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均衡問題と解の近似について
An equilibrium problem and approximation of its solutions

東京工業大学・大学院情報理工学研究科
木村泰紀 (Yasunori Kimura)
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology

1 Introduction

Let X be a set and $f : X \times X \to \mathbb{R}$. The equilibrium problem for f is to find a point $x \in X$ such that

$$f(x,y) \geq 0 \quad \text{for all } y \in X,$$

and the set of its solutions is denoted by $EP(f)$. It is known that the equilibrium problem includes many kinds of important problems in various fields of applied mathematics such as minimization problems, saddle point problems, Nash equilibria in noncooperative games, fixed point problems, and others; see [4].

The existence of the solution for equilibrium problem has been discussed; for instance, see Fan [7], Takahashi [16], Blum and Oettli [4], Iusem and Sosa [11], and others. On the other hand, various types of approximating the solution has been proposed; see Flåm and Antipin [8], Combettes and Hirstaga [6], Iiduka and Takahashi [10], Tada and Takahashi [15], and others.

In this paper, we deal with a sequence of functions as an approximate of the function appearing in the original equilibrium problem. We assume convergence of a sequence of corresponding sets of solutions of equilibrium problems in the sense of Mosco. We obtain weak and strong convergence of a sequence of resolvents to a generalized projection onto the original set of solutions under certain conditions. Our main results are a generalized version of the results discussed in [12].

2 Preliminaries

Throughout this paper, we always deal with a real Banach space and denote it by E. We denote its norm by $\|\cdot\|$, its dual space by E^*, and for $x^* \in E^*$, the value of x^*

at $x \in E$ by $\langle x, x^* \rangle$. The norm of E^* is also denoted by $\|\cdot\|$.

A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for every $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in E$ satisfying that $\|x\| = \|y\| = 1$, it holds that $(\|x+ty\| - \|x\|)/t$ converges as $t \to 0$, and in this case E is said to be smooth. E is said to have the Kadec-Klee property if a weakly convergent sequence $\{x_n\}$ of E with a limit x_0 converges strongly to x_0 whenever $\{\|x_n\|\}$ converges to $\|x_0\|$. For more details, see [9, 17].

The normalized duality mapping $J : E \rightrightarrows E^*$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$. We know that J is single-valued if E is smooth. In this case, $J : E \to E^*$ is norm-to-weak* continuous. Moreover, if E is reflexive and strictly convex, then J is a bijection from E onto E^*.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a reflexive Banach space E. We define two subsets $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $n \in \mathbb{N}$. We define the Mosco convergence [13] of $\{C_n\}$ as follows: If C_0 satisfies that

$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n,$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. For more details, see [3].

Let C be a nonempty closed convex subset of a smooth, reflexive and strictly convex Banach space E. We consider a function $\phi : E \times E \to \mathbb{R}$ defined as

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$. It is easy to show that $\phi(x, y) \geq 0$ for all $x, y \in E$. From strict convexity of E, the function $\phi(\cdot, y)$ is a strictly convex function for every $y \in E$. Therefore, for arbitrarily fixed $y \in E$, a function $\phi(\cdot, y)|_C$ has a unique minimizer, say $x_y \in C$. Using this point, we define the generalized projection Π_C such that $\Pi_C y = x_y$ for all $y \in E$. Notice that if E is a Hilbert space, Π_C coincides with the metric projection onto C since $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$ in this case. For more details, see, for example, [1, 5, 14].

3 Convergence of resolvents for a sequence of functions

Let E be a real Banach space and C a nonempty convex subset of E. We assume that a function $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(E1) $f(x,x) = 0$ for every $x \in C$;
(E2) $f(x,y) + f(y,x) \leq 0$ for every $x, y \in C$;
(E3) $f(x,\cdot)$ is convex and lower semicontinuous for every $x \in C$.

In [12], we assume the following condition which is called upper hemicontinuity of f with respect to the first variable;

(E4) $\limsup_{t\downarrow 0} f(ty + (1-t)x, y) \leq f(x,y)$ for every $x, y \in C$.

We shall assume the maximal monotonicity [4, 2] of f instead of (E4) as follows:

(E5) for each $x \in C$ and $x^* \in E^*$, if $\langle z - x, x^* \rangle - f(z,x) \geq 0$ for all $z \in C$, then $\langle y - x, x^* \rangle + f(x,y) \geq 0$ for all $y \in C$.

Theorem 1 (Aoyama-Kimura-Takahashi [2]). *Let E be a reflexive, smooth, and strictly convex Banach space and C a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions* (E1), (E2), (E3), *and* (E5). *Then for every $x \in E$, there exists a unique $u \in C$ such that*

$$0 \leq f(u,y) + \langle y - u, Ju - Jx \rangle$$

for all $y \in C$.

This theorem guarantees that a resolvent T_{rf} for $f : C \times C \to \mathbb{R}$ and $r > 0$ defined by

$$T_{rf} \colon E \ni x \mapsto \{u \in C : 0 \leq rf(u,y) + \langle y - u, Ju - Jx \rangle \,,\ \forall y \in C\} \subset C$$

is well defined as a single-valued mapping of E into C. Namely, for every $x \in E$, $T_{rf}x$ is a unique point of C which satisfies that

$$0 \leq rf(T_{rf}x, y) + \langle y - T_{rf}x, JT_{rf}x - Jx \rangle$$

for all $y \in C$.

On the other hand, it is easy to see that if f satisfies the conditions (E1), (E2), (E3), and (E4), then f also satisfies the condition (E5). Indeed, let $x \in C$, $x^* \in E^*$, and suppose that for f satisfying (E5), $\langle z - x, x^* \rangle - f(z,x) \geq 0$ for all $z \in C$. Then, for arbitrarily chosen $y \in C$ and $0 < t < 1$, it follows that

$$\begin{aligned} 0 &= f(tx + (1-t)y, tx + (1-t)y) \\ &\leq tf(tx + (1-t)y, x) + (1-t)f(tx + (1-t)y, y) \\ &\leq t\langle tx + (1-t)y - x, x^* \rangle + (1-t)f(tx + (1-t)y, y) \\ &= t(1-t)\langle y - x, x^* \rangle + (1-t)f(tx + (1-t)y, y), \end{aligned}$$

and thus $t\langle y - x, x^* \rangle + f(tx + (1-t)y, y) \geq 0$. Tending $t \to 1$, we have that

$$\langle y - x, x^* \rangle + f(x,y) \geq 0$$

by using (E4). Hence f satisfies (E5).

Therefore, we obtain the following results, which generalize the results shown by the author in [12]. The proofs are the same as in [12].

Theorem 2. *Let E be a reflexive, smooth, and strictly convex Banach space and C a nonempty closed convex subset of E. Let $\{r_n\}$ be a positive real sequence such that $\lim_{n\to\infty} r_n = \infty$. Let $\{f_n\}$ be a sequence of functions of $C \times C$ into $\mathbb{R}$ satisfying the conditions* (E1), (E2), (E3), *and* (E5). *Let C_0 be a nonempty closed convex subset of C satisfying the following conditions:*

(i) $C_0 \subset \text{s-Li}_n\, EP(f_n)$;

(ii) $\text{w-Ls}_n\, EP(f_n + g_{u_n^*}) \subset C_0$ *for every* $\{u_n^*\} \subset E^*$ *converging strongly to* 0,

where $g_{u^*} : C \times C \to \mathbb{R}$ is defined by $g_{u^*}(x, y) = \langle y - x, u^* \rangle$ for $x, y \in C$. Then, a sequence of resolvents $\{T_{r_n f_n} x\}$ converges weakly to $\Pi_{C_0} x \in C_0$ for every $x \in C$.

Theorem 3. *Let E be a reflexive, smooth, and strictly convex Banach space having the Kadec-Klee property. Let C, $\{r_n\}$, $\{f_n\}$ be the same as Theorem 2. Then, a sequence of resolvents $\{T_{r_n f_n} x\}$ converges strongly to $\Pi_{C_0} x \in C_0$ for every $x \in C$.*

Letting $f_n = f$ for all $n \in \mathbb{N}$, we deduce the following corollary.

Corollary 1. *Let E be a reflexive, smooth, and strictly convex Banach space and C a nonempty closed convex subset of E. Let $\{r_n\}$ be a positive real sequence such that $\lim_{n\to\infty} r_n = \infty$. Let f be a function of $C \times C$ into $\mathbb{R}$ satisfying the conditions* (E1), (E2), (E3), *and* (E5). *Then, a sequence of resolvents $\{T_{r_n f} x\}$ converges weakly to $\Pi_{EP(f)} x \in EP(f)$ for every $x \in C$. Moreover, if E has the Kadec-Klee property, then $\{T_{r_n f} x\}$ converges strongly to $\Pi_{EP(f)} x \in EP(f)$ for every $x \in C$.*

Proof. Let $f_n = f$ for all $n \in \mathbb{N}$ and $C_0 = EP(f)$. Then, it is obvious that the condition (i) in Theorem 2 is satisfied. For (ii), Let $\{u_n^*\}$ be a sequence of E^* converging strongly to 0 and $v \in \text{w-Ls}_n\, EP(f + g_{u_n^*})$. Then, there exist a subsequence $\{n_i\}$ of $\mathbb{N}$ and a sequence $\{v_i\} \subset E$ such that $v_i \in EP(f + g_{u_{n_i}^*})$ and that $\{v_i\}$ converges weakly to v. Then, we have that

$$f(v_i, z) + g_{u_{n_i}^*}(v_i, z) = f(v_i, z) + \langle z - v_i, u_{n_i}^* \rangle \geq 0$$

for all $z \in C$. By (E2), it follows that $\langle z - v_i, u_{n_i}^* \rangle - f(z, v_i) \geq 0$ for $z \in C$ and using (E5), we obtain that

$$\langle y - v_i, u_{n_i}^* \rangle + f(v_i, y) \geq 0$$

for all $y \in C$. As $i \to \infty$, we have that

$$f(v, y) = \langle y - v, 0 \rangle + f(v, y) \geq 0$$

for all $y \in C$ and hence $v \in EP(f)$. Therefore $\text{w-Ls}_n\, EP(f + g_{u_n^*}) \subset EP(f) = C_0$ and (ii) holds. Using Theorems 2 and 3, we obtain the desired result. □

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