イデアルとする. C は, $\dim C/MC < \infty$ であるような, 有限生成 $\mathcal{E}_n$-加群とする. さらに, A を C の $\mathcal{E}_n$-部分加群とする. そのとき、以下が成り立つ.

$$C \subset A + MC \Rightarrow C \subset A.$$

3 Notions and notations

Most notions and notations introduced in this section are due to Mather ([12], [13]) and already common in singularity theory of differentiable mappings. For details of them, we recommend an excellent survey [22] to the readers.

For a C^∞ map-germ $g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, T)$, where S (resp. T) is a finite subset of $\mathbf{R}^n$ (resp. $\mathbf{R}^p$), let $\theta_S(g)$ be the $\mathcal{E}_n$-module of germs of vector fields along g. We may identify $\theta_S(g)$ with $\underbrace{\mathcal{E}_n^p \times \cdots \times \mathcal{E}_n^p}_{r \text{ tuples}}$. We put $\theta_S(n) = \theta_S(id._{\mathbf{R}^n})$ and $\theta_T(p) = \theta_T(id._{\mathbf{R}^p})$, where $id._{\mathbf{R}^n}$ is the identity map-germ of $(\mathbf{R}^n, S)$ and $id._{\mathbf{R}^p}$ is the identity map-germ of $(\mathbf{R}^p, T)$. For any $k \in \{0, 1, \cdots, \infty\}$, an element of $m_n^k \theta_S(n)$ or $m_p^k \theta_{\{0\}}(p)$ is a map-germ such that the the Taylor polynomial of degree $(k-1)$ is zero. For a given C^∞ map-germ $f:(\mathbf{R}^n,S)\to(\mathbf{R}^p,0)$, Mather's two homomorphisms tf (tf is an $\mathcal{E}_n$-homomorphism) and ωf (ωf is an $\mathcal{E}_p$-homomorphism via f^*) is defined as follows.

$$tf:\theta_S(n)\to\theta_S(f),\qquad tf(a)=df\circ a,$$
$$\omega f:\theta_{\{0\}}(p)\to\theta_S(f),\qquad \omega f(b)=b\circ f,$$

where df is the differential of f. We put

$$\begin{aligned}
T\mathcal{R}(f) &= tf(m_n\theta_S(n)),\\
T\mathcal{L}(f) &= \omega f(m_p\theta_{\{0\}}(p)),\\
T\mathcal{C}(f) &= f^*m_p\theta_S(f),\\
T\mathcal{A}(f) &= tf(m_n\theta_S(n))+\omega f(m_p\theta_{\{0\}}(p)),\\
T\mathcal{K}(f) &= tf(m_n\theta_S(n))+f^*m_p\theta_S(f).
\end{aligned}$$

For a given C^∞ map-germ $f:(\mathbf{R}^n,S)\to(\mathbf{R}^p,0)$ we define

$$Q(f)=\frac{\mathcal{E}_n}{f_1^*m_p\mathcal{E}_n}\times\cdots\times\frac{\mathcal{E}_n}{f_r^*m_p\mathcal{E}_n},$$

whre recall that f_i is the restriction of f to $(\mathbf{R}^n,s_i)$ and $S=\{s_1,\cdots,s_r\}$. The dimension $\dim_{\mathbf{R}}Q(f)$ is called the *multiplicity* of f and denoted by $\delta(f)$. Note that $Q(f)^n$ (resp. $Q(f)^p$) may be identified with $\theta_S(n)/f^*m_p\theta_S(n)$ (resp. $\theta_S(f)/f^*m_p\theta_S(f)$). For a given map-germ f such that $\delta(f)<\infty$, Wall's homomorphism of $Q(f)$-modules ([22]) is the following:

$$\bar{t}f:Q(f)^n\to Q(f)^p,\quad \bar{t}f([g])=[tf(g)],$$

where $[g]=g+f^*m_p\theta_S(n)$ and $[tf(g)]=g+f^*m_p\theta_S(f)$. Let $\gamma(f)$ be the dimension of the kernel of $\bar{t}f$.

For a given C^∞ map-germ $f:(\mathbf{R}^n,S)\to(\mathbf{R}^p,0)$ such that $\delta(f)<\infty$, we put ${}_i\delta(f)=\frac{1}{p}\dim_{\mathbf{R}}{}_iQ(f)^p$. Note that ${}_i\delta(f)$ must be an integer. Let ${}_i\gamma(f)$ be the dimension of the kernel of the following homomorphism of $Q(f)$-modules.

$${}_i\bar{t}f:{}_iQ(f)^n\to{}_iQ(f)^p,\quad {}_i\bar{t}f([g])=[tf(g)].$$

Then, we see easily that $\delta(f)\le{}_i\delta(f)\le p^i\delta(f)$, and thus ${}_i\delta(f)<\infty$ if $\delta(f)<\infty$. Similarly $\gamma(f)\le{}_i\gamma(f)\le p^i\gamma(f)$. Note that ${}_iQ(f)$ is not isomorphic to ${}_iQ(F)$, where F is an unfolding of f. However, we see easily that ${}_1\delta(F)=(1+q)\,{}_1\delta(f)$ and ${}_1\gamma(F)=(1+q)\,{}_1\gamma(f)$, where q is the number of parameters for the unfolding F.

The Taylor polynomials of degree k at all points of S for a map-germ $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ is called *k jet of f at S* and is denoted by $j^k f(S)$. We put

$$J^k(n,p) = \{j^k f(0) \mid f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)\}.$$

Two map-germs $f, g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ are said to be $\mathcal{A}$-equivalent if there exist germs of C^∞ diffeomorfisms $\varphi : (\mathbf{R}^n, S) \to (\mathbf{R}^n, S)$ such that $\varphi(s_j) = s_j$ for $S = \{s_1, \cdots, s_k\}$ ($s_i \neq s_j$ if $i \neq j$) and $\psi : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $f = \psi \circ g \circ \varphi^{-1}$. For a C^∞ map-germ f $\mathcal{A}$-equivalence class of it is denoted by $\mathcal{A}(f)$. Two map-germs $f, g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ are said to be $\mathcal{L}$-equivalent if there exists a germ of C^∞ diffeomorfism $\psi : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $f = \psi \circ g$. For a C^∞ map-germ f $\mathcal{L}$-equivalence class of it is denoted by $\mathcal{L}(f)$. A C^∞ map-germ $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ is said to be k-$\mathcal{A}$-determined (resp. k-$\mathcal{L}$-determined) if f is $\mathcal{A}$-equivalent (resp. $\mathcal{L}$-equivalent) to any g with $j^k f(0) = j^k g(0)$.

Next we define jet space suitable for multi-germs $(\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ and their equivalence classes, which is the following multi-jet space:

$${}_rJ^k(n,p) = \{(j^k f_1(s_1), \cdots, j^k f_r(s_r)) \mid f_1(s_1) = \cdots = f_r(s_r), s_i \neq s_j \text{ if } i \neq j\}.$$

For a C^∞ map-germ $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ where $S = \{s_1, \cdots, s_r\}$ ($s_i \neq s_j$ if $i \neq j$), the quotient space $\frac{m_n \theta_S(f)}{m_n^{k+1}\theta_S(f)}$ can be identified with the multi-jet space ${}_rJ^k(n,p)$. Under this identification we put

$$\begin{aligned}
T\mathcal{R}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{R}(f)\}, \\
T\mathcal{L}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{L}(f)\}, \\
T\mathcal{C}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{C}(f)\}, \\
T\mathcal{A}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{A}(f)\}, \\
T\mathcal{K}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{K}(f)\},
\end{aligned}$$

where $[g] = g + m_n^{k+1}\theta_S(f)$. These are tangent spaces to orbits of actions of well-defined Lie groups corresponding to Mather's groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{C}$, $\mathcal{A}$ and $\mathcal{K}$. (for details, see [22]).

We recall the definitions of $\mathcal{A}$-simple map-germ and $\mathcal{L}$-simple map-germ. A map-germ $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ is said to be *$\mathcal{A}$-simple* (resp. *$\mathcal{L}$-simple*) if there exists a finite number of $\mathcal{A}$-equivalence classes (resp. $\mathcal{L}$-equivalence classes) such that for any positive integer d and any C^∞ mapping $F : U \to V$ where $U \subset \mathbf{R}^n \times \mathbf{R}^d$ is a neighbourhood of $S \times 0$, $V \subset \mathbf{R}^p \times \mathbf{R}^d$ is a neighbourhood of $(0,0)$, $F(x,\lambda) = (f_\lambda(x), \lambda)$ and the germ of f_0 at S is f, there exists a sufficiently small neighbourhood $W_i \subset U$ of $(s_i, 0)$ $(1 \leq i \leq r)$ such that for every $\{(x_1, \lambda), \cdots, (x_r, \lambda)\}$ with $(x_i, \lambda) \in W_i$ and $F(x_1, \lambda) =$

$\cdots = F(x_r, \lambda)$ the map-germ $f_\lambda : (\mathbf{R}^n, \{x_1, \cdots, x_r\}) \to (\mathbf{R}^p, f_\lambda(x_i))$ lies in one of these finite $\mathcal{A}$-equivalence classes.

Finally, a C^∞ map-germ $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ is said to be of *corank at most one* if $\max\{n - \mathrm{rank} Jf_i(s_i) \mid 1 \le i \le r\} \le 1$ holds, where recall that $S = \{s_1, \cdots, s_r\}$ ($s_i \neq s_j$ if $i \neq j$) is a finite set of $\mathbf{R}^n$, and f_i is the restriction of f to $(\mathbf{R}^n, s_i)$ and $Jf_i(s_i)$ is the Jacobian matrix of f_i at s_i.

4 Sketch of proof of theorem 1

In this section, we give a sketch of proof of theorem 1 given in [17]. Since theorem 1 concerns only a mono-germ (that is to say, a map-germ such that the number of branches r is 1), for the sake of clearness in this section we use the simplified notations $\theta(f)$, $\theta(n)$ and $\theta(p)$ instead of $\theta_S(f)$, $\theta_S(n)$ and $\theta_{\{0\}}(f)$.

By using appropriate coordinate transformations, from the first we may assume that f has the following form:

$$f(x) = (x^\delta, f_2(x), \cdots, f_p(x)),$$

where $\delta = \delta(f)$ and $f_j(x) = o(x^\delta)$ for $j = 2, \cdots, p$. Since f is a-$\mathcal{A}$-determined, there exists a positive integer k ($k \ge 2$) such that $a + 1 \le k\delta \le a + \delta$ and

$$T\mathcal{A}(f) \supset m_1^{k\delta}\theta(f) = f^* m_p^k \theta(f).$$

By Sylvester's duality on Frobenius number ([21], see also [1, 2]. for a complete proof of Sylvester's duality, see the comment to problem 1999-8 of [2]) we see that for any integer $c \ge (k\delta - 1)k\delta$ there exist non-negative integers ℓ_1, ℓ_2 such that $c = \ell_1 k\delta + \ell_2(k\delta + 1)$. Therefore, in the case that there exists a j ($2 \le j \le p$) such that the order of $f_j(x)$ is $\delta + 1$, we have that

$${}_{(k\delta-1)k}Q(f)^p = \{[g] \mid g \in \omega f(m_p^2 \theta(p))\}.$$

Next, we consider the case that $f_j(x) = o(x^{\delta+1})$ for any j ($2 \le j \le p$).

Lemma 1 *If $f_j(x) = o(x^{\delta+1})$ for any j ($2 \le j \le p$), then ${}_kT\mathcal{R}(f) \subset V$,* *where V is the vector subspace of ${}_kQ(f)^p$ spanned by the following vectors:*

$$\begin{array}{lllll}
[x^{k\delta}\frac{\partial}{\partial X_1}], & [x^{k\delta+1}\frac{\partial}{\partial X_1}], & [x^{k\delta+2}\frac{\partial}{\partial X_1}], & \cdots, & [x^{k\delta+(\delta-1)}\frac{\partial}{\partial X_1}], \\
[x^{k\delta}\frac{\partial}{\partial X_2}], & & [x^{k\delta+2}\frac{\partial}{\partial X_2}], & \cdots, & [x^{k\delta+(\delta-1)}\frac{\partial}{\partial X_2}], \\
\vdots & & \vdots & \vdots & \vdots \\
[x^{k\delta}\frac{\partial}{\partial X_p}], & & [x^{k\delta+2}\frac{\partial}{\partial X_p}], & \cdots, & [x^{k\delta+(\delta-1)}\frac{\partial}{\partial X_p}].
\end{array}$$

For the proof of lemma 1, see [17]. By lemma 1, even in the case that $f_j(x) = o(x^{\delta+1})$ for any j $(2 \leq j \leq p)$ we have

$${}_{(k\delta-1)k}Q(f)^p = \{[g] \mid g \in \omega f(m_p^2\theta(p))\}.$$

Thus, we have

$$\omega f(m_p^2\theta(p)) \supset f^* m_p^{(k\delta-1)k}\theta(f) = m_1^{(k\delta-1)k\delta}\theta(f).$$

Since $a+1 \leq k\delta \leq a+\delta$ we have

$$m_1^{(k\delta-1)k\delta}\theta(f) \supset m_1^{(a+\delta-1)(a+\delta)}\theta(f).$$

We see easily that $\delta \leq a+1$. Thus, we have

$$\omega f(m_p^2\theta(f)) \supset m_1^{2(a-1)(2a-1)}\theta(f).$$

Put $\ell = 2r(2r+1) - 1 = 4r^2 + 2r - 1$ and let g be a C^∞ map-germ such that $j^\ell f(0) = j^\ell g(0)$. Then, as in [4], since $(g-f) \in m_1^{\ell+1}\theta(f)$, from the above inclusion we see that there exists a germ of C^∞ diffeomorphisms $h : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $g = h \circ f$. □

5 Sketch of proofs of theorem 2 and proposition 2

In this section, we give a sketch of proofs of theorem 2 and proposition 2 given in [18].

Lemma 2 *Let $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ such that $\delta(f) < \infty$.*

1. *Suppose that $T\mathcal{C}(f) = T\mathcal{L}(f)$. Then, the following inequality holds:*

$$ {}_1\delta(f) \leq p. $$

2. *Suppose that $T\mathcal{K}(f) = T\mathcal{A}(f)$. Then, the following inequality holds:*

$$ (p-n)\,{}_1\delta(f) + {}_1\gamma(f) - \gamma(f) \leq p^2. $$

For the proof of lemma 2, see [18].

[*Sketch of proofs of theorem 2 and proposition 2*] If f is $\mathcal{A}$-simple (resp. $\mathcal{L}$-simple), there must exist an $\mathcal{A}$-simple (resp. $\mathcal{L}$-simple) map-germ g such that both of $Q(f) \cong Q(g)$ (resp. $Q(f) = Q(g)$) and $T\mathcal{K}(g) = T\mathcal{A}(g)$ (resp. $T\mathcal{C}(g) = T\mathcal{L}(g)$) are satisfied. For the g we see that ${}_1\delta(g) = n\delta(g), {}_1\gamma(g) = n\gamma(g) = n(\delta(g) - r)$ and thus theorem 2 and proposition 2 follow from lemma 2. □

6 Sketch of proofs of theorem 3 and corollary 1

In this section, we give a sketch of proofs of theorem 3 and corollary 1 given in [18].

First, note that for an immersive map-germ $g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ the following hold:

$$
\begin{aligned}
\dim_{\mathbf{R}} T\mathcal{R}^1(j^1g(S)) &= n^2r, \\
\dim_{\mathbf{R}} T\mathcal{C}^1(j^1g(S)) &= npr, \\
\dim_{\mathbf{R}} T\mathcal{K}^1(j^1g(S)) &= npr.
\end{aligned}
$$

If f is $\mathcal{A}$-simple (resp. $\mathcal{L}$-simple), near f there must exist $g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ which is immersive, $\mathcal{A}$-simple (resp. $\mathcal{L}$-simple) and satisfies the equality $\dim_{\mathbf{R}} T\mathcal{K}^1(j^1g(S)) = \dim_{\mathbf{R}} T\mathcal{A}^1(j^1g(S))$ (resp. $\dim_{\mathbf{R}} T\mathcal{C}^1(j^1g(S)) = \dim_{\mathbf{R}} T\mathcal{L}^1(j^1g(S))$). For the $\mathcal{L}$-simple g, by proposition 2 we have $r \leq p/n$. Thus, corollary 1 holds. For the $\mathcal{A}$-simple g, since we have that $\dim_{\mathbf{R}} T\mathcal{L}^1(j^1g(S)) \leq p^2$ (equality holds if and only if $\sum_{i=1}^r j^1g_i(s_i)(\mathbf{R}^n) = \mathbf{R}^p$, where $s_i \in S$) and the equality $\dim_{\mathbf{R}} T\mathcal{K}^1(j^1g(S)) = \dim_{\mathbf{R}} T\mathcal{A}^1(j^1g(S))$ holds for the g, we have $npr \leq n^2r + p^2$. However, we see that the equality does not hold for $\mathcal{A}$-simple map-germ g since the scalar multiple operation gives a trivial 1-dimensional intersection of $T\mathcal{R}^1(j^1g(S))$ and $T\mathcal{L}^1(j^1g(S))$. Thus, theorem 3 holds. □

Remark. We know phenomena similar as theorem 3 in several situations where $\mathcal{K}^2$-moduli sets occur naturally as investigated in [14]. For instance, consider the following set:

$$\Sigma_4^2(4,8) = \{j^2f(0) \mid f : (\mathbf{R}^4,0) \to (\mathbf{R}^8,0) \quad \mathrm{corank}(f) = 4\}.$$

We see that $\dim_{\mathbf{R}} \Sigma_4^2(4,8) = 80$ and the dimension of $T\mathcal{R}^2(j^2f(0))$ (resp. $T\mathcal{C}^2(j^2f(0))$ is less than or equal to 16 (resp. 64); and for a generic $f \in \Sigma_4^2(4,8)$ the equality holds for each inequality. Thus, for a generic $f \in \Sigma_4^2(4,8)$, we have that

$$\dim_{\mathbf{R}} \Sigma_4^2(4,8) = \dim_{\mathbf{R}} T\mathcal{R}^2(j^2f(0)) + \dim_{\mathbf{R}} T\mathcal{C}^2(j^2f(0)).$$

However, we see that

$$\dim_{\mathbf{R}}(T\mathcal{R}(j^2f(0))) \cap T\mathcal{C}(j^2f(0))) > 0$$

by Euler's relation $tf(\Sigma_{i=1}^n x_i \frac{\partial}{\partial x_i}) = 2\omega f(\Sigma_{j=1}^p X_i \frac{\partial}{\partial X_j}))$. Thus, even for a generic $f \in \Sigma_4^2(4,8)$, we have that

$$\dim_{\mathbf{R}} \Sigma_4^2(4,8) > \dim_{\mathbf{R}} T\mathcal{K}^2(j^2f(0)).$$

Not only for $\Sigma_4^2(4,8)$ we encounter the similar phenomina as theorem 3 but also for $\Sigma_3^2(3,3)$, which can be seen as follows. Put

$$\Sigma_3^2(3,3) = \{j^2f(0) \mid f : (\mathbf{R}^3,0) \to (\mathbf{R}^3,0) \quad \mathrm{corank}(f) = 3\}.$$

We see that $\dim_{\mathbf{R}} \Sigma_3^2(3,3) = 18$ and each of the dimension of $T\mathcal{R}^2(j^2f(0))$ and $T\mathcal{C}^2(j^2f(0))$ is less than or equal to 9; and for a generic $f \in \Sigma_3^2(3,3)$ the equality holds for each inequality. Thus, for a generic $f \in \Sigma_3^2(3,3)$, we have that

$$\dim_{\mathbf{R}} \Sigma_3^2(3,3) = \dim_{\mathbf{R}} T\mathcal{R}^2(j^2f(0)) + \dim_{\mathbf{R}} T\mathcal{C}^2(j^2f(0)).$$

However, we see that

$$\dim_{\mathbf{R}}(T\mathcal{R}(j^2f(0))) \cap T\mathcal{C}(j^2f(0))) > 0$$

again by Euler's relation $tf(\Sigma_{i=1}^n x_i \frac{\partial}{\partial x_i}) = 2\omega f(\Sigma_{j=1}^p X_j \frac{\partial}{\partial X_j}))$. Thus, even for a generic $f \in \Sigma_3^2(3,3)$, we have that

$$\dim_{\mathbf{R}} \Sigma_3^2(3,3) > \dim_{\mathbf{R}} T\mathcal{K}^2(j^2f(0)).$$

The author does not know a reasonable explanation why such resembling phenomena occur in diferent situations. Note that by the fact that $\Sigma_4^2(4,8)$ (resp. $\Sigma_3^2(3,3)$) is a $\mathcal{K}^2$-moduli set we can calculate $\sigma_4^2(4,8) = 32$ (resp. $\sigma_3^2(3,3) = 9$), which yields that $\sigma(n,p) = 6(p-n)+8 \quad (p-n \geq 4,\ n \geq 4)$ (resp. $\sigma(n,p) = 6(p-n)+9 \quad (p \geq n \geq 3)$) as shown in [14].

7 Proof of proposition 3

By composing linear transformations of $\mathbf{R}^p$ and germs of non-linear transformations of $\mathbf{R}$ if necessary, from the first we may assume that $g(x) = (g_1(x), g_2(x) \cdots, g_p(x))$ has the following form:

$$\begin{aligned} g_1(x) &= a_1x^9 + o(x^9), \\ g_2(x) &= a_2x^8 + o(x^9), \\ g_3(x) &= a_3x^7 + o(x^9), \\ g_i(x) &= o(x^9) \quad (i \geq 4), \end{aligned}$$

where $a_i = 1$ or 0.

Suppose that $a_2 = 0$. Then, by (1) in §1 we see that for any sufficiently small $t \neq 0$ there exist germs of C^∞ diffeomorphisms $\varphi_t : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ and $\psi_t : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $(g+tf_{(1,t)})\circ\varphi_t = \psi_t \circ f_{(1,t)}$. Furthermore, by (2) in §2 we see that

$$\frac{\partial f_{(1,t)}}{\partial t} \notin T\mathcal{A}(f_{(1,t)}).$$

Thus, proposition 3 holds in the case that $a_2 = 0$.

Next, suppose that $a_2 = 1$. Then, by (1) in §1 we see that for any sufficiently small $t \neq 0$ there exist germs of C^∞ diffeomorphisms $\varphi_t : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ and $\psi_t : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $(g + tf_{(0,1)}) \circ \varphi_t = \psi_t \circ f_{(\frac{1}{t},1)}$. By (2) in §1, we see that

$$\frac{\partial f_{(\frac{1}{t},1)}}{\partial t} \notin T\mathcal{A}(f_{(\frac{1}{t},1)}).$$

Thus, proposition 3 holds in any case. □

8 Sketch of proofs of propositions 4 and 5

Here, we give a sketch of proofs of propositions 4 and 5 given in [18].

We consider 1 of proposition 4 or 5. In the following in this section, we are putting $n = 1$, $p \geq 3$ if we are considering 1 of proposition 4; and we are putting $p = 2n$ if we are considering 1 of proposition 5. By calculations we see that $\dim_{\mathbf{R}} T\mathcal{C}^1(j^1f(S)) = \dim_{\mathbf{R}} T\mathcal{L}^1(j^1f(S))$, which implies ${}_1Q(f)^p = {}_1Q(f, \mathcal{L})^p$. By proposition 1 for $\omega f(m_p\theta_{\{0\}}(f))$, we see that $T\mathcal{C}(f) = T\mathcal{L}(f)$. Note that the $\mathcal{C}$-equivalence class of f is the largest $\mathcal{C}$-equivalence class in the set $\{g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)\ C^\infty\}$ for the given S. Thus, there are no $\mathcal{L}$-equivalence classes to which f is adjacent, which implies that f is $\mathcal{L}$-simple. We can obtain 2 of propositions 4 and 5 similarly. By theorem 3, 3 of propositions 4 and 5 are obtained immediately. □

9 Assertion 1 の証明

M は $\mathcal{E}_n$ の有限生成イデアルであるので, $f_1, \cdots, f_p$ $(p < \infty)$ で生成されるとしてよい. 仮定より,

$$\frac{C}{MC} = \frac{C}{f^* m_p C} = \left\{ [g] \in \frac{C}{f^* m_p C} \;\middle|\; g \in A \right\}$$

を得る. よって, マルグランジュの予備定理より $C \subset A$ を得る. □

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