数理解析研究所講究録
第 1612 巻 2008 年 76-79

ADJOINTNESS IN THE CATEGORY OF PARTIAL ABELIAN MONOIDS

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1. Introduction

Partial abelian monoid is a topological space equipped with a partial sum. In [5] and [3], partial abelian monoid is used as a recipe for generalized homology thories. It suggests that the category of partial abelian monoid (denoted by PAM) has a structure rich enough for describing stable homotopy theory. On the other hand, PAM contains the category Top_* of pointed topological spaces as a full sub-category as well as the category AM of topological abelian monoids. This shows that every phenomena of unstable homotopy theory occurs in PAM. With these facts as a back ground, it should be valuable if suitable homotopy theory of partial abelian monoid is established. In this paper, we start algebraic topology of partial abelian monoids by showing that some adjointness holds for appropriate product and mapping space construction in PAM. In the case of abelian monoids, a very successful theory is developed by M. C. McCord [2]. It is also notable that N. J. Kuhn investigates[1] various facts in homotopy theory in a unified way being based on the McCord model.

2. Partial abelian monoid

A pointed topological space M is a partial abelian monoid if there exists a subspace $M_2 \subset M \times M$ and a map $\mu : M_2 \to M$ satisfying following conditions :

(1) $M \vee M \subset M_2$ and μ coincides with the folding map on $M \vee M$; $\mu(m,*) = m = \mu(*,m)$ for any $m \in M$.
(2) If $(m_1, m_2) \in M_2$ then $(m_2, m_1) \in M_2$ and $\mu(m_1, m_2) = \mu(m_2, m_1)$.
(3) $(m_1, m_2) \in M_2$ and $(\mu(m_1, m_2), m_3) \in M_2$ iff $(m_2, m_3) \in M_2$ and $(m_1, \mu(m_2, m_3) \in M_2$ and $\mu(\mu(m_1, m_2), m_3) = \mu(m_1, \mu(m_2, m_3))$.

When M and N are partial abelian monoids, a map $f : M \to N$ is called a homomorphism if $(f \times f)(M_2) \subset N_2$ and $f(m_1+m_2) = f(m_1)+f(m_2)$ for any $(m_1, m_2) \in M_2$. We denote the space of homomorphisms from M to N by $\mathrm{Hom}(M, N)$. We give $\mathrm{Hom}(M, N)$ a PAM structure by letting $\mathrm{Hom}(M, N)_2 = \{(f, g) \mid (f(m), g(m)) \in N_2 \text{ for any } m \in M\}$ and $f+g$ be the pointwise sum of f and g for $(f, g) \in \mathrm{Hom}(M, N)_2$. We denote by PAM the category with partial abelian monoids as objects and homomorphisms between them as morphisms.

Examples 1. (1) Any based space X can be considered as a PAM by letting $X_2 = X \vee X$ and μ be the folding map. This PAM is called a trivial PAM. Moreover, a homomorphism from a trivial PAM to another PAM is just a based map. Especially, we observe that Top_* is a full sub-category of PAM.

(2) Any abelian monoid G can be considered as a PAM in an obvious way. The category of topological abelian monoids, denoted by AM, is also a full subcategory of PAM.
(3) Configuration space $C(V, M)$ is a space of configuration of finite number of points in V labelled by M, where V is a space and M is a PAM. An element of $C(V, M)$ can be represented by a pair (S, m), where S is a finite subset of V and m is a map $S \to M$. It can be viewed as a PAM by superimposition : two configurations represented by (S, v) and (T, w) are summable if $(v(x), w(x))$ is summable in M for any $x \in S \cap T$. We keep these examples in our mind when we speak of PAM as well as the following one, which is relevant with connective K-homology ([4]).
(4) Let Gr^∞ denote the inifinite Grassmannian manifold, the space of finite dimensional subspaces of $\mathbb{R}^\infty$. Two subvector spaces V and W in $\mathbb{R}^\infty$ are summable if V and W are orthogonal.

3. Product

Before giving a definition of the product, we observe the classifying space construction, which indicates our choice of product in PAM.

We can associate a simplicial space to a partial abelian monoid in a natural way. Given a partial abelian monoid M, let M_k denote the summable k-tuples in M. Then a simplicial space is given by a sequence $M_0, M_1, \dots$ of spaces and the degeneracy maps given by insertion of the unit and the face maps given by taking a sum. The classifying space BM is defined by the geometric realization of this simplicial space. As the name shows, BM coincides with the usual classifying space when M is an abelian monoid. On the other hand, BM has a homotopy type of ΣX when $M = X$ is a space.

Now we define a product in PAM so that it is a generalization of BM. Recall that the infinite symmetric product $\mathrm{SP}^\infty X$ of a space X is a free abelian monoid generated by X with appropriate topology. We denote the formal sum in $\mathrm{SP}^\infty X$ by $\dotplus$, thus any element of $\mathrm{SP}^\infty X$ can be written as a formal sum $x_1 \dotplus \cdots \dotplus x_k$ for some $x_1, \dots, x_k \in X$. We have a monoid, denoted M^{mon} associated to a partial abelian monoid M and a PAM map $\iota : M \to M^{\mathrm{mon}}$ which satisfies the following universality : for any abelian monoid G and a PAM map $f : M \to G$, there exists a unique homomorphism $f^{\mathrm{mon}} : M^{\mathrm{mon}} \to G$ such that $f^{\mathrm{mon}} \circ \iota = f$.

We set up some terminologies on summability. By the associativity, it makes sense to say that a finite multiset in a partial abelian monoid M is summable or not, where a finite multiset in M is a function $M \to \mathbb{N}$ with finite support. Note that a finite multiset in M can be identified with an element of $\mathrm{SP}^\infty(M)$. When we have a map $f : X \to M$ with finite support, we can speak of a multiset $\mathrm{Im} f$ in M given by a function $m \mapsto \#\{x \in X \mid f(x) = m\}$. It also makes sense to say that an element of M^{mon} is summable or not, since the summability is independent of the choice of representative in $\mathrm{SP}^\infty(M)$. For an element $\alpha \in M^{\mathrm{mon}}$, we denote by $|\alpha|$ the minimum length of the representative in $\mathrm{SP}^\infty(M)$.

Let $M \otimes N$ be a monoid defined by taking the quotient of $\mathrm{SP}^\infty(M \times N)$ by an equivalence relation generated by $(m_1, n) \dotplus (m_2, n) \sim (m_1 + m_2, n)$ and $(m, n_1) \dotplus (m, n_2) \sim (m, n_1 + n_2)$. Note that $M \otimes N$ is canonically isomorphic to $M^{\mathrm{mon}} \otimes N^{\mathrm{mon}}$. We denote by $\pi : \mathrm{SP}^\infty(M \times N) \to M \otimes N$ the projection. Let $M \tilde{\ltimes} N$ be a subspace of $\mathrm{SP}^\infty(M \times N)$ consisting of elements with summable M-coordinates : $M \tilde{\ltimes} N = \{(m_1, n_1) \dotplus \cdots \dotplus (m_k, n_k) \mid (m_1, \dots m_k) \in M_k, n_i \in N\}$. We define $M \ltimes N$ be the image of $M \tilde{\ltimes} N$ in $M \otimes N$ under the projection π.

A subspace $(M \ltimes N)_2$ of $(M \ltimes N)^2$ is defined as the subspace consisting of pairs whose sum in $M \otimes N$ is in $M \ltimes N$:

$$(M \ltimes N)_2 = \{(\alpha, \beta) \mid \alpha \dot{+} \beta \in M \ltimes N\}.$$

We can define $\mu : (M \ltimes N)_2 \to M \ltimes N$ by $\mu(\alpha, \beta) = \alpha + \beta$. Now μ is unital and commutative, but we need some condition on M or N to show that it is associative.

A partial abelian monoid is said to have the unique factorization property (UF for short) if any element have unique decomposition into a sum of irreducible elements. It is said to be non-invertible if it has no invertible elements.

Theorem 1. Assume one of the following conditions :

(C1) M is UF,
(C2) M is a monoid, or
(C3) N is trivial (a space) .

Assume also that N is non-invertible. Then the product μ is associative, so that $M \ltimes N$ is a partial abelian monoid with this multiplication.

To prove the above theorem, we introduce the following Lemma which shows that under a suitable condition, a subspace X of an abelian monoid G has a structure of partial abelian monoid. For any (based) subspace $X \subset G$, we can define a subspace $X_2 \subset X^2$ by $X_2 = \{(x_1, x_2) \mid x_1 + x_2 \in X\}$ and a partial sum by $\mu(x_1, x_2) = x_1 + x_2$. Then μ is unital and commutative, thus associativity is only the problem.

Lemma 2. Let X be a subspace of an abelian monoid G. Assume for any x and $y \in G$, x and $x + y \in X$ implies $y \in X$. Then μ is associative so that X is a sub-partial abelian monoid of G.

A proof of the above theorem and lemma is give in the next section.

Note that this choice of the 'tensor product' can be justified by the the following examples :

Examples 2. (1) When $M = G$ is an abelian monoid, $G \ltimes N = G \otimes N^{\mathrm{mon}}$.

(2) When $M = X$ is a space, $X \ltimes N = X \wedge N$.

(3) When $N = X$ is a space, $M \ltimes X \subset C(X, M)$ is a configuration space of finite number of points in X which has a totally summable labels in M. Thus giving an element of $M \ltimes X$ amounts to giving a pair (S, m) with S a finite subset of X and $m : S \to M$ is a map such that $\mathrm{Im}(m)$ considered as a multiset in M is summable. Especially, $M \ltimes S^1$ coincides with the classifying space BM.

In the following, when we speak of $M \ltimes N$ (or $\ltimes$-product of other PAMs), we assume that one of the conditions (C1) $\sim$ (C3) in Theorem 1 holds. Note that if M and N both satisfy one of (C1) $\sim$ (C3) simultaneously, then $M \ltimes N$ satisfy the same condition.

Unfortunately, we should change the PAM structure of the function space to have an adjointness. If $(a, b), (b, c), (c, a) \in M_2$ implies $(a, b, c) \in M_3$, we say that M is pairwise determined. Let $\hom(M, N) = \mathrm{Hom}(M, N)$ as a space and

$$\hom(M, N)_2 = \{(f, g) \mid (f(m_1), f(m_2)) \in N_2 \text{ for any } (m_1, m_2) \in M^2\}.$$

Then we can define a partial sum on $\hom(M, N)$ by the pointwise sum. This partial sum is unital and commutative. It is associative if N is pairwise determined. Thus $\hom(M, N)$ is in PAM if N is pairwise determined.

Then we have the following adjointness in the category of partial abelian monoids.

Theorem 3. Assume one of (C1)~(C3) in Theorem 1. Assume also that N is non-invertible and K is pairwise determined. Then we have an isomorphism in PAM :

$$\hom(M \ltimes N, K) \cong \hom(M, \hom(N, K)).$$

4. Proof of Lemmas and Theorems

Proof of Lemma 2. The problem is only to show that $(y, z) \in X_2$ when $(x, y) \in X_2$ and $(x + y, z) \in X_2$. But $(x + y, z) \in X_2$ is equivalent to $x + y + z \in X$. Thus we have $y + z \in X_2$ by the assumption. □

Proof of Theorem 1. It is clear that $M \ltimes N$ satisfies the assumption of Lemma 2 if M is a monoid or N is a space(see Examples). Assume that M is UF. Suppose $\alpha, \alpha+\beta \in M \ltimes N$ for $\alpha, \beta \in M \otimes N$. Since M is UF, we can write uniquely as

$$\alpha = m_1 \otimes \alpha_1 + \cdots + m_k \otimes \alpha_k$$

and

$$\beta = m_1 \otimes \beta_1 + \cdots + m_k \otimes \beta_k$$

as elements in $M \otimes N = M \otimes N^{\text{mon}}$, where m_i are irredicible for each i and mutually distinct. Since $\alpha \in M \ltimes N$, $|\alpha_1|m_1 \dotplus \cdots \dotplus |\alpha_k|m_k \in M^{\text{mon}}$ is summable. (For any $a \in N^{\text{mon}}$ and $m \in M$, we mean by $|a|m$ the $\dotplus$-sum in M^{mon} of $|a|$ copies of m.) Similarly, $|\alpha_1 \dotplus \beta_1|m_1 \dotplus \cdots \dotplus |\alpha_k \dotplus \beta_k|m_k \in M^{\text{mon}}$ is also summable. Since N is non-invertible, $|\beta_i| \leq |\alpha_i \dotplus \beta_i|$ for each i, $|\beta_i|m_1 \dotplus \cdots \dotplus |\beta_k|m_k$ is also summable and we have $\beta \in M \ltimes N$. □

Proof of Theorem 3. We define

$$\Phi : \text{Hom}(M \ltimes N, K) \to \text{Hom}(M, \hom(N, K))$$

by letting

$$\Phi(\alpha)(m) : N \to K$$

be a map $n \mapsto \alpha(m \ltimes n)$, where $\alpha : M \ltimes N \to K$ is a homomorphism and $m \in M, n \in N$. Also, we define

$$\Psi : \text{Hom}(M, \hom(N, K)) \to \text{Hom}(M \ltimes N, K)$$

by letting

$$\Psi(\beta)([m_1 \ltimes n_1 \dotplus \cdots \dotplus m_k \ltimes n_k]) = \sum_i \beta(m_i)(n_i)$$

where $\beta : M \to \hom(N, K)$ is a homomorphism and $m_i \in M, n_i \in N$. It is straightforward to check that Φ and Ψ are well-defined maps, which are inverse to each other. □

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