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ON THE EXISTENCE OF CONTINUOUS SELECTIONS AVOIDING EXTREME POINTS

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Throughout this note, all spaces are assumed to be T_1 and λ stands for an infinite cardinal number. For undefined terminology, we refer to [3]. The purpose of this note is to introduce some results of [15] and [16].

Let X be a space and $(Y, \|\cdot\|)$ a Banach space. By 2^Y, we denote the set of all non-empty subsets of Y. For a mapping $\varphi : X \to 2^Y$, a mapping $f : X \to Y$ is called a *selection* if $f(x) \in \varphi(x)$ for each $x \in X$.

For $K \in \mathcal{F}_c(Y)$, a point $y \in K$ is called an *extreme point* if every open line segment containing y is not contained in K. For $K \in \mathcal{F}_c(Y)$, the *weak convex interior* $\mathrm{wci}(K)$ of K ([5]) is the set of all non-extreme points of K, that is,

$$\mathrm{wci}(K) = \{y \in K \mid y = \delta y_1 + (1-\delta) y_2 \text{ for some } y_1, y_2 \in K \setminus \{y\} \text{ and } 0 < \delta < 1\}.$$

Our concern of this note is to obtain theorems on continuous selections avoiding extreme points, which is motivated by Problem 3 below posed by V. Gutev, H. Ohta and K. Yamazaki [5].

1. A problem of Gutev, Ohta and Yamazaki

A Hausdorff space X is called *countably paracompact* if every countable open cover of X is refined by a locally finite open cover of X. Let $\mathbf{R}$ be the space of real numbers with the usual topology. The following insertion theorem due to C. H. Dowker [2] and M. Katětov [7] is fundamental in our study.

Theorem 1 (Dowker [2, Theorem 4], Katětov [7, Theorem 2]). *A T_1-space X is normal and countably paracompact if and only if for every upper semicontinuous function $g : X \to \mathbf{R}$ and every lower semicontinuous function $h : X \to \mathbf{R}$ with $g(x) < h(x)$ for each $x \in X$, there exists a continuous function $f : X \to \mathbf{R}$ such that $g(x) < f(x) < h(x)$ for each $x \in X$.*

The cardinality of a set S is denoted by $\mathrm{Card}\, S$. A T_1-space X is called *λ-collectionwise normal* if for every discrete collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of X with $\mathrm{Card}\, A \leq \lambda$, there exists a disjoint collection $\{G_\alpha \mid \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$. The space $c_0(\lambda)$ is the Banach space consisting of functions $s : D(\lambda) \to \mathbf{R}$, where $D(\lambda)$ is a set with $\mathrm{Card}\, D(\lambda) = \lambda$, such that for each $\varepsilon > 0$ the set $\{\alpha \in D(\lambda) \mid |s(\alpha)| \geq \varepsilon\}$ is finite, where the linear operations are defined pointwise and $\|s\| = \sup\{|s(\alpha)| \mid \alpha \in D(\lambda)\}$ for each $s \in c_0(\lambda)$. In order to connect insertion theorems with selection theorems, V. Gutev, H. Ohta and K. Yamazaki [5] introduced lower and upper semicontinuity of a mapping to the Banach space $c_0(\lambda)$ and, with the aid of these concepts, they proved sandwich-like characterizations of paracompact-like properties. Moreover, they introduced generalized $c_0(\lambda)$-spaces for Banach spaces and established the following theorem. A mapping $\varphi : X \to 2^Y$ is called *lower semicontinuous* (*l.s.c.* for short) if for every open subset V of Y, the set $\varphi^{-1}[V] = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$ is open in X. By $\mathcal{C}_c(Y)$ we denote the set of all non-empty compact convex subsets of Y and let $\mathcal{C}'_c(Y) = \mathcal{C}_c(Y) \cup \{Y\}$.

Theorem 2 (Gutev, Ohta and Yamazaki [5, Theorem 4.5]). *A T_1-space X is countably paracompact and λ-collectionwise normal if and only if for every generalized $c_0(\lambda)$-space Y, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

Note that the "only if" part of Theorem 2 implies that of Theorem 1. By $w(Y)$ we denote the weight of a space Y. Since generalized $c_0(\lambda)$-space is a special Banach space with $w(Y) \leq \lambda$, Gutev, Ohta and Yamazaki [5] posed the following problem.

Problem 3 (Gutev, Ohta and Yamazaki [5, Problem 4.7]). *Can the phrase "every generalized $c_0(\lambda)$-space Y" in Theorem 2 be replaced with "every Banach space Y with $w(Y) \leq \lambda$"?*

It is proved in [15] that the answer of Problem 3 is affirmative.

Theorem 4 ([15]). *A T_1-space X is countably paracompact and λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

In particular, we have the following.

Corollary 5. *A T_1-space X is countably paracompact and normal if and only if for every separable Banach space Y, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

Corollary 6. *A T_1-space X is countably paracompact and collectionwise normal if and only if for every Banach space Y, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

A Hausdorff space X is called *λ-paracompact* if every open cover $\mathcal{U}$ of X with $\operatorname{Card}\mathcal{U} \leq \lambda$ is refined by a locally finite open cover of X. The set of all non-empty closed convex subsets of a Banach space Y is denoted by $\mathcal{F}_c(Y)$. The following theorem is a λ-paracompact analogue of Theorems 2 and 4.

Theorem 7 ([15]). *A T_1-space X is normal and λ-paracompact if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

Thus we have the following variation of [11, Theorem 3.2″].

Corollary 8. *A T_1-space X is paracompact if and only if for every Banach space Y, every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ such that* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

2. The role of countable paracompactness for continuous selections avoiding extreme points

The following selection theorem is due to E. Michael [11] and S. Nedev [12].

Theorem 9 (E. Michael [11, Theorem 3.2′], S. Nedev [12, Theorem 4.2]). *A T_1-space X is λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ admits a continuous selection.*

Although the existence itself of a continuous selection is guaranteed by Theorem 9, the assumption in Theorem 4 that X is countably paracompact can not be dropped. Suggested by this fact, we are next concerned with the role of countable paracompactness to obtain a continuous selections avoiding extreme points. Our study has two directions; one is to obtain an l.s.c. set-valued selection avoiding extreme points under a separation axiom of X weaker than λ-collectionwise normality, another is to drop countable paracompactness instead of imposing a condition to set-valued mappings.

2.1. **L.s.c. set-valued selections avoiding extreme points.** For a mapping $\varphi : X \to 2^Y$, a mapping $\theta : X \to 2^Y$ is called a *set-valued selection* if $\theta(x) \subset \varphi(x)$ for each $x \in X$. A topological space X is called *countably metacompact* if every countable open cover $\mathcal{U}$ of X is refined by a point-finite open cover of X. We have the following characterization of countably metacompact spaces without any separation axiom.

Theorem 10 ([16]). *A topological space X is countably metacompact if and only if for every normed space Y, every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathcal{C}_c(Y)$ such that $\phi(x) \subset \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

If the mappings $\varphi, \phi : X \to \mathcal{C}_c(Y)$ can be replaced with mappings $\varphi, \phi : X \to \mathcal{C}'_c(Y)$, then Theorem 4 follows from Theorem 9 and the replaced statement. But the author does not know whether Theorem 10 remains valid even if the mappings $\varphi, \phi : X \to \mathcal{C}_c(Y)$ are replaced with $\varphi, \phi : X \to \mathcal{C}'_c(Y)$.

A topological space X is *almost λ-expandable* ([9], [14]) if for every locally finite collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of X with $\operatorname{Card} A \leq \lambda$, there exists a point-finite collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$. Note that every countably paracompact λ-collectionwise normal space is almost λ-expandable ([8]), and every almost λ-expandable space is countably metacompact ([9, Theorem 2.6]). For compact-valued l.s.c. set-valued selections of mappings $\varphi : X \to \mathcal{C}'_c(Y)$, we have the following.

Theorem 11 ([16]). *A normal space X is almost λ-expandable if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathcal{C}_c(Y)$ such that $\phi(x) \subset \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

A T_1-space X is *λ-PF-normal* if every point-finite open cover is normal. A T_1-space X is *PF-normal* if X is λ-PF-normal for each infinite cardinal λ. PF-normal spaces are first investigated by E. Michael [10], and the name "PF-normal" is due to J. C. Smith [13]. Note that every λ-collectionwise normal space is λ-PF-normal and ω-PF-normality coincides with the normality, where ω is the first infinite cardinal number. T. Kandô [6] and S. Nedev [12] proved the following selection theorem for λ-PF-normal spaces (PF-normal spaces are called pointwise-paracompact and normal in [6], while λ-PF-normal spaces are called λ-pointwise-$\aleph_0$-paracompact and normal in [12]).

Theorem 12 (T. Kandô [6, Theorem IV], S. Nedev [12, Theorem 4.1]). *A T_1-space X is λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ admits a continuous selection.*

A space is countably paracompact and λ-collectionwise normal if and only if it is almost λ-expandable and λ-PF-normal. Thus Theorem 4 follows from Theorems 11 and Theorem 12. Also, by Theorems 10 and 12, we have the following.

Theorem 13 ([16]). *A T_1-space X is countably paracompact and λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$.*

A topological space X is called *λ-metacompact* if every open cover $\mathcal{U}$ of X with $\operatorname{Card}\mathcal{U} \leq \lambda$ is refined by a point-finite open cover of X. M. M. Čoban [1, Theorem 6.1] characterized λ-metacompactness in terms of l.s.c. set-valued selections. For λ-metacompact analogue of Theorem 11, we have the following.

Theorem 14 ([16]). *A regular space X is λ-metacompact if and only if for every Banach space Y with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with*

$\operatorname{Card}\varphi(x) > 1$ *for each* $x \in X$ *admits an l.s.c. set-valued selection* $\phi : X \to \mathcal{C}_c(Y)$ *such that* $\phi(x) \subset \operatorname{wci}(\varphi(x))$ *for each* $x \in X$.

2.2. **Dropping countable paracompactness.** Next, we drop countable paracompactness of Theorem 4 instead of imposing a condition to set-valued mappings. In fact, the additional condition for set-valued mappings is that the values of them has uniformly large diameters. For a subset A of a metric space (Y, d), let $\operatorname{diam} A = \sup\{d(y_1, y_2) \mid y_1, y_2 \in A\}$.

Theorem 15 ([16]). *A* T_1*-space* X *is* λ*-collectionwise normal if and only if for every Banach space* Y *with* $w(Y) \leq \lambda$, *every l.s.c. mapping* $\varphi : X \to \mathcal{C}'_c(Y)$ *with* $\inf\{\operatorname{diam}\varphi(x) \mid x \in X\} > 0$ *admits a continuous selection* $f : X \to Y$ *such that* $f(x) \in \operatorname{wci}(\varphi(x))$ *for each* $x \in X$.

We also have the following characterization of λ-PF-normal spaces.

Theorem 16 ([16]). *A* T_1*-space* X *is* λ*-PF-normal if and only if for every Banach space* Y *with* $w(Y) \leq \lambda$, *every l.s.c. mapping* $\varphi : X \to \mathcal{C}_c(Y)$ *with* $\inf\{\operatorname{diam}\varphi(x) \mid x \in X\} > 0$ *admits a continuous selection* $f : X \to Y$ *such that* $f(x) \in \operatorname{wci}(\varphi(x))$ *for each* $x \in X$.

Let X be a topological space and (Y, d) a metric space. A mapping $\varphi : X \to 2^Y$ is said to be *d-upper semicontinuous* (*d-u.s.c.* for short) if for each $x \in X$ and $\varepsilon > 0$, there exists a neighborhood U of x such that $\varphi(x') \subset B(\varphi(x), \varepsilon)$ for each $x' \in U$. A mapping $\varphi : X \to 2^Y$ is called *d-proximal continuous* if φ is l.s.c. and d-u.s.c. If $\varphi : X \to 2^Y$ is d-proximal continuous for some metric d compatible with the topology of Y, then φ is called *proximal continuous*. Note that all continuous mappings $f : X \to (\mathcal{F}(Y), \tau_V)$ and $f : X \to (\mathcal{F}(Y), \tau_{H(d)})$ are proximal continuous, where τ_V is the Vietoris topology on $\mathcal{F}(Y)$ and $\tau_{H(d)}$ is the topology on $\mathcal{F}(Y)$ induced by the Hausdorff distance with respect to some compatible metric d of Y (see [4, Section 2]). V. Gutev [4, Theorem 6.1] proved that for every topological space X and for every Banach space Y, every proximal continuous mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection. For continuous selections avoiding extreme points, we have the following.

Theorem 17 ([16]). *Let* X *be a topological space,* Y *a Banach space and* $\varphi : X \to \mathcal{F}_c(Y)$ *a proximal continuous mapping. Then there exists a continuous selection* $f : X \to Y$ *of* φ *such that* $f(x) \in \operatorname{wci}(\varphi(x))$ *whenever* $\operatorname{Card}\varphi(x) > 1$.

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