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# CHARACTERIZING SET CONTAINMENTS WITH QUASICONVEX INEQUALITIES

島根大学大学院 総合理工学研究科 鈴木 聡 (SATOSHI SUZUKI)
INTERDISCIPLINARY GRADUATE SCHOOL OF SCIENCE AND ENGINEERING, SHIMANE UNIVERSITY

島根大学 総合理工学部 黒岩 大史 (DAISHI KUROIWA)
INTERDISCIPLINARY FACULTY OF SCIENCE AND ENGINEERING, SHIMANE UNIVERSITY

ABSTRACT: In this paper, we show the set containment characterization with quasiconvex inequalities, and give a few examples of this characterization.

KEYWORDS: quasiconvex function, quasiconjugate function, levelsets of function

## 1. INTRODUCTION

Recently, various extensions of the containment problem to more general situations have been obtained in [1, 3, 4, 5], by means of mathematical programming theory, conjugacy theory and quasiconjugacy theory. Mangasarian[4] established dual characterization of containment of a polyhedral set in an arbitrary polyhedral set, and of general closed convex set, defined by finite convex constraints, in a reverse-convex set, defined by convex constraints. Jeyakumar[1, 3] also established dual characterizations of the containment of a closed convex set, defined by infinite convex constraints, in an arbitrary polyhedral set, in a reverse-convex set, defined by convex constraints, and in another convex set, defined by finite convex constraints. The dual characterizations are provided in terms of epigraphs of the Fenchel conjugate functions. Suzuki and Kuroiwa[5] established dual characterizations of the containment of a convex set, defined by quasiconvex constraints, in an open half space, in a reversepace, in a reverse-convex set, defined by quasiconvex constraints. The dual characterizations are provided in terms of levelsets of $H$-quasiconjugate and $R$-quasiconjugate functions.

In this paper, we show the set containment characterization with quasiconvex inequalities, and give some examples of this characterization in [5].

## 2. NOTATION AND PRELIMINARIES

Throughout this paper, let $f$ be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Remember that $f$ is said to be quasiconvex if, for all $x_1$, $x_2 \in \mathbb{R}^n$ and $\alpha \in (0,1)$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Define

$$L(f, \diamond, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$, symbol $\diamond$ means any binary relations on $\mathbb{R}$, then $f$ is quasiconvex if and only if for any $\alpha \in \mathbb{R}$,

$$L(f, \leq, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$$

is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$,

$$L(f, <, \alpha) = \{x \in \mathbb{R}^n \mid f(x) < \alpha\}$$

is a convex set. We know that the quasiconvexity for functions is a notion that is weaker than the convexity.

**Definition 1.** A subset $S$ of $\mathbb{R}^n$ is said to be evenly convex if there exists a family of open halfspaces such that $S$ is equal to the intersection of the family of open halfspaces.

**Definition 2.** A subset $S$ of $\mathbb{R}^n$ is said to be $H$-evenly convex if there exists a family of open halfspaces such that $S$ is equal to the intersection of the family of open halfspaces, and all open halfspaces contain 0.

Note that the whole space and empty set are evenly convex and $H$-evenly convex. Also, any open convex set and any closed convex set are evenly convex. Clearly, every evenly convex set is convex, A nonempty subset is $H$-evenly convex if and only if it is an evenly convex set which contains 0.

Given a set $S \subset \mathbb{R}^n$, we shall denote by int$S$, cl$S$, and co$S$ the interior, the closure, and the convex hull generated by $S$ respectively. The evenly convex hull of $S$, denoted by ec$S$, is the smallest evenly convex set which contains $S$ (i.e., it is the intersection of all open halfspaces which contain $S$). The $H$-evenly convex hull of $S$, denoted by Hec$S$, is the smallest $H$-evenly convex set which contains $S$. Note that co$S \subset$ ec$S \subset$ clco$S$, and these differences are slight because clco$S$ = clec$S$. Moreover if $S$ is nonempty, then Hec$S$ = ec$(S \cup \{0\})$.

**Definition 3.** A function $f$ is said to be evenly quasiconvex if $L(f, \leq, \alpha)$ is evenly convex for all $\alpha \in \mathbb{R}$.

**Definition 4.** A function $f$ is said to be strictly evenly quasiconvex if $L(f, <, \alpha)$ is evenly convex for all $\alpha \in \mathbb{R}$.

**Definition 5.** A function $f$ is said to be $H$-evenly quasiconvex if $L(f, \leq, \alpha)$ is $H$-evenly convex for all $\alpha \in \mathbb{R}$.

**Definition 6.** A function $f$ is said to be strictly $H$-evenly quasiconvex if $L(f, <, \alpha)$ is $H$-evenly convex for all $\alpha \in \mathbb{R}$.

Clearly, every evenly quasiconvex function is quasiconvex, every lower semicontinuous (lsc for short) quasiconvex function is evenly quasiconvex, and every upper semicontinuous (usc for short) quasiconvex function is strictly evenly quasiconvex. Also $f$ is $H$-evenly quasiconvex if and only if $f$ is evenly quasiconvex and $f(0) = \inf_{x \in \mathbb{R}^n} f(x)$. Furthermore, we can check every strictly ($H$-)evenly quasiconvex function is ($H$-)evenly quasiconvex, see [5].

We introduce $H$-quasiconjugacy, which is simply called quasiconjugate in [6]. To distinguish the notion of $R$-quasiconjugate, which is seen in [5, 7], we call it by $H$-quasiconjugate.

**Definition 7** ([6]). $H$-quasiconjugate of $f$ is the functional $f^H : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$f^H(\xi) = \begin{cases} -\inf\{f(x) \mid \langle \xi, x\rangle \geq 1\} & \text{if} \quad \xi \neq 0 \\ -\sup\{f(x) \mid x \in \mathbb{R}^n\} & \text{if} \quad \xi = 0. \end{cases}$$

The $H$-quasiconjugate of the function $f^H$ is called the $H$-biquasiconjugate of $f$ and denoted by $f^{HH}$.

**Definition 8** ([6]). We say that $f$ achieves the maximum value at infinity if $f(x_k) \to \sup\{f(x) \mid x \in \mathbb{R}^n\}$ for any sequence $\{x_k\}$ with $\|x_k\| \to \infty$.

**Definition 9** ([5]). We say that $f$ achieves the minimum value at the origin if $f(x_k) \to \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}$ for any sequence $\{x_k\} \subset \mathbb{R}^n\backslash\{0\}$ with $x_k \to 0$.

Let $\Gamma^\infty$ and $\gamma^0$ be the set of all functions which achieves the maximum value at infinity, and the set of all functions which achieves the minimum value at the origin, respectively; That is,

$$\begin{aligned} \Gamma^\infty &= \{g : \mathbb{R}^n \to \overline{\mathbb{R}} \mid g \text{ achieves the maximum value at infinity}\}, \\ \gamma^0 &= \{g : \mathbb{R}^n \to \overline{\mathbb{R}} \mid g \text{ achieves the minimum value at the origin}\}. \end{aligned}$$

## 3. CHARACTERIZATION OF SET CONTAINMENTS

Jeyakumar[1, 3] established dual characterizations of the containment of a closed convex set, defined by infinite convex constraints, in an arbitrary polyhedral set. The dual characterizations are provided in terms of epigraphs of the Fenchel conjugate functions.

**Theorem 1** ([3]). Let $I$ be an arbitrary index set, and $g_i$ be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$ for all $i \in I$, $u \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, $\{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\} \neq \emptyset$. Then the following (i) and (ii) are equivalent.
(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\} \subseteq \{x \in \mathbb{R}^n \mid \forall j \in J, \langle u_j, x\rangle \leq \alpha_j\}$
(ii)$(u, \alpha) \in \text{cl cone co} \bigcup_{i\in I} \text{epi} g_i^*$

In this paper, we show the characterization of the set contaiment with quasiconvex inequalities.

**Theorem 2** ([5]). $v \in \mathbb{R}^n\backslash\{0\}$, $\alpha, \beta \in \mathbb{R}, \alpha > 0$, then

$$L(f, <, \beta) \subseteq \{x \mid \langle v, x\rangle < \alpha\} \Longleftrightarrow \frac{v}{\alpha} \in L(f^H, \leq -\beta)$$

**Example 1.** Let $f$ be a function from $\mathbb{R} \to \mathbb{R}$ as follows;

$$f(x) = \begin{cases} -x-2 & \text{if} \quad x \leq -1 \\ x & \text{if} \quad -1 \leq x \leq 0 \\ 0 & \text{if} \quad 0 \leq x \leq 1 \\ x-1 & \text{if} \quad 1 \leq x \end{cases}$$

Then, $L(f,<,0)=(-2,0)$, and $H$-quasiconjugate of $f$ is

$$f^H(\xi)=\begin{cases}1 & \text{if} \quad \xi\le -1\\ 2+\frac{1}{\xi} & \text{if} \quad -1\le \xi<0\\ -\infty & \text{if} \quad \xi=0\\ -\frac{1}{\xi}+1 & \text{if} \quad 0<\xi\le 1\\ 0 & \text{if} \quad 1\le \xi\end{cases}$$

One has $L(f^H,\le,0)=[-\frac{1}{2},\infty)$, and

$$L(f,<,0)\subset\{x\in\mathbb{R}\mid\langle v,x\rangle<1\}\iff v\in L(f^H,\le,0).$$

**Example 2.** Let $f$ be a function from $\mathbb{R}^2\to\mathbb{R}$ as follows;

$$f(x_1,x_2)=(x_1+1)^2+x_2^2-4.$$

Then, $L(f,<,0)=\{(x_1,x_2)\mid(x_1+1)^2+x_2^2\le 4\}$, and $H$-quasiconjugate of $f$ is

$$f^H(\xi_1,\xi_2)=\begin{cases}4 & \text{if} \quad \xi_1\le -1\\ -\infty & \text{if} \quad (\xi_1,\xi_2)=(0,0)\\ -\dfrac{(\xi_1+1)^2}{\xi_1^2+\xi_2^2}+4 & \textit{otherwise.}\end{cases}$$

One has $L(f^H,\le,0)=\{(\xi_1,\xi_2)\mid 9(\xi_1-\frac{1}{3})^2+12\xi_2^2\le 4\}$, and

$$L(f,<,0)\subset\{x\in\mathbb{R}^2\mid\langle v,x\rangle<1\}\iff v\in L(f^H,\le,0).$$

**Theorem 3** ([5]). *Let $I$ be an arbitrary index set, and $f_i$ be a strict evenly quasiconvex function from $\mathbb{R}^n$ to $\mathbb{R}$ for all $i\in I$. If $\sup_{i\in I} f_i(x)>\sup_{i\in I} f_i(0)$ for all $x\in\mathbb{R}^n\backslash\{0\}$, then*

$$(\sup_{i\in I} f_i)^H(x)=(\inf_{i\in I} f_i^H)^{HH}(x),$$

*for all $x\in\mathbb{R}^n\backslash\{0\}$.*

**Theorem 4.** *Let $I$ be an arbitrary index set, and $f_i$ be a strict evenly quassiconvex function from $\mathbb{R}^n$ to $\mathbb{R}$ for all $i\in I$, $v\in\mathbb{R}^n\backslash\{0\}$, $\alpha>0$, $\beta\in\mathbb{R}$, $\forall x\in\mathbb{R}^n\backslash\{0\}$, $\sup_{i\in I} f_i(x)>\sup_{i\in I} f_i(0)$, $\inf_{i\in I} f_I^H\in\Gamma^\infty$ be l.s.c. Then, the following (i) and (ii) are equivalent:*
*(i)$L(\sup_{i\in I} f_i,<,\beta)\subseteq\{x\in\mathbb{R}^n\mid\langle v,x\rangle<\alpha\}$,*
*(ii)$\frac{v}{\alpha}\in\operatorname{Hec}\bigcup_{i\in I} L(f_i^H,\le-\beta)$.*

**Example 3.** Let $I=\{1,2,3,4\}$, and $f_1$, $f_2$, $f_3$ and $f_4$ be the functions from $\mathbb{R}^2\to\mathbb{R}$ as follows;

$$\begin{aligned}f_1(x_1,x_2)&=(x_1+1)^2+x_2^2-4,\\ f_2(x_1,x_2)&=(x_1-1)^2+x_2^2-4,\\ f_3(x_1,x_2)&=x_1^2+(x_2-1)^2-4,\\ f_4(x_1,x_2)&=x_1^2+(x_2+1)^2-4.\end{aligned}$$

Then, for each $i \in I$, $f_i$ is a strictly evenly quasiconvex function and for all $x \in \mathbb{R}^2 \setminus \{0\}$, $\sup_{i\in I} f_i(x) > \sup_{i\in I} f_i(0)$. Then by using Theorem 3, $(\sup_{i\in I} f_i)^H = (\inf_{i\in I} f_i^H)^{HH}$. One has $L(f_1^H, \leq, 0) = \{(\xi_1,\xi_2) \mid 9(\xi_1 - \frac{1}{3})^2 + 12\xi_2^2 \leq 4\}$, $L(f_2^H, \leq, 0) = \{(\xi_1,\xi_2) \mid 9(\xi_1+\frac{1}{3})^2+12\xi_2^2 \leq 4\}$, $L(f_3^H, \leq, 0) = \{(\xi_1,\xi_2) \mid 12\xi_1^2+9(\xi_2+\frac{1}{3})^2 \leq 4\}$ and $L(f_4^H, \leq, 0) = \{(\xi_1,\xi_2) \mid 12\xi_1^2 + 9(\xi_2 - \frac{1}{3})^2 \leq 4\}$. Moreover $\inf_{i\in I} f_i^H$ is lsc and in $\Gamma^\infty$, then by using Theorem 4,

$$L(\sup_{i\in I} f_i, <, 0) \subset \{x \in \mathbb{R}^2 \mid \langle v, x\rangle < 1\} \iff v \in \operatorname{Hec} \bigcup_{i\in I} L(f_i^H, \leq, 0).$$

Also we can check this equivalence relation directly.

**Example 4.** Let $I = \{1,2\}$, and $f_1$, $f_2$ be the same functions in Example 3. Then, $f_1$, $f_2$ are strictly evenly quasiconvex function and for all $x \in \mathbb{R}^2\setminus\{0\}$, $\sup_{i\in I} f_i(x) > \sup_{i\in I} f_i(0)$. Then by using Theorem 3, $(\sup_{i\in I} f_i)^H = (\inf_{i\in I} f_i^H)^{HH}$. Moreover $\inf_{i\in I} f_i^H$ is not in $\Gamma^\infty$, but

$$\operatorname{Hec} \bigcap_{i\in I} L(f_i^H, \leq, 0) = L((\inf_{i\in I} f_i^H)^{HH}, \leq, 0) = \bigcap_{\varepsilon>0} \operatorname{Hec} \bigcap_{i\in I} L(f_i^H, <, 0+\varepsilon).$$

Therefore the following equivalence relation holds (see [5]).

$$L(\sup_{i\in I} f_i, <, 0) \subset \{x \in \mathbb{R}^2 \mid \langle v, x\rangle < 1\} \iff v \in \operatorname{Hec} \bigcup_{i\in I} L(f_i^H, \leq, 0).$$

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