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WEAK AND STRONG CONVERGENCE THEOREMS FOR A FAMILY OF RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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1. INTRODUCTION

Let H be a Hilbert space and let $\{C_i\}$ be a family of closed convex subsets of H such that $F=\bigcap_{i\in I}C_i$ is nonempty. Then the convex feasibility problem is to find an element of F by using the metric projections P_i from H onto C_i. Each P_i is a nonexpansive mapping, that is,

$$\|P_ix-P_iy\|\leqq\|x-y\|$$

for all $x,y\in H$. We also know that $C_i=F(P_i)$, where $F(P_i)$ denotes the set of fixed points of P_i. Thus, the convex feasibility problem in the setting of Hilbert spaces is reduced the problem of finding a common fixed point of a given finite family of nonexpansive mappings. Matsushita and Takahashi [12, 13, 14] introduced the notion of relatively nonexpansive mapping (see [6]). They also obtained weak and strong convergence theorems to approximate a fixed point of a relatively nonexpansive mapping.

In this paper, we introduce an iterative process of finding a common fixed point of a finite family of relatively nonexpansive mappings in a Banach space by the hybrid method which is used in the mathematical programming and then prove a strong convergence theorem for the family in a Banach space (see [13, 16]). Further, we also prove weak convergence theorems for the family by an iterative process. Using the obtained results, we study the convex feasibility problem.

2. PRELIMINARIES AND LEMMAS

Throughout this paper, E is a real Banach space and E^* is the dual space of E. We denote by $\langle y,x^*\rangle$ the value of $x^*\in E^*$ at $y\in E$. We write $x_n\rightharpoonup x$ (or $w\text{-}\lim_{n\to\infty}x_n=x$) to indicate that the sequence $\{x_n\}$ of vectors converges weakly to x. Similarly, $x_n\to x$ (or $\lim_{n\to\infty}x_n=x$) will symbolize strong convergence. In addition, we denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real numbers and all nonnegative integers, respectively.

A Banach space E is said to be strictly convex if $\dfrac{\|x+y\|}{2}<1$ for $x,y\in E$ with $\|x\|=\|y\|=1$ and $x\neq y$. In a strictly convex Banach space, we have that if $\|x\|=\|y\|=$

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$\|(1-\lambda)x+\lambda y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every real number ε with $0 \le \varepsilon \le 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. A closed convex subset C of a Banach space E is said to have normal structure if for each bounded closed convex subset K of C which contains at least two points, there exists an element of K which is not a diametral point of K. It is well-known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. The following result was proved in [7].

Theorem 2.1. Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then, $F(T)$ is nonempty.

The multi-valued mapping J from E into E^* defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for every} \quad x \in E$$

is called the duality mapping of E. From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$. A Banach space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth, then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E.

Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping from E into E^*, and let C be a nonempty closed convex subset of E. Define the real valued function ϕ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in E$. Following Alber [1], the generalized projection P_C from E onto C is defined by

$$P_C x = \arg\min_{y \in C} \phi(y, x)$$

for all $x \in E$. If E is a Hilbert space, we have that $\phi(y, x) = \|y - x\|^2$ for all $y, x \in E$ and hence P_C is reduced to the metric projection. We know the following lemma concerning generalized projections.

Lemma 2.2 ([1, 8]). Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let P_C be the generalized projection from E onto C. Then,

$$\phi(x, P_C y) + \phi(P_C y, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in E$.

Lemma 2.3 ([1, 8]). Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let P_C be a generalized projection from E onto C. Let $x \in E$, and let $z \in C$. Then, $z = P_C x$ is equivalent to

$$\langle y - z, Jx - Jz \rangle \leq 0$$

for all $y \in C$.

We also know the following four lemmas.

Lemma 2.4 ([8]). Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.5 ([8]). Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.6 ([22, 23, 24]). Let E be a uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r$ and $t \in [0, 1]$.

Lemma 2.7 ([9]). Let E be a smooth, strictly convex and reflexive Banach space, let $z \in E$ and let $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite set in E such that

$$\phi\left(z, J^{-1}\left(\sum_{j=1}^m t_j J x_j\right)\right) = \phi(z, x_i)$$

for all $i \in \{1, 2, \ldots, m\}$, then $x_1 = x_2 = \ldots = x_m$.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let T be a mapping from C into itself and let $F(T)$ be the set of all fixed points of T. Then, a point $z \in C$ is said to be an asymptotic fixed point of T (see [17]) if there exists a sequence $\{z_n\}$ in C such that $z_n \rightharpoonup z$ and $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. Following Matsushita and Takahashi [12, 13, 14], we say that T: $C \to C$ is relatively nonexpansive if the following conditions are satisfied:

(i) $F(T)$ is nonempty;
(ii) $\phi(u, Tx) \leq \phi(u, x)$ for each $u \in F(T)$ and $x \in C$;
(iii) $\hat{F}(T) = F(T)$.

A mapping $T : C \to C$ is called strongly relatively nonexpansive if T is relatively nonexpansive and $\phi(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a bounded sequence in C such that $\phi(p, x_n) - \phi(p, Tx_n) \to 0$ for some $p \in F(T)$.

The following lemma was proved by Matsushita and Takahashi [14].

Lemma 2.8 ([14]). Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let T be a relatively nonexpansive mapping of C into itself. Then, $F(T)$ is closed and convex.

We also know the following two lemmas.

Lemma 2.9 ([9, 10]). Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable. Let C be a nonempty closed convex subset of E and let $S: C \to C$ and $T: C \to C$ be relatively nonexpansive mappings such that $F(S) \cap F(T) \neq \emptyset$. Suppose that S or T is strongly relatively nonexpansive. Then $\hat{F}(ST) = F(ST) = F(S) \cap F(T)$ and $ST: C \to C$ is relatively nonexpansive. Moreover, if both S and T are strongly relatively nonexpansive, then $ST: C \to C$ is also strongly relatively nonexpansive.

Lemma 2.10 ([9, 10]). Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let P_C be the generalized projection from E onto C. Let $S: C \to C$ be a strongly relatively nonexpansive mapping, let $T: C \to C$ be a relatively nonexpansive mapping and let $U : C \to C$ be a mapping defined by $U = P_C J^{-1}(\lambda JS + (1-\lambda)JT)$, where $\lambda \in (0,1)$. Suppose $F(S) \cap F(T) \neq \emptyset$. Then $\hat{F}(U) = F(U)$ and U is strongly relatively nonexpansive.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $T_1, T_2, \dots, T_r$ be mappings of C into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be a real numbers such that $0 \leq \alpha_i \leq 1$ for every $i \in \{1, 2, \dots, r\}$. Let P_C be the generalized projection from E onto C. Then, Takahashi [20] defined a mapping W of C into itself as follows:

$$
\begin{aligned}
U_1 &= P_C J^{-1}(\alpha_1 JT_1 + (1-\alpha_1)J), \\
U_2 &= P_C J^{-1}(\alpha_2 JT_2 U_1 + (1-\alpha_2)J), \\
&\vdots \\
U_{r-1} &= P_C J^{-1}(\alpha_{r-1} JT_{r-1} U_{r-2} + (1-\alpha_{r-1})J), \\
W = U_r &= P_C J^{-1}(\alpha_r JT_r U_{r-1} + (1-\alpha_r)J).
\end{aligned}
\tag{1}
$$

Such a mapping W is called the W-mapping generated by $P_C, T_n, T_{n-1}, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$. Using Lemmas 2.9 and 2.10 , we obtain the following three lemmas.

Lemma 2.11. Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be relatively nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be a real numbers such that $0 < \alpha_i < 1$ for every $i \in \{1, 2, \dots, r\}$. Let P_C be the generalized projection from E onto C. Let $U_1, U_2, U_3, \dots, U_{r-1}$ and W be the mappings defined by (1). Let $k \in \{1, 2, \dots, r\}$. Then,

$$\phi(u, Wx) \le \phi(u, x) \quad \text{and} \quad \phi(u, U_k x) \le \phi(u, x)$$

for all $u \in \bigcap_{i=1}^{r} F(T_i)$ and $x \in C$.

Lemma 2.12. Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be relatively nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i \in \{1, 2, \dots, r\}$. Let P_C be the generalized projection from E onto C. Let W be the W-mapping of C into itself generated by $P_C, T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Then, $F(W) = \bigcap_{i=1}^{r} F(T_i)$.

Lemma 2.13. Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be relatively nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i \in \{1, 2, \dots, r\}$. Let P_C be the generalized projection from E onto C. Let $U_1, U_2, U_3, \dots, U_{r-1}$ and W be the the mapping defined by (1). Then, for each $k \in \{1, 2, \dots, r\}$, $T_k U_{k-1}$ and U_k are relatively nonexpansive mapping, where $U_0 = I$.

3. Strong Convergence Theorems

In this section, we study an iterative process of finding common fixed points of a family of relatively nonexpansive mappings by the hybrid method in the mathematical programming (see also [15, 16, 18, 19]). Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $T_1, T_2, \dots, T_r$ be relatively nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and let P_C be the generalized projection from E onto C. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be a real numbers such that $0 \le \alpha_i \le 1$ for every $i \in \{1, 2, \dots, r\}$. Let W be the W-mapping of C into itself generated by $P_C, T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Consider the following iteration scheme (see also [13]):

$$\begin{aligned}
&x_0 = x \in C, \\
&C_n = \{z \in C : \phi(z, Wx_n) \le \phi(z, x_n)\}, \\
&Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\
&x_{n+1} = P_{C_n \cap Q_n} x
\end{aligned}$$

for each $n \in \mathbb{N}$, where $P_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$. Now, we can prove a strong convergence theorem for a family of relatively nonexpansive mappings.

Theorem 3.1 ([5]). Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be relatively nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i \in \{1, 2, \dots, r\}$. Let P_C be the generalized projection from E onto C. Let W be the W-mapping of C into itself generated by P_C, $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Suppose that $\{x_n\}$ is given by

$$\begin{aligned}
x_0 &= x \in C,\\
C_n &= \{z \in C : \phi(z, Wx_n) \le \phi(z, x_n)\},\\
Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\},\\
x_{n+1} &= P_{C_n \cap Q_n}(x)
\end{aligned}$$

for each $n \in \mathbb{N}$, where $P_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$. Then, $\{x_n\}$ converges strongly to the element $P_F x$, where P_F is the generalized projection from E onto F.

As a direct consequence of Theorem 3.1, we have the following.

Theorem 3.2 ([5]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^r F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for each $i \in \{1, 2, \dots, r\}$. Let W be the W-mapping of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Consider the following iteration scheme:

$$\begin{aligned}
x_0 &= x \in C,\\
C_n &= \{z \in C : \phi(z, Wx_n) \le \phi(z, x_n)\},\\
Q_n &= \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}$$

for each $n \in \mathbb{N}$, where $P_{C_n \cap Q_n}$ is the metric projection of E onto $C_n \cap Q_n$. Then, $\{x_n\}$ converges strongly to the element $P_F x$, where P_F is the metric projection from E onto F.

Theorem 3.3 ([5]). Let E be a uniformly smooth and uniformly convex Banach space and let $\{C_i\}$ be a countable family of nonempty closed convex subsets of E such that $C = \bigcap_{i=1}^r C_i \neq \emptyset$. Let $P_{C_1}, P_{C_2}, \dots, P_{C_r}$ be the generalized projection from E onto C_i for each $i \in \mathbb{N}$. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for each $i \in 1, 2, \dots, r$. Let W be the W-mapping of C into itself generated by $P_{C_1}, P_{C_2}, \dots, P_{C_r}$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Suppose that $\{x_n\}$ is given by

$$\begin{aligned}
x_0 &= x \in C,\\
D_n &= \{z \in C : \phi(z, Wx_n) \le \phi(z, x_n)\},\\
Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\},\\
x_{n+1} &= P_{D_n \cap Q_n} x
\end{aligned}$$

for each $n \in \mathbb{N}$, where $P_{D_n \cap Q_n}$ is the generalized projection from E onto $D_n \cap Q_n$. Then, $\{x_n\}$ converges strongly to the element $P_{\bigcap_{i=1}^r C_i} x$, where $P_{\bigcap_{i=1}^r C_i}$ is the generalized projection from E onto $\bigcap_{i=1}^r C_i$.

4. Weak Convergence Theorems

In this section, we prove weak convergence theorems for finite family of relatively nonexpansive mappings in Banach spaces. For the sake of simplicity, we write $F =$

$\bigcap_{i=1}^{r} F(T_i)$. Throughout this paper, P_C is the generalized projection from E onto C. We can prove the following result by using the idea of [9, 12].

Theorem 4.1 ([4]). Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be relatively nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i \in \{1, 2, \dots, r\}$. Let P_C be the generalized projection from E onto C. Let W be the W-mapping of C into itself generated by $P_C, T_1, T_2, \dots, T_r$ and and $\alpha_1, \alpha_2, \dots, \alpha_r$. Suppose that $\{x_n\}$ is given by $x_0 = x \in C$ and $x_{n+1} = Wx_n$ for every $n = 0, 1, 2, \dots$. Then, $\{P_F x_n\}$ converges strongly to the unique element z of F such that

$$\lim_{n\to\infty} \phi(z, x_n) = \min\left\{ \lim_{n\to\infty} \phi(y, x_n) : y \in F \right\},$$

where P_F is the generalized projection from E onto F.

The following result is essential in the proof of Theorem 4.3.

Theorem 4.2 ([4]). Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be relatively nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i \in \{1, 2, \dots, r\}$. Let P_C be the generalized projection from E onto C. Let W be the W-mapping of C into itself generated by $P_C, T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Let $\{z_n\}$ be a bounded sequence in C such that $\phi(u, z_n) - \phi(u, Wz_n) \to 0$ for some $u \in F$ and $z_{n_k} \rightharpoonup z$. Then, $z \in F$.

Using theorems 4.1 and 4.2, we can prove the following weak convergence theorem.

Theorem 4.3 ([4]). Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be relatively nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i \in \{1, 2, \dots, r\}$. Let P_C be the generalized projection from E onto C. Let W be the W-mapping of C into itself generated by $P_C, T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Suppose that $\{x_n\}$ is given by $x_0 = x \in C$ and $x_{n+1} = Wx_n$ for every $n = 0, 1, 2, \dots$. Then, following hold:

(a) The sequence $\{x_n\}$ is bounded and each weak subsequentially limit of $\{x_n\}$ belongs to $\bigcap_{i=1}^{r} F(T_i)$;

(b) if the duality mapping J from E into E^* is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the element z of $\bigcap_{i=1}^{r} F(T_i)$, where $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^{r} F(T_i)} x_n$.

As a direct consequence of Theorem 4.3, we have the following.

Theorem 4.4. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for each $i \in \{1, 2, \dots, r\}$. Let P_C be a metric projection from E onto C. Let W be the W-mapping of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Suppose that $\{x_n\}$ is given by $x_0 = x \in C$ and $x_{n+1} = Wx_n$ for every $n = 0, 1, 2, \dots$. Then, $\{x_n\}$ converges weakly to the element z of $\bigcap_{i=1}^{r} F(T_i)$, where $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^{r} F(T_i)} x_n$.

Using Theorem 4.3, we also obtain the following theorems (see [12]).

Theorem 4.5. Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of a Banach space E. Let T be a relatively nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let α be a real number such that $0 < \alpha < 1$. Suppose that $\{x_n\}$ is given by $x_0 = x \in C$ and $x_{n+1} = P_C J^{-1}(\alpha JTx_n + (1-\alpha)Jx_n)$ for every $n = 0, 1, 2, \ldots$. Then, the following hold:

(a) The sequence $\{x_n\}$ is bounded and each weak subsequentially limit of $\{x_n\}$ belongs to $F(T)$.

(b) If the duality mapping J from E into E^* is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the element z of $F(T)$, where $z = \lim_{n\to\infty} P_{F(T)}x_n$.

Theorem 4.6. Let E be a uniformly smooth and uniformly convex Banach space and let $\{C_i\}$ be a finite family of nonempty closed convex subsets of E such that $C = \bigcap_{i=1}^r C_i \neq \emptyset$. Let $P_{C_1}, P_{C_2}, \ldots, P_{C_r}$ be the generalized projections from E onto C_i for $i \in \{1, 2, \ldots\}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for each $i \in 1, 2, \ldots, r$. Let W be the W-mapping of C into itself generated by $P_{C_1}, P_{C_2}, , \ldots, P_{C_r}$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$. Suppose that $\{x_n\}$ is given by $x_0 = x \in E$ and $x_{n+1} = Wx_n$ for every $n = 0, 1, 2, \ldots$. Then, the followingG hold:

(a) The sequence $\{x_n\}$ is bounded and each weak subsequentially limit of $\{x_n\}$ belongs to $\bigcap_{i=1}^r C_i$.

(b) If the duality mapping J from E into E^* is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the element z of $\bigcap_{i=1}^r C_i$, where $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^r C_i}x_n$.

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