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# Nonlinear mappings and the theory of reproducing kernels (非線形システムの同定と逆を求める方法への再生核の理論の応用)

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First we recall a basic relation between linear mappings in the framework of Hilbert spaces and reproducing kernels. In particular, we can see here why we meet ill-posed problems, indeed, we can see the idea and method for the avoidance of the ill-posed problems in the framework of Hilbert spaces. However, this will be a mathematical theory and for the purpose of developing numerical methods, we will need the idea of Tikhonov regularization. However we will need essentially the applications of the theory of reproducing kernels to both mathematical and numerical theories for bounded linear operator equations in the framework of Hilbert spaces.

We consider any *positive matrix* $K(p,q)$ on a fixed set $E$; that is, for an abstract set $E$ the complex-valued function $K(p,q)$ on $E\times E$ satisfies, for any finite points $\{p_j\}$ of $E$ and for any complex numbers $\{C_j\}$,

$$\sum_j \sum_{j'} C_j \overline{C_{j'}} K(p_{j'}, p_j) \geq 0.$$

Then, by the fundamental theorem by Moore–Aronszajn, we have:

**Proposition 0.0.1** *([1]) For any positive matrix $K(p,q)$ on $E$, there exists a uniquely determined functional Hilbert space (abbreviated* RKHS*) $H_K$ comprising functions $\{f\}$ on $E$ and admitting the reproducing kernel $K(p,q)$ satisfying and characterized by*

$$K(\cdot,q)\in H_K \quad \text{for any} \quad q\in E \tag{1}$$

*and, for any $q\in E$ and for any $f\in H_K$*

$$f(q)=(f(\cdot),K(\cdot,q))_{H_K}. \tag{2}$$

For some general properties of reproducing kernel Hilbert spaces and for various constructions of the RKHS $H_K$ from a positive matrix $K(p,q)$, see the book [16] and its Chapter 2, Section 5, respectively.

## CONNECTIONS WITH LINEAR MAPPINGS

Let us connect linear mappings in the framework of Hilbert spaces with reproducing kernels ([9]).

For an abstract set $E$ and for any Hilbert (possibly finite–dimensional) space $\mathcal{H}$, we shall consider an $\mathcal{H}$–valued function $\boldsymbol{h}$ on $E$

$$\boldsymbol{h}: \quad E \longrightarrow \mathcal{H} \tag{3}$$

and the linear mapping from $\mathcal{H}$ into a linear space comprising functions on $E$, given by $\boldsymbol{f} \longrightarrow f$, where

$$f(p) = (\boldsymbol{f}, \boldsymbol{h}(p))_{\mathcal{H}} \quad \text{for} \quad \boldsymbol{f} \in \mathcal{H}. \tag{4}$$

This represents, in particular, the Fredholm integral equations of the first kind in the framework of Hilbert spaces.

For this linear mapping (4), we form the positive matrix $K(p,q)$ on $E$ defined by

$$K(p,q) = (\boldsymbol{h}(q), \boldsymbol{h}(p))_{\mathcal{H}} \quad \text{on} \quad E \times E, \tag{5}$$

which is, by Proposition 0.0.1, a reproducing kernel.

Then, we have the following fundamental results:

(I) For the RKHS $H_K$ admitting the reproducing kernel $K(p,q)$ defined by (5), the images $\{f(p)\}$ by (4) for $\mathcal{H}$ are characterized as the members of the RKHS $H_K$.

(II) In general, we have the inequality in (4)

$$\|f\|_{H_K} \leq \|\boldsymbol{f}\|_{\mathcal{H}}, \tag{6}$$

however, for any $f \in H_K$ there exists a uniquely determined $\boldsymbol{f}^* \in \mathcal{H}$ satisfying

$$f(p) = (\boldsymbol{f}^*, \boldsymbol{h}(p))_{\mathcal{H}} \quad \text{on } E \tag{7}$$

and

$$\|f\|_{H_K} = \|\boldsymbol{f}^*\|_{\mathcal{H}}. \tag{8}$$

In (6), the isometry holds if and only if $\{\boldsymbol{h}(p); p \in E\}$ is complete in $\mathcal{H}$.

(III) We can obtain the inversion formula for (4) in the form

$$f \longrightarrow \boldsymbol{f}^*, \tag{9}$$

by using the RKHS $H_K$.

However, this inversion formula will depend on, case by case, the realizations of the RKHS $H_K$.

(IV) Conversely, if we have an isometric mapping $\tilde{L}$ from the RKHS $H_K$ admitting a reproducing kernel $K(p,q)$ on $E$ onto a Hilbert space $\mathcal{H}$, then the mapping is linear and its isometric inversion $\tilde{L}^{-1}$ is represented in the form (4). Here, the Hilbert space $\mathcal{H}$–valued function $\boldsymbol{h}$ satisfying (3) and (4) is given by

$$\boldsymbol{h}(p) = \tilde{L}K(\cdot,p) \quad \text{on} \quad E \tag{10}$$

and, $\{\boldsymbol{h}(p) : p \in E\}$ is complete in $\mathcal{H}$.

When (4) is isometrical, sometimes we can use the isometric mapping for a realization of the RKHS $H_K$, conversely — that is, if the inverse $L^{-1}$ of the linear mapping (4) is known, then we have $\|f\|_{H_K} = \|L^{-1}f\|_{\mathcal{H}}$.

## GENERAL APPLICATIONS

We shall state some general applications of the results (I)$\sim$(IV) to several wide subjects and their basic references:

**(1)** Linear mappings ([11,13,16,19]).

**(2)** Linear mappings among smooth functions ([21]).

**(3)** Nonharmonic linear mappings ([11]).

**(4)** Various norm inequalities ([14]).

**(5)** Nonlinear mappings ([14],[17]).

**(6)** Linear (singular) integral equations ([22],[6]).

**(7)** Linear differential equations with variable coefficients ([29]).

**(8)** Approximation theory ([3],[16]).

**(9)** Representations of inverse functions ([15]).

**(10)** Various operators among Hilbert spaces ([18]).

**(11)** Sampling theorems ([16], Chapter 4, Section 2).

**(12)** Interpolation problems of Pick-Nevanlinna type ([12]).

**(13)** Analytic extension formulas and their applications ([23],[26]).

**(14)** Inversions of a family of bounded linear operators on a Hilbert space into various Hilbert spaces ([28]).

**(15)** Applications of the reproducing kernel theory to inverse problems ([24]).

**(16)** Principle of telethoscope ([25]).

**(17)** Applications to the Tikhonov regularization ([2,7,8-33]).

In a very general nonlinear mapping of a reproducing kernel Hilbert space, we can look for a natural reproducing kernel Hilbert space containing the image space and furthermore, we can derive a natural norm inequality in the nonlinear mapping. What is a basic relation between linear mappings and non-linear mappings in the framework of reproducing kernel Hilbert spaces? It seems that the theory of reproducing kernels gives a fundamental and interesting answer for this question.

As our new research topics and results, we shall present the identification problems and inversion formulas in very general nonlinear mappings.

## IDENTIFICATIONS OF NON-LINEAR SYSTEMS

Some nonlinear problems had been discussed in [15] and [14,17]. For nonlinear cases, we have basically the identification problems and inversion problems, of course. For inversion formulas, we start with from the general idea in [15], however, the problems are, of course, very involved and so, we shall discuss step by step them, see, for example, [35]. Here, we shall discuss the identification problems for nonlinear systems by using the theory of reproducing kernels based on [36]. We will be able to obtain a very natural general theory if we apply the theory of reproducing kernels. The identification problems may be stated as follows:

We assume a function $f$ on a set $E$ is an input function of a function space and a nonlinear mapping $\varphi$ of $f$

$$\varphi : \quad f \longrightarrow \quad \varphi(f)$$

is given. For a finite number of points $\{p_j\}_{j=1}^{N}$ of the set $E$, we have the observation data as follows:

$$\varphi(f(p_j)) = \alpha_j; \quad j = 1, 2, ..., N. \tag{11}$$

Then, we wish to determine all the out puts of the system: For any $p \in E$

$$\varphi(f(p)).$$

For example, for the typical nonlinear system

$$\varphi(f) = \sum_{n=0}^{\infty} C_n f^n, \tag{12}$$

from (11) we must determine all the coefficients $\{C_j\}$ and so the identification problem will be very involved, See, for example, [5] for the Volterra series idea for non-linear systems. For this identification, a very fairly simple method exists when the function space is a reproducing kernel Hilbert space. Note that the reproducing kernel Hilbert space is a very general and natural Hilbert space; because a function Hilbert space admits a reproducing kernel if and only if the point evaluation $f \longrightarrow f(p) \quad (p \in E)$ is a bounded linear operator from the space into $\mathbf{C}$. In order to challenge to the problem, we shall recall that for a very general nonlinear transform of a reproducing kernel Hilbert space, its image space belongs to a natural reproducing kernel Hilbert space and there exists a natural norm inequality in this nonlinear mapping. These facts will be very important for our present identification problem. See [36] for the details.

Recall that the identification problems may be directly related to interpolation problems, approximations of functions and the theory of learning. See, for example, [4] and [34].

However, the true identification problems will mean that we must determine $\{C_j\}$ in (12) independently of the members of a function space of $f$, not fixed a function $f$. We referred to this more difficult problem in [36].

## REPRESENTATIONS OF INVERSE FUNCTIONS BY THE INTEGRAL TRANSFORM WITH THE SIGN KERNEL

We shall consider some representation of the inversion $\phi^{-1}$ in terms of some integral form - at this moment, we shall need a natural assumption for the mapping $\phi$ . Then, we shall transform the integral representation by the mapping $\phi$ to the original space that is the defined domain of the mapping $\phi$. Then, we will be able to obtain the representation of the inverse $\phi^{-1}$ in terms of the direct mapping $\phi$. In [15], we considered the representation of the inverse $\phi^{-1}$ in some reproducing kernel Hilbert spaces, however, here, we shall consider the representations of the inverse $\phi^{-1}$ for a very concrete situation and we shall give a very fundamental representation of the inverse for some general functions on 1 dimensional spaces. At this moment, indeed, in [35], we considered the problems by using a simple Sobolev and reproducing kernel space. By using the representation of the functions in the reproducing kernel Hilbert space, we will be able to obtain very natural representation formulas of the inverses of some general and reasonable functions.

Note that

$$K(y_1, y_2) = \frac{1}{2} e^{-|y_1 - y_2|} \qquad y_1, \ y_2 \in [A, B] \tag{13}$$

is the reproducing kernel in the Sobolev Hilbert space $H_K$ whose members are real-valued and absolutely continuous functions on $[A, B]$ and whose inner product is given by

$$(f_1, f_2)_{H_K} = \int_A^B (f_1'(y) f_2'(y) + f_1(y) f_2(y)) dy + f_1(A) f_2(A) + f_1(B) f_2(B). \tag{14}$$

For a function $y = f(x)$ that is of $C^1$ class and a strictly increasing function and $f'(x)$ is not vanishing on $[a, b]$ $(f(a) = A, f(b) = B)$. Then, of course, the inverse function $f^{-1}(y)$ is a single-valued function and it belongs to the space $H_K$ and from the reproducing property, we obtain the representation, for any $y_0 \in [f(a), f(b)]$

$$f^{-1}(y_0) = \left(f^{-1}(\cdot), K(\cdot, y_0)\right)_{H_K}$$

$$= \int_{f(a)}^{f(b)} \left((f^{-1})'(y) K_y(y, y_0) + f^{-1}(y) K(y, y_0)\right) dy + aK(f(a), y_0) + bK(f(b), y_0). \tag{15}$$

Surprisingly enough, from this identity we derived the very simple representation

$$f^{-1}(y_0) = \frac{a+b}{2} + \frac{1}{2} \int_a^b \operatorname{sign}\left(y_0 - f(x)\right) dx. \tag{16}$$

By using the several reproducing kernel Hilbert spaces from [16] as in (15), we calculated similarly with the related assumptions, however, surprisingly enough, we obtain the same formula (16). For the formula (16), we note directly that we do not need any smoothness assumptions for the function $f(x)$, indeed, we need only the strictly increasing assumption. The assumption of integrability does not, even, need for the formula (16).

For some multi-dimesional versions of this simple representation, we have the fundamental open problem.

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