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Dynamical properties of equivariant holomorphic maps

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Abstract

This paper is a resume of [10]. We consider complex dynamics of a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, which is S_{k+2}-equivariant and *critically finite*, for each $k \geq 1$. Here S_{k+2} is the $k+2$-th symmetric group. The Fatou set of each map of this family consists of attractive basins of superattracting points. Each map of this family satisfies Axiom A.

1 Introduction

For a finite group G acting on $\mathbf{P}^k$ as projective transformations, we say that a rational map f on $\mathbf{P}^k$ is *G-equivariant* if f commutes with each element of G. That is, $f \circ r = r \circ f$ for any $r \in G$, where $\circ$ denotes the composition of maps. P. Doyle and C. McMullen [3] introduced the notion of *equivariant* maps on $\mathbf{P}^1$ to solve quintic equations. See also [11] for *equivariant* maps on $\mathbf{P}^1$. In the study of extending P. Doyle and C. McMullen's result to higher dimensions, S. Crass [2] found a good family of finite groups and *equivariant* maps for which one may say something about global dynamics. S. Crass [2] conjectured that the Fatou set of each map of this family consists of attractive basins of superattracting points. Our results [10] give affirmative answers for the conjectures in [2].

In section 2 we shall explain an action of the symmetric group S_{k+2} on $\mathbf{P}^k$ and properties of our S_{k+2}-equivariant map. In section 3 and 4 we shall denote our results about the Fatou sets and hyperbolicity of our maps. We need the properties of our maps and Kobayashi metrics for the proofs.

2 S_{k+2}-equivariant maps on $\mathbf{P}^k$

S. Crass [2] selected the symmetric group S_{k+2} as a finite group acting on $\mathbf{P}^k$ and found an S_{k+2}-*equivariant* map which is holomorphic and *critically finite* for each $k \geq 1$. We denote by $C = C(f)$ the critical set of f and say that f is *critically finite* if each irreducible component of $C(f)$ is periodic or preperiodic. More precisely, S_{k+2}-*equivariant* map g_{k+3} defined in section 2.2 preserves each irreducible component of $C(g_{k+3})$, which is a projective hyperplane. The complement of $C(g_{k+3})$ is Kobayashi hyperbolic. Furthermore restrictions of g_{k+3} to invariant projective subspaces have the same properties as above. See section 2.3 for details.

2.1 S_{k+2} acts on $\mathbf{P}^k$

An action of the $(k+2)$-th symmetric group S_{k+2} on $\mathbf{P}^k$ is induced by the permutation action of S_{k+2} on $\mathbf{C}^{k+2}$ for each $k \geq 1$. The transposition (i, j) in S_{k+2} corresponds with the transposition "$u_i \leftrightarrow u_j$" on $\mathbf{C}_u^{k+2}$, which pointwise fixes the hyperplane $\{u_i = u_j\} = \{u \in \mathbf{C}_u^{k+2} \mid u_i = u_j\}$. Here $\mathbf{C}^{k+2} = \mathbf{C}_u^{k+2} = \{u = (u_1, u_2, \cdot\cdot, u_{k+2}) \mid u_i \in \mathbf{C} \text{ for } i = 1, \cdot\cdot, k+2\}$.

The action of S_{k+2} preserves a hyperplane H in $\mathbf{C}_u^{k+2}$, which is identified with $\mathbf{C}_x^{k+1}$ by projection $A : \mathbf{C}_u^{k+2} \to \mathbf{C}_x^{k+1}$,

$$H = \left\{ \sum_{i=1}^{k+2} u_i = 0 \right\} \overset{A}{\simeq} \mathbf{C}_x^{k+1} \text{ and } A = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}.$$

Here $\mathbf{C}^{k+1} = \mathbf{C}_x^{k+1} = \{x = (x_1, x_2, \cdot\cdot, x_{k+1}) \mid x_i \in \mathbf{C} \text{ for } i = 1, \cdot\cdot, k+1\}$.

Thus the permutation action of S_{k+2} on $\mathbf{C}_u^{k+2}$ induces an action of "S_{k+2}" on $\mathbf{C}_x^{k+1}$. Here "S_{k+2}" is generated by the permutation action S_{k+1} on $\mathbf{C}_x^{k+1}$ and a $(k+1, k+1)$-matrix T which corresponds to the transposition $(1,\ k+2)$ in S_{k+2},

$$T = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \dots & 1 \end{pmatrix}.$$

Hence the hyperplane $\{u_i = u_j\}$ corresponds to $\{x_i = x_j\}$ for $1 \leq i < j \leq k+1$. The hyperplane $\{u_i = u_{k+2}\}$ corresponds to $\{x_i = 0\}$ for $1 \leq i \leq k+1$. Each element in "S_{k+2}" which corresponds to some transposition in S_{k+2} pointwise fixes one of these hyperplanes in $\mathbf{C}_x^{k+1}$.

The action of "S_{k+2}" on $\mathbf{C}^{k+1}$ projects naturally to the action of "S_{k+2}" on $\mathbf{P}^k$. These hyperplanes on $\mathbf{C}^{k+1}$ projects naturally to projective hyperplanes on $\mathbf{P}^k$. Here $\mathbf{P}^k = \{x = [x_1 : x_2 : \cdot\cdot : x_{k+1}] \mid (x_1, x_2, \cdot\cdot, x_{k+1}) \in \mathbf{C}^{k+1} \setminus \{0\}\}$. Each element in the action of "S_{k+2}" on $\mathbf{P}^k$ which corresponds to some transposition in S_{k+2} pointwise fixes one of these projective hyperplanes. We denote "S_{k+2}" also by S_{k+2} and call these projective hyperplanes transposition hyperplanes.

2.2 Existence of our maps

One way to get S_{k+2}-*equivariant* maps on $\mathbf{P}^k$ which are *critically finite* is to make S_{k+2}-*equivariant* maps whose critical sets coincide with the union of the transposition hyperplanes.

Theorem 1 ([2]). *For each $k \geq 1$, g_{k+3} defined below is the unique S_{k+2}-equivariant holomorphic map of degree $k+3$ which is doubly critical on each transposition hyperplane.*

$$g = g_{k+3} = [g_{k+3,1} : g_{k+3,2} : \cdot\cdot : g_{k+3,k+1}] : \mathbf{P}^k \to \mathbf{P}^k,$$

$$\textit{where } g_{k+3,l}(x) = x_l^3 \sum_{s=0}^{k} (-1)^s \frac{s+1}{s+3} x_l^s A_{k-s}, \ A_0 = 1,$$

and A_{k-s} is the elementary symmetric function of degree k-s in $\mathbf{C}^{k+1}$.

Then the critical set of g coincides with the union of the transposition hyperplanes. Since g is S_{k+2}-*equivariant* and each transposition hyperplane is pointwise fixed by some element in S_{k+2}, g preserves each transposition hyperplane. In particular g is *critically finite*. Although Crass [2] used this explicit formula to prove Theorem 1, we shall only use properties of the S_{k+2}-*equivariant* maps described below.

2.3 Properties of our maps

Let us look at properties of the S_{k+2}-*equivariant* map g on $\mathbf{P}^k$ for a fixed k, which is proved in [2] and shall be used to prove our results. Let L^{k-1} denote one of the transposition hyperplanes, which is isomorphic to $\mathbf{P}^{k-1}$. Let L^m denote one of the intersections of $(k-m)$ or more distinct transposition hyperplanes which is isomorphic to $\mathbf{P}^m$ for $m = 0, 1, \cdot\cdot, k-1$.

First, let us look at properties of g itself. The critical set of g consists of the union of the transposition hyperplanes. By S_{k+2}-*equivariance*, g preserves each transposition hyperplane. Furthermore the complement of the critical set of g is Kobayashi hyperbolic.

Next, let us look at properties of g restricted to L^m for $m = 1, 2, \cdot\cdot, k-1$. Let us fix any m. Since g preserves each L^m, we can also consider the dynamics of g restricted to any L^m. Each restricted map has the same properties as above. Let us fix any L^m and denote by $g|_{L^m}$ the restricted map of g to the L^m. The critical set of $g|_{L^m}$ consists of the union of intersections of the L^m and another L^{k-1} which does not include the L^m. We denote it by L^{m-1}, which is an irreducible component of the critical set of $g|_{L^m}$. By S_{k+2}-*equivariance*, $g|_{L^m}$ preserves each irreducible component of the critical set of $g|_{L^m}$. Furthermore the complement of the critical set of $g|_{L^m}$ in L^m is Kobayashi hyperbolic.

Finally, let us look at a property of superattracting fixed points of g. The set of superattracting points, where the derivative of g vanishes for all directions, coincides with the set of L^0's.

Remark 1. *For every $k \geq 1$ and every m, $1 \leq m \leq k$, a restricted map of g_{k+3} to any L^m is not conjugate to g_{m+3}.*

3 The Fatou sets of the S_{k+2}-equivariant maps

Let us recall theorems about *critically finite* holomorphic maps. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. The Fatou set of f is defined to be the maximal open subset where the iterates $\{f^n\}_{n\geq 0}$ is a normal family. The Julia set of f is defined to be the complement of the Fatou set of f. Each connected component of the Fatou set is called a Fatou component. Let U be a Fatou component of f. A holomorphic map h is said to be a limit map on U if there is a subsequence $\{f^{n_s}|_U\}_{s\geq 0}$ which locally converges to h on U. We say that a point q is a Fatou limit point if there is a limit map h on a Fatou component U such that $q \in h(U)$. The set of all Fatou limit points is called the Fatou limit set. We define the ω-limit set $E(f)$ of the critical points by

$$E(f) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{n=j}^{\infty} f^n(C)}.$$

Theorem 2. *([9, Proposition 5.1]) If f is a critically finite holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, then the Fatou limit set is contained in the ω-limit set $E(f)$.*

Let us recall the notion of Kobayashi metrics. Let M be a complex manifold and $K_M(x, v)$ the Kobayashi quasimetric on M,

$$\inf\left\{ |a| \,\middle|\, \varphi : \mathbf{D} \to M : \text{holomorphic}, \varphi(0) = x, D\varphi\left(a\left(\frac{\partial}{\partial z}\right)_0\right) = v, a \in \mathbf{C}\right\}$$

for $x \in M$, $v \in T_x M$, $z \in \mathbf{D}$, where $\mathbf{D}$ is the unit disk in $\mathbf{C}$. We say that M is Kobayashi hyperbolic if K_M becomes a metric.

Let us recall theorems about dynamics of *critically finite* holomorphic maps in low dimensions. Theorem 5 is a corollary of Theorem 3 and Theorem 4 for $k = 1$ and 2.

Theorem 3. *([7, Corollary 14.5]) If f is a critically finite holomorphic map from $\mathbf{P}^1$ to $\mathbf{P}^1$, then the only Fatou components of f are attractive components of superattracting points. Moreover if the Fatou set is not empty, then the Fatou set has full measure in $\mathbf{P}^1$.*

Theorem 4. *([4, Theorem 7.7]) If f is a critically finite holomorphic map from $\mathbf{P}^2$ to $\mathbf{P}^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of f are attractive components of superattracting points.*

We get our first result by using Theorem 2, Kobayashi metrics and the properties of our maps.

Theorem 5. *For each $k \geq 1$, the Fatou set of the S_{k+2}-equivariant map g consists of attractive basins of superattracting fixed points which are intersections of k or more distinct transposition hyperplanes.*

4 The S_{k+2}-equivariant maps satisfy Axiom A

Let us define hyperbolicity of non-invertible maps and the notion of Axiom A. See [5] for details. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$ and K a compact subset such that $f(K) = K$. Let $\widehat{K}$ be the set of histories in K and $\widehat{f}$ the induced homeomorphism on $\widehat{K}$. We say that f is hyperbolic on K if there exists a continuous decomposition $T_{\widehat{K}} = E^u + E^s$ of the tangent bundle such that $D\widehat{f}(E^{u/s}_{\widehat{x}}) \subset E^{u/s}_{\widehat{f}(\widehat{x})}$ and if there exists constants $c > 0$ and $\lambda > 1$ such that for every $n \geq 1$,

$$|D\widehat{f}^n(v)| \geq c\lambda^n |v| \text{ for all } v \in E^u \text{ and}$$

$$|D\widehat{f}^n(v)| \leq c^{-1}\lambda^{-n} |v| \text{ for all } v \in E^s.$$

Here $|\cdot|$ denotes the Fubini-Study metric on $\mathbf{P}^k$. If a decomposition and inequalities above hold for f and K, then it also holds for $\widehat{f}$ and $\widehat{K}$. In particular we say that f is expanding on K if f is hyperbolic on K with unstable dimension k. Let Ω be the non-wandering set of f, i.e., the set of points for any neighborhood U of which there exists an integer n such that

$f^n(U)$ intersects with U. By definition, Ω is compact and $f(\Omega) = \Omega$. We say that f satisfies Axiom A if f is hyperbolic on Ω and periodic points are dense in Ω.

Let us introduce a theorem which deals with repelling part of dynamics. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. We define the k-th Julia set J_k of f to be the support of the measure with maximal entropy, in which repelling periodic points are dense. It is a fundamental fact that in dimension 1 the 1st Julia set J_1 coincides with the Julia set J. Let K be a compact subset such that $f(K) = K$. We say that K is a repeller if f is expanding on K.

Theorem 6. *([6]) Let f be a holomorphic map on $\mathbf{P}^k$ of degree at least 2 such that the ω-limit set $E(f)$ is pluripolar. Then any repeller for f is contained in J_k. In particular,*

$$J_k = \overline{\{repelling\ periodic\ points\ of\ f\}}$$

If f is *critically finite*, then $E(f)$ is pluripolar. Hence our maps satisfies the condition in the theorem above.

We get our second result by using Theorem 3, Kobayashi metrics and the properties of our maps.

Theorem 7. *For each $k \geq 1$, the S_{k+2}-equivariant map g satisfies Axiom A.*

Since g satisfies Axiom A, [1, Theorem 4.11] and [8] induces the following corollary.

Corollary 1. *The Fatou set of the S_{k+2}-equivariant map g has full measure in $\mathbf{P}^k$ for each $k \geq 1$.*

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