数理解析研究所講究録
第 1611 巻 2008 年 186-194

On some constants in Banach spaces and uniform normal structure*

S. Dhompongsa[‡] and A. Kaewkhao[§,†]

[‡]Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, THAILAND

[§]Department of Mathematics, Faculty of Science, Burapha University, Chonburi 20131, THAILAND

Abstract. We investigate some constants in Banach spaces related to uniform normal structure. We also provide a simple proof of Theorem 2.1 of [9].

2000 *Mathematics Subject Classification.* primary 46B20; secondary 46B08.

Keywords : Nonexpansive mapping; Banach space; weakly convergent sequence coefficient; James constant; normal structure

1 Introduction

As it is well-known, the notions of normal structure and uniform normal structure play important role in metric fixed point theory (see Goebel and Kirk [21]). Some parameters and constants defined on Banach spaces can be used to verify whether a specific Banach space enjoys uniform normal structure. These constants include the James constants and the Jordan-von Neumann constants, which are introduced by Gao and Lau [17] and Clarkson [7], respectively.

In this article we investigate some constants defined in Banach spaces and their relationship with uniform normal structure. We also provide a simple proof of Theorem 2.1 of [9].

2 Preliminaries and Notations

Throughout the paper we let X and X^* stand for a Banach space and its dual space, respectively. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y\in A} \|x-y\| : x \in A\}$ is

*This work was completed with the support of the Commission on Higher Education under the project: "Fixed Point Theory on Banach and Metric Spaces".

†Corresponding author. E-mail address : akaewkhao@yahoo.com Tel.: +66 38 745900 Ext. 3036; fax: +66 38 745900 Ext. 3078

called the Chebyshev radius of A. The number $\operatorname{diam}(A) = \sup\{\|x-y\| : x, y \in A\}$ is called the diameter of A. A Banach space X has normal structure (resp. weak normal structure) if

$$r(A) < \operatorname{diam}(A)$$

for every bounded closed (resp. weakly compact) convex subset A of X with $\operatorname{diam}(A) > 0$. The normal structure coefficient $N(X)$ of X [5, Bynum] is the number

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r(A)}\right\},$$

where the infimum is taken over all bounded closed convex subsets A of X with $\operatorname{diam}(A) > 0$. The weakly convergent sequence coefficient $WCS(X)$ [5] of X is the number

$$WCS(X) = \inf\left\{\frac{A(\{x_n\})}{r_a(\{x_n\})}\right\},$$

where the infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent, $A(\{x_n\}) = \limsup_n \{\|x_i - x_j\| : i, j \geq n\}$ is the asymptotic diameter of $\{x_n\}$, and $r_a(\{x_n\}) = \inf\{\limsup_n \|x_n - y\| : y \in \overline{\operatorname{co}}\{x_n\}\}$ is the asymptotic radius of $\{x_n\}$. A space X with $N(X) > 1$ (resp. $WCS(X) > 1$) is said to have uniform (resp. weak uniform) normal structure. For a Banach space X, the James constant, or the nonsquare constant is defined by Gao and Lau [17] as

$$J(X) = \sup\{\|x+y\| \wedge \|x-y\| : x, y \in B_X\}.$$

It is known that $J(X) < 2$ if and only if X is uniformly nonsquare. Dhompongsa et. al [9, Theorem 3.1] showed that if $J(X) < \frac{1+\sqrt{5}}{2}$, then X has uniform normal structure. The Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of X, which is introduced by Clarkson [7], is defined by

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both zero}\right\}.$$

A relation between these two constants is

$$\frac{(J(X))^2}{2} \leq C_{\mathrm{NJ}}(X) \leq \frac{(J(X))^2}{(J(X)-1)^2+1} \text{([24, Kato et. al]).}$$

From this relation, it is easy to conclude that $C_{\mathrm{NJ}}(X) < 2$ is equivalent to $J(X) < 2$. Recently, Dhompongsa and Kaewkhao [10, Theorem 3.16] obtained the latest upper bound of the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ at $\frac{1+\sqrt{3}}{2}$ for X to have uniform normal structure. However, it is still not clear that if the upper bounds of the James constants and of the Jordan-von Neumann constants are sharp for having uniform normal structure (see a conjecture in [9]). The constant $R(a, X)$, which is a generalized Garcí a-Falset coefficient [19], is introduced by Domínguez [13] : for a given positive real number a

$$R(a, X) := \sup\{\liminf_n \|x + x_n\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequence $\{x_n\}$ in the unit ball of X such that

$$D(x_n) = \limsup_n \left(\limsup_m \|x_n - x_m\|\right) \leq 1.$$

Concerning with this coefficient, Domínguez obtained a fixed point theorem which states that if X is a Banach space with $R(a, X) < 1 + a$ for some a, then X has the weak fixed point property (for details about the (weak) fixed point property, the readers are referred to Goebel and Kirk [22]). In [29], Mazcuñán-Navarro showed that

$$R(1, X) \leq J(X), \tag{2.1}$$

and then combined it with the fixed point theorem of Domínguez to solve a long stand open problem. Indeed, it was proved that the uniform nonsquareness implies the weak fixed point property. One last concept we need to mention is ultrapowers of Banach spaces. We recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I} X_i$ equipped with the norm $\|\{x_i\}\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) above and the definition of the quotient norm that

$$\|\{x_i\}_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an ultrapower of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see Aksoy and Khamsi [1] or Sims [31]).

3 The James constants

Theorem 3.1. [9, Theorem 2.1] If $J(X) < \dfrac{1+\sqrt{5}}{2}$, then X has uniform normal structure.

Proof. Since $J(X) < 2$, X is uniformly nonsquare, and consequently, X is reflexive. Thus, normal structure and weak normal structure coincide. By [18, Theorem 5.2], it suffices to prove that X has weak normal structure.

Suppose on the contrary that X does not have weak normal structure. Thus, there exists a weak null sequence $\{x_n\}$ in S_X such that for $C := \overline{co}\{x_n : n \geq 1\}$,

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam}C = 1 \text{ for all } x \in C \tag{3.1}$$

(cf. [32]). By the definition of $R(1, X)$ and inequality (2.1) We obtain

$$\limsup_{n\to\infty} \|x_n + x\| \leq R(1, X) \leq J(X). \tag{3.2}$$

Convexity of C and equation (3.1) imply that

$$\begin{aligned} \lim_{n\to\infty} \left\|(x_n - x) + \left(\frac{x_n + x}{J(X)}\right)\right\| &= \left(1 + \frac{1}{J(X)}\right) \lim_{n\to\infty} \left\|x_n - \left(\frac{J(X) - 1}{J(X) + 1}\right) x\right\| \\ &= \left(1 + \frac{1}{J(X)}\right). \end{aligned} \tag{3.3}$$

On the other hand, we have, by the weak lower semicontinuity of the norm $\|\cdot\|$

$$\begin{aligned} \limsup_{n\to\infty} \|(x_n - x) - \left(\frac{x_n + x}{J(X)}\right)\| &= \left(1 + \frac{1}{J(X)}\right) \lim_{n\to\infty} \| \left(\frac{R(1,X) - 1}{R(1,X) + 1}\right) x_n - x\| \\ &\geq \left(1 + \frac{1}{J(X)}\right). \end{aligned} \tag{3.4}$$

We can assume, passing through a subsequence if necessary, that "lim sup" in equation (3.4) can be replaced by "lim". Now let $\widetilde{X}$ be a Banach space ultrapower of X over an ultrafilter $\mathcal{U}$ on N. Set

$$\tilde{x} = \{x_n - x\}_{\mathcal{U}} \text{ and } \tilde{y} = \left\{\frac{x_n + x}{J(X)}\right\}_{\mathcal{U}}.$$

(3.1) and (3.2) guarantee that $\tilde{x}$, $\tilde{y} \in B_{\widetilde{X}}$. Then , by (3.3) and (3.4),

$$(\|\tilde{x} + \tilde{y}\| \wedge \|\tilde{x} - \tilde{y}\|) \geq \left(1 + \frac{1}{J(X)}\right),$$

that is

$$J(X) \geq \left(1 + \frac{1}{J(X)}\right).$$

Hence, this contradicts to the assumption that $J(X) < \frac{1+\sqrt{5}}{2}$. The proof is now complete. □

4 Conclusions and Open Problems

The objective of this section is to examine what is known, and not known, about fixed point results for several kinds of mappings related to the two constants. In the notion of geometric properties in Banach spaces especially the notions of normal and uniform normal structure, four important kinds of mappings are involved. Let recall their definitions. Let C be a subset of a Banach space X and $T : C \to C$ be a mapping. Firstly, T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers satisfying $\lim_n k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ $\forall x, y \in C, \forall n \in \mathbb{N}$ [20, Goebel and Kirk]. Secondly, if $k_n \equiv 1, \forall n \in \mathbb{N}$, then T is called a nonexpansive mapping. Thirdly, if there exists a constant k such that $k_n \equiv k, \forall n \in \mathbb{N}$, then T is said to be uniformly Lipschitzian. The final one is an asymptotically regular mapping. A mapping $T : X \to X$ is called asymptotically regular if

$$\lim_n \|T^n x - T^{n+1} x\| = 0 \quad \text{for all } x \in X.$$

The concept of asymptotically regular mappings is due to Browder and Petryshyn [2]. We set

$$s(T) = \liminf_n |T^n|,$$

where $|T^n| = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x-y\|} : x, y \in C, \ x \neq y\right\}$. Fixed point results for asymptotically regular mappings can be found in [3, 4, 16, 14, 15, 23, 26, 27]. Most of these results are related to geometric coefficients in Banach spaces. We state here the one using the weak convergent sequence coefficients.

Theorem 4.1. *[16, Theorem 3.2] Suppose X is a Banach space with $WCS(X) > 1$, C is a nonempty weakly compact convex subset of X, and $T : C \to C$ is a uniformly Lipschitzian mapping such that $s(T) < \sqrt{WCS(X)}$. Suppose in addition that T is asymptotically regular on C. Then T has a fixed point.*

In the following, let C be a closed bounded convex subset of a Banach space X.

Fact 4.1 [29, Mazcuñán-Navarro] If $J(X) < 2$, equivalently $C_{NJ}(X) < 2$, then every nonexpansive mapping $T : C \to C$ has a fixed point.

In [25, Theorem 1], Kim and Xu proved that if a Banach space X has uniform normal structure, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point. By combining this theorem with Theorem 3.1 and Theorem 3.6 of [10], we obtain the following

Fact 4.2 If $J(X) < \frac{1+\sqrt{5}}{2}$, or $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point.

In [6], Casini and Maluta proved the existence of fixed points of a uniformly k−Lipschitzian mapping T with $k < \sqrt{N(X)}$ in a space X with uniform normal structure. (As before, $N(X)$ is the normal structure coefficient of X.) Prus showed in [30] that $N(X) \geq J(X) + 1 - \sqrt{(J(X)+1)^2 - 4}$ (see also Llorens-Fuster [28]). On the other hand, Kato et al. [24] showed that $N(X) \geq \frac{1}{\sqrt{C_{NJ}(X) - \frac{1}{4}}}$ (see also [28]). By using the results just mentioned, we now conclude the following results.

Fact 4.3 (1) Suppose $J(X) < \frac{3}{2}$, and $T : C \to C$ is a uniformly k−Lipschitzian mapping such that

$$k < \sqrt{J(X) + 1 - \sqrt{(J(X)+1)^2 - 4}}.$$

Then T has a fixed point.
(2) Suppose $C_{\mathrm{NJ}}(X) < \frac{5}{4}$, and $T : C \to C$ is a uniformly $k-$Lipschitzian mapping such that

$$k < \frac{1}{\sqrt{\sqrt{C_{\mathrm{NJ}}(X) - \frac{1}{4}}}}.$$

Then T has a fixed point.

In [11], it was shown that

$$WCS(X) \geq \frac{J(X)+1}{(J(X)^2}.$$

For the Jordan von Neumann constant [8], we have

$$[WCS(X)]^2 \geq \frac{2C_{\mathrm{NJ}}(X)+1}{2C_{\mathrm{NJ}}(X)}.$$

By combining these results with Theorem 4.1, we obtain the following

Fact 4.4 Suppose X is a Banach space with $J(X) < \frac{1+\sqrt{5}}{2}$ or $C_{\mathrm{NJ}}(X) < \frac{1+\sqrt{3}}{2}$ C is a nonempty weakly compact convex subset of X, and $T : C \to C$ is a uniformly Lipschitzian mapping such that

$$s(T) < \frac{\sqrt{J(X)+1}}{J(X)^2},$$

or

$$s(T) < \sqrt{\sqrt{\frac{2C_{\mathrm{NJ}}(X)+1}{2C_{\mathrm{NJ}}(X)}}},$$

Suppose in addition that T is asymptotically regular on C. Then T has a fixed point.

We end this paper by posing some open questions about these concepts.

Problem 4.5 Are the upper bounds of the James constants and of the Jordan-von Neumann constants sharp for a space to have uniform normal structure ?

Problem 4.6 Can the upper bounds of $J(X)$ and $C_{NJ}(X)$ appearing in Fact 4.2 be improved ?

Problem 4.7 Can the upper bounds of k appearing in Fact 4.3 be improved ?

Problem 4.8 Can the upper bounds of $s(T)$ appearing in Fact 4.4 be improved ?

Acknowledgement

The authors are grateful to Kyoto University, Professor Shigeo Akashi, and Professor Wataru Takahashi for their kind hospitality. The second author would like to thank the faculty of science, Burapha University for the financial support.

References

[1] A.G. Aksoy and M.A. Khamsi, Nonstandard methods in fixed point theory, Springer-Verlag, 1990.

[2] F. E. Browder and W. V. Petryshyn, The solutions by iteration of nonlinear functional equations in Banach spaces, Bull. Amer. Math. Soc. 72 (1966), 571-576.

[3] M. Budzynska, T. Kuczumow and S. Reich, Uniform asymptotic normal structure, the uniform semi-Opial property, and fixed points of asymtotically regular uniformly Lipschitzian semigroups : part I, preprint.

[4] M. Budzynska, T. Kuczumow and S. Reich, Uniform asymptotic normal structure, the uniform semi-Opial property, and fixed points of asymtotically regular uniformly Lipschitzian semigroups : part II, preprint.

[5] W. L. Bynum, Normal structure coefficients for Banach spaces, Pac. J. Math. 86 (1980), 427-436.

[6] E. Casini and E. Maluta, Fixed points of uniformly Lipschitzian mappings in spaces with uniformly normal structure, Nonlinear Analysis, 9 (1985), 103-108.

[7] J. A. Clarkson, The von Neumann-Jordan constant for the Lebesgue spaces, Ann. of Math. 38 (1973), 114-115.

[8] S. Dhompongsa, T. Domínguez Benavides, A Kaewcharoen, A. Kaewkhao and B. Panyanak, The Jordan-von Neumann constant and fixed points for multivalued non-expansive mappings, J. Math. Anal. Appl., J. Math. Anal. Appl. 2006; 320; 2: 916-927.

[9] S. Dhompongsa, A. Kaewkhao, and S. Tasena, On a generalized James constant, J. Math. Anal. Appl.285(2003) 2, 419-435.

[10] S. Dhompongsa and A. Kaewkhao, A note on properties that imply the weak fixed point property, Abstract and Applied Analysis, 2006; doi:10.1155/AAA/2006/34959

[11] Dhompongsa S., Kaewkhao A, An inequality concerning the James constant and the weakly convergent sequence coefficient . Journal of Nonlinear and Convex Analysis. 2007 ; 8 : 325-333.

[12] T. Domínguez Benavides, Lower bounds for normal structure coefficients, Proc. Royal Soc. Edinburgh 121A (1992), 254-252.

[13] T. Domínguez Benavides, Stability of the fixed point property for nonexpansive mappings, Houston J. of Math. (4) 22 (1996), 835-849.

[14] T. Domínguez Benavides, Fixed point theorems for uniformly Lipschitzian mappings and asymptotically regular mappings, Nonlinear Analysis vol. 32 (1) (1998) 15-27.

[15] T. Domínguez Benavides and M. A. Japón Pineda, Opial modulus, moduli of noncompact convexity, and fixed points for asymtotically regular mappings, Nonlinear Analysis, 41 (2000), 617-630.

[16] T. Domínguez Benavides and H. K. Xu, A new geometrical coefficient for Banach spaces and its applications in fixed point theory, Nonlinear Analysis vol. 25 no. 3 (1995), 311-325.

[17] J. Gao and K. S. Lau, On the geometry of spheres in normed linear spaces, J. Austral. Math. Soc. Ser. A 48 (1990), 101–112.

[18] J. Gao and K. S. Lau, On two classes of Banach spaces with uniform normal structure, Studia Math. (1) 99 (1991), 41–56.

[19] J. Garcia-Falset, Stability and fixed point for nonexpansive mappings, Houston J. Math. 99 (1) (1991), 40-56.

[20] K. Goebel and W. A. Kirk, A fixed point theorem for asymtotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972) 171-174.

[21] K. Goebel and W. A. Kirk, "Topics in metric fixed point theory", Cambridge University Press, Cambridge, 1990.

[22] K. Goebel and W. A. Kirk, Classical theory of nonexpansive mappings, Handbook of Metric Fixed Point Theory, W. A. Kirk and B. Sims, Editors, Kluwer Academic Publishers, Dordrecht, 2001.

[23] J. Gorinicki, A fixed point theorem for asymtotically regular mappings, Colloq. Math. LXIV (1993) 55-57.

[24] M. Kato, L. Maligranda and Y. Takahashi, On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces, Studia Math. (3) 144 (2001), 275–295.

[25] T. H. Kim and H. K. Xu, Remarks on asymtotically nonexpansive mappings, Nonlinear Analysis, 41 (2000), 405-415.

[26] T. Kuczumow, Opial's modulus and fixed points of semigroups of mappings, Proc. Amer. Math. Soc., in press.

[27] T. Kuczumow and S. Reich, An application of Opial's modulus to fixed point theory of semigroup of Lipschipzian mappings, preprint.

[28] A. Llorens-Fuster, Some moduli and constants related to metric fixed point theory, Handbook of Metric Fixed Point Theory, W. A. Kirk and B. Sims, Editors, Kluwer Academic Publishers, Dordrecht, 2001.

[29] E. M. Mazcuñán-Navarro, Geometry of Banach spaces in Metric Fixed Point Theory, Ph. D. Thesis.

[30] S. Prus, Some estimates for the normal structure coefficient in Banach spaces, Rendiconti del Cir. Mat. di Palermo, 40 (1991), 128-135.

[31] B. Sims, "Ultra-techniques in Banach space theory," Queen's Papers in Pure and Applied Mathematics, vol. 60, Queen's University, Kingston, 1982.

[32] B. Sims, A class of spaces with weak normal structure, Bull. Austral. Math. Soc. 49 (1994) 523 · 28.