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COMMENTS ON MEIR-KEELER'S FIXED POINT THEOREM

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ABSTRACT. We give some comments on Meir-Keeler's fixed point theorem. First we give a proof of the theorem. We next compare the theorem with the Banach contraction principle, Edelstein's and Branciari's fixed point theorems. Also, we discuss Lim's characterization and state recent generalizations of the theorem.

1. INTRODUCTION

In 1969, Meir and Keeler [9] proved the following, very interesting and excellent fixed point theorem.

Theorem 1 (Meir and Keeler [9]). *Let (X, d) be a complete metric space and let T be a Meir-Keeler contraction (MKC, for short) on X, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon$$

for all $x, y \in X$. Then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

Recently Suzuki [18] gave a proof of Theorem 1 in which we use reductio ad absurdum only once. The following proof is slightly better than that in [18].

Proof. For $x, y \in X$, putting $\varepsilon := d(x, y)$, we obtain $d(Tx, Ty) \leq d(x, y)$. So $\{d(T^n x, T^n y)\}$ is nonincreasing and thus converges to some nonnegative real number α. Assume $\alpha > 0$. Then from the assumption, there exists $\delta_1 > 0$ such that

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- $d(u,v) < \alpha + \delta_1$ implies $d(Tu, Tv) < \alpha$.

We can choose $\nu \in \mathbb{N}$ such that $d(T^\nu x, T^\nu y) < \alpha + \delta_1$. Then we have

$$\alpha = \lim_{n\to\infty} d(T^n x, T^n y) \leq d(T^{\nu+1}x, T^{\nu+1}y) < \alpha,$$

which is a contradiction. Therefore we obtain

$$\lim_{n\to\infty} d(T^n x, T^n y) = 0$$

for all $x, y \in X$. Fix $x \in X$ and $\varepsilon > 0$. Then there exists $\delta_2 > 0$ such that

- $d(u,v) < \varepsilon + \delta_2$ implies $d(Tu, Tv) < \varepsilon$.

Since $\lim_n d(T^n x, T^{n+1}x) = 0$, $d(T^\ell x, T^{\ell+1}x) < \delta_2$ holds for sufficiently large $\ell \in \mathbb{N}$. We shall show

$$d(T^{\ell+1}x, T^{\ell+m}x) < \varepsilon \tag{1}$$

for $m \in \mathbb{N}$ by induction. It is obvious that (1) holds when $m = 1$. We assume that (1) holds for some $m \in \mathbb{N}$. Then we have

$$d(T^\ell x, T^{\ell+m}x) \leq d(T^\ell x, T^{\ell+1}x) + d(T^{\ell+1}x, T^{\ell+m}x) < \delta_2 + \varepsilon$$

and hence $d(T^{\ell+1}x, T^{\ell+m+1}x) < \varepsilon$ holds. So, by induction, (1) holds for every $m \in \mathbb{N}$. Therefore we have shown

$$\lim_{n\to\infty} \sup_{m>n} d(T^n x, T^m x) = 0.$$

This implies that $\{T^n x\}$ is Cauchy. Since X is complete, $\{T^n x\}$ converges to some point $z \in X$. Since T is continuous, we obtain

$$Tz = T\left(\lim_{n\to\infty} T^n x\right) = \lim_{n\to\infty} T \circ T^n x = z.$$

That is, z is a fixed point of T. For every $y \in X$, we have

$$\lim_{n\to\infty} d(z, T^n y) = \lim_{n\to\infty} d(T^n z, T^n y) = 0.$$

This implies that the fixed point is unique. □

In this paper, we give some comments on the Meir-Keeler fixed point theorem. We have given a proof of the theorem. Next, we compare the theorem with the Banach contraction principle, Edelstein's and Branciari's fixed point theorems. Also, we discuss Lim's characterization and state recent generalizations of the theorem.

2. The Banach Contraction Principle

It is well known that the Meir-Keeler theorem is a generalization of the Banach contraction principle [1] and Edelstein's fixed point theorem [4].

Theorem 2 (Banach [1]). *Let (X, d) be a complete metric space let T be a contraction on X, i.e., there exists $r \in (0, 1)$ such that*

$$d(Tx, Ty) \le r\, d(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Fix $\varepsilon > 0$ and put $\delta = (1/r - 1)\,\varepsilon$. Then if $d(x, y) < \varepsilon + \delta$ and $x \neq y$, we have

$$d(Tx, Ty) \le r\, d(x, y) < r\,\varepsilon + r\,\delta = \varepsilon.$$

Thus, T is an MKC. By Theorem 1, we obtain the desired result. □

Theorem 3 (Edelstein [4]). *Let (X, d) be a compact metric space and let T be a mapping on X. Suppose that*

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.

Proof. Assume that T is not an MKC. Then there exist $\varepsilon > 0$, sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$d(x_n, y_n) < \varepsilon + 1/n \quad \text{and} \quad d(Tx_n, Ty_n) \ge \varepsilon. \tag{2}$$

Since X is compact, without loss of generality, we may assume $\{x_n\}$ and $\{y_n\}$ converge to some points x_0 and y_0 in X, respectively. Since T is continuous, we have

$$d(x_0, y_0) \le \varepsilon \le d(Tx_0, Ty_0) < d(x_0, y_0).$$

This is a contradiction. Therefore T is an MKC. By Theorem 1, we obtain the desired result. □

3. Branciari's Fixed Point Theorem

In 2002, Branciari extended the Banach contraction principle in another direction. The theorem can be proved by Theorem 1; see [20].

Theorem 4 (Branciari [2]). *Let (X,d) be a complete metric space and let T be a Branciari contraction on X, i.e., there exist $r \in [0,1)$ and a locally integrable function f from $[0,\infty)$ into itself such that*

$$\int_0^s f(t)\,dt > 0 \quad \textit{and} \quad \int_0^{d(Tx,Ty)} f(t)\,dt \le r \int_0^{d(x,y)} f(t)\,dt$$

for all $s > 0$ and $x, y \in X$. Then T has a unique fixed point.

Proof. Assume that T is not an MKC. Then there exist $\varepsilon > 0$, sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying (2). We have

$$\int_0^{\varepsilon} f(t)\,dt \le \int_0^{d(Tx_n,Ty_n)} f(t)\,dt \le r \int_0^{d(x_n,y_n)} f(t)\,dt \le r \int_0^{\varepsilon+1/n} f(t)\,dt$$

and hence

$$\int_0^{\varepsilon} f(t)\,dt \le r \int_0^{\varepsilon} f(t)\,dt.$$

This contradicts $\int_0^{\varepsilon} f(t)\,dt > 0$. Therefore T is an MKC. By Theorem 1, we obtain the desired result. □

From the above proof, we know that contractions of integral type are MKC. So, it is natural to consider MKC of integral type. Our answer is that MKC of integral type are still MKC. That is, the following holds.

Theorem 5 ([20]). *Let (X,d) be a metric space and let T be a mapping on X. Let f be a locally integrable function from $[0,\infty)$ into itself satisfying $\int_0^s f(t)\,dt > 0$ for all $s > 0$. Assume that for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_0^{d(x,y)} f(t)\,dt < \varepsilon + \delta \quad \textit{implies} \quad \int_0^{d(Tx,Ty)} f(t)\,dt < \varepsilon$$

for all $x, y \in X$. Then T is an MKC.

4. Lim's Characterization

In 1977, Wong [24] characterized MKC. Lim [8] gave another characterization of MKC, using the notion of L-functions. However, Lim's proof is very difficult. Recently Suzuki [17, 18] improved Lim's theorem and gave a simple proof. Proinov [11] also gave a simple proof.

Definition (Lim [8]). A function φ from $[0,\infty)$ into itself is called an *L-function* if $\varphi(0)=0$, $\varphi(s)>0$ for $s\in(0,\infty)$, and for every $s\in(0,\infty)$ there exists $\delta>0$ such that $\varphi(t)\le s$ for all $t\in[s,s+\delta]$.

Theorem 6 ([8, 18]). *Let (X,d) be a metric space and let T be a mapping on X. Then the following are equivalent:*

(i) *T is an MKC.*

(ii) *There exists an L-function φ such that*

$$x,y\in X,\ x\neq y \quad \textit{implies} \quad d(Tx,Ty)<\varphi\big(d(x,y)\big). \tag{3}$$

(iii) *There exists a nondecreasing, Lipschitz continuous L-function φ satisfying* (3).

Sketch of proof. It is obvious that (iii) implies (ii). We can easily prove that (ii) implies (i). Let us prove that (i) implies (iii). Assume that T is an MKC. Then from the assumption, we can define a function $\alpha:(0,\infty)\to(0,\infty)$ such that

$$d(x,y)<\varepsilon+\alpha(\varepsilon) \quad \text{implies} \quad d(Tx,Ty)<\varepsilon$$

for $\varepsilon\in(0,\infty)$. We also define functions $\beta:(0,\infty)\to[0,\infty)$, $\psi:[0,\infty)\to[0,\infty)$ and $\varphi:[0,\infty)\to[0,\infty)$ as follows:

$$\beta(t)=\inf\big\{\varepsilon>0:\ t<\varepsilon+\delta(\varepsilon)\big\},$$

$$\psi(t)=\begin{cases}0 & \text{if } t=0,\\ \beta(t) & \text{if } t>0 \text{ and } \min\big\{\varepsilon>0: t<\varepsilon+\delta(\varepsilon)\big\} \text{ exists},\\ \big(\beta(t)+t\big)/2 & \text{otherwise},\end{cases}$$

$$\varphi(t)=\sup\Big\{\psi(t)+\min\{2\,(t-u),\ 0\}:\ u\in(0,\infty)\Big\}.$$

Then such φ satisfies (iii). □

5. Generalizations

We finally state recent generalizations of the Meir-Keeler theorem. The following theorem is also a generalization of Kirk's theorem for asymptotic contractions [7].

Theorem 7 ([17]). *Let (X,d) be a complete metric space and let T be a continuous mapping on X. Assume that T is an asymptotic contraction of Meir-Keeler type (ACMK, for short), i.e., there exists a sequence $\{\varphi_n\}$ of functions from $[0,\infty)$ into itself satisfying the following:*

(i) $\limsup_n \varphi_n(\varepsilon) \leq \varepsilon$ *for all* $\varepsilon \geq 0$.

(ii) *For each* $\varepsilon > 0$, *there exist* $\delta > 0$ *and* $\nu \in \mathbb{N}$ *such that* $\varphi_\nu(t) \leq \varepsilon$ *for all* $t \in [\varepsilon, \varepsilon + \delta]$.

(iii) $d(T^n x, T^n y) < \varphi_n(d(x,y))$ *for all* $n \in \mathbb{N}$ *and* $x, y \in X$ *with* $x \neq y$.

Then T *has a unique fixed point.*

In 2001, Suzuki [12] introduced the notion of τ-distances.

Definition ([12]). Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

(τ1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x, y, z \in X$.

(τ2) $\eta(x, 0) = 0$ and $\eta(x,t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable.

(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for all $w \in X$.

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$.

(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. Many useful examples and propositions are stated in [5, 12–16, 19, 22] and references therein. Using the notion of τ-distances, Suzuki [14] proved the following. See also [23].

Theorem 8 ([14]). *Let* X *be a complete metric space with a τ-distance* p, *and let* T *be a mapping on* X. *Suppose that* T *is a Meir-Keeler contraction with respect to* p, *i.e., for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$p(x,y) < \varepsilon + \delta \quad \textit{implies} \quad p(Tx, Ty) < \varepsilon$$

for all $x, y \in X$. *Then* T *has a unique fixed point.*

Kikkawa and Suzuki [6] proved the following theorem, which is also a generalization of Park-Bae's theorem [10]. See also [21].

Theorem 9 ([6]). *Let* (X, d) *be a complete metric space. Let* S *and* T *be mappings on* X *satisfying the following:*

(i) S *is continuous.*

(ii) $T(X) \subset S(X)$.

(iii) *S and T commute.*

Assume that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{2}\,d(Sx,Tx) < d(Sx,Sy) \ \text{ and } \ d(Sx,Sy) < \varepsilon + \delta \quad \text{imply} \quad d(Tx,Ty) < \varepsilon$$

for all $x, y \in X$. Then there exists a unique common fixed point of S and T.

Di Bari, Suzuki and Vetro proved the following, which is also a generalization of Theorem 1 though Theorem 10 is not a fixed point theorem.

Theorem 10 ([3]). *Let X be a uniformly convex Banach space and let A and B be nonempty subsets of X. Suppose that A is closed and convex. Let T be a cyclic Meir-Keeler contraction on $A \cup B$, that is,*

(i) *$T(A) \subset B$ and $T(B) \subset A$.*

(ii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(x,y) < d(A,B) + \varepsilon + \delta \quad \text{implies} \quad d(Tx,Ty) < d(A,B) + \varepsilon$$

for all $x \in A$ and $y \in B$, where $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$.

Then there exists a unique best proximity point z in A, that is, $d(z,Tz) = d(A,B)$.

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