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ON THE IMAGE OF THE BURAU REPRESENTATION OF THE IA-AUTOMORPHISM GROUP OF A FREE GROUP

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ABSTRACT. In this paper we study the graded quotients of the lower central series of the image of the IA-automorphism group of a free group by the Burau representation. In particular, we determine their structures for degrees 1 and 2.

1. INTRODUCTION

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, x_2, \ldots, x_n$, and $\Gamma_n(1) := F_n$, $\Gamma_n(2), \ldots$ its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n which induce the identity on the quotient group $F_n/\Gamma_n(k+1)$. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, which is called the Johnson filtration of $\mathrm{Aut}\, F_n$. The Johnson filtration of $\mathrm{Aut}\, F_n$ was originally introduced in 1963 with a remarkable pioneer work by Andreadakis [1] who showed that $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, $\ldots$ is a central series of $\mathcal{A}_n(1)$, and that the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank for each $k \geq 1$. Furtheremore, he [1] also showed that $\mathcal{A}_2(1)$, $\mathcal{A}_2(2)$, $\ldots$ coincides with the lower central series of $\mathcal{A}_2(1)$.

The group $\mathcal{A}_n(1)$ is called the IA-automorpshim group which is also denoted by IA_n. Magnus [15] showed that IA_n is finitely generated. Furthermore, recently, Cohen-Pakianathan [5, 6], Farb [7] and Kawazumi [13] inedepedently determined the abelianization of IA_n. (See Subsection 2.2.) In general, however, the group structure of IA_n is far from being well understood. For example, a presentation of IA_n is still not known. For $n = 3$, Krstić and McCool [14] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not. In addition to this, even the structures of the low dimensional (co)homology of IA_n are not completely determined.

Since each of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ is considered as a one by one approximation of IA_n, to determine the structure of $\mathrm{gr}^k(\mathcal{A}_n)$ plays very important roles on study of the group structure and the (co)homology groups of IA_n. In order to investigate each of $\mathrm{gr}^k(\mathcal{A}_n)$, certain injective homomorphisms

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

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are defined. These homomorphisms are called the Johnson homomorphisms of $\mathrm{Aut}\,F_n$. (For definition, see [20] and [26].) Recently, the study of the Johnson filtration and the Johnson homomorphisms of $\mathrm{Aut}\,F_n$ are made good progress by many authors, for example, [5], [6], [7], [13], [18], [19], [20], [24] and [26]. Here, we are interested in the following two problems. One is to determine whether $\mathcal{A}_n(k)$ coincides with the k-th term $\mathcal{A}'_n(k)$ of the lower central series of $\mathrm{IA}_n = \mathcal{A}_n(1)$ or not. Andreadakis [1] showed that $\mathcal{A}_3(3) = \mathcal{A}'_3(3)$. Cohen-Pakianathan [5, 6], Farb [7] and Kawazumi [13] independently showed that $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ for any $n \geq 3$. Furthermore, recently, Pettet [24] obtained that $\mathcal{A}'_n(3)$ has a finite index in $\mathcal{A}_n(3)$. However, it seems that there are few results for higher degrees. The other problem is to detremine the abelianization of each $\mathcal{A}_n(k)$ for $k \geq 2$. By a contribution from the study of the Johnson homomorphisms of $\mathrm{Aut}\,F_n$, we see that it contains a free abelian group of finite rank. However, it is not known even whether each of $H_1(\mathcal{A}_n(k), \mathbf{Z})$ is finitely generated or not.

In this paper, we study the images of $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$ through the Burau representation, which is one of the most important Magnus representations of $\mathrm{Aut}\,F_n$ defined on IA_n. (For definition, see subsection 2.4.) In general, the Magnus representations of $\mathrm{Aut}\,F_n$ are representations of various subgroups of $\mathrm{Aut}\,F_n$ by making use of the Fox's free differential calculus. (See [4] for details.) In this paper, we denote the Burau representation by τ_B, and write $\mathcal{B}_n(k) := \tau_B(\mathcal{A}_n(k))$ and $\mathcal{B}'_n(k) := \tau_B(\mathcal{A}'_n(k))$. First, we determine the abelianization of $\tau_B(\mathrm{IA}_n)$.

Theorem 1. *For any $n \geq 2$, $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \cong \mathbf{Z}^{\oplus n(n-1)}$.*

Next, to study $\mathcal{B}'_n(k)$ and its graded quotients $\mathrm{gr}^k(\mathcal{B}'_n) := \mathcal{B}'_n(k)/\mathcal{B}'_n(k+1)$ for $k \geq 2$, we consider a certain normal subgroup of $\tau_B(\mathrm{IA}_n)$. For $1 \leq i \neq j \leq n$, let L_{ij} be an automorphism of F_n defined by

$$L_{ij} : \begin{cases} x_i & \mapsto x_j x_i x_j^{-1}, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i). \end{cases}$$

We denote by $\mathcal{Y}_n$ a subgroup of $\tau_B(\mathrm{IA}_n)$ generated by L_{in} and L_{nj} for $1 \leq i, j \leq n-1$. Let $\mathcal{Y}'_n(k)$ be the lower central series of $\mathcal{Y}_n$. Then we prove:

Theorem 2. *For any $n \geq 2$ and $k \geq 2$, $\mathcal{Y}'_n(k) = \mathcal{B}'_n(k)$.*

Using this, we show:

Theorem 3. *For $n \geq 2$, $\mathrm{gr}^2(\mathcal{B}'_n) \cong \mathbf{Z}^{\oplus(n^2-n-1)}$.*

Observing the proof of the theorem above, as a corollary, we obtain:

Corollary 1. *For $n \geq 2$, $\mathcal{B}_n(3) = \mathcal{B}'_n(3)$.*

To show these, for $1 \leq l \leq k$, we define certain homomorphisms $\psi_{k,l}$ from $\mathcal{B}_n(k)$ to a free abelian group, and determine its image in Section 3. Using these homomorphisms, we detect a free abelian subgroup of $\mathrm{gr}^k(\mathcal{B}_n)$ and $\mathrm{gr}^k(\mathcal{B}'_n)$. We also show:

Corollary 2. *For $n \geq 2$, $k \geq 2$ and $1 \leq l \leq k$, $\psi_{k,l}(\mathcal{A}_n(k)) = \psi_{k,l}(\mathcal{A}'_n(k))$.*

This shows that the difference between $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$ is characterized by the kernel of the homomorphisms $\psi_{k,l}$. Furthermore, observing the image of $\psi_{k,k}$, we obtain:

Corollary 3. *For $n \geq 2$ and $k \geq 2$, $H_1(\mathcal{A}_n(k), \mathbf{Z}) \supset \mathbf{Z}^{\oplus k(n^2-n-1)}$.*

We remark that we can not detect all of $\mathbf{Z}^{\oplus k(n^2-n-1)} \subset H_1(\mathcal{A}_n(k), \mathbf{Z})$ by the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$ since some part of $\mathbf{Z}^{\oplus k(n^2-n-1)}$ is contained in $\mathcal{A}_n(k+1)$.

As an application, using a result $\mathrm{gr}^2(\mathcal{B}'_n) \cong \mathbf{Z}^{\oplus n^2-n-1}$, we can determine the image of the cup product $\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$. We show:

Theorem 4. *For $n \geq 2$, $\mathrm{Im}(\cup) \cong \mathbf{Z}^{\oplus(n-2)(n+1)(n^2-n-1)/2}$*

Finally, we consider the case where $n = 2$. In particular, we show

Theorem 5. *For any $k \geq 2$, $\mathrm{gr}^k(\mathcal{B}'_2) \cong \mathbf{Z}$.*

Here we remark that by a result of Andreadakis [1], we have $\mathrm{gr}^k(\mathcal{B}_2) = \mathrm{gr}^k(\mathcal{B}'_2)$ for each $k \geq 1$.

In Section 2, we show the definition and some properties of the IA-automorphism group, the Johnson filtration and the Magnus representations of the automorphism group of a free group. In Section 3, to study the $\mathrm{gr}^k(\mathcal{B}_n)$ and $\mathrm{gr}^k(\mathcal{B}'_n)$, we define homomorphisms $\psi_{k,l}$ and determine their images. In Section 4, we consider the lower central series $\mathcal{B}'_n(k)$ of $\tau_B(\mathrm{IA}_n)$. In particular, we determine the structure of the graded quotients $\mathrm{gr}^k(\mathcal{B}'_n)$ for $k = 1$ and 2. In Section 5, we determine the image of the cup product map $\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$. Finally, In Section 6, we consider the case where $n = 2$.

Contents

2. Preliminaries

In this section, we recall the definition and some properties of the IA-automorphism group and the Magnus representations of the automorphism group of a free group.

2.1. Notation.

Throughout the paper, we use the following notation and conventions.

- For a group G, the abelianization of G is denoted by G^{ab}.

- For a group G, the group $\mathrm{Aut}\, G$ acts on G from the right. For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, the action of σ on x is denoted by x^{σ}.
- For a group G, and its quotient group G/N, we also denote the coset class of an element $g \in G$ by $g \in G/N$ if there is no confusion.
- For elements x and y of a group, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. **IA-automorphism group.**

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, \ldots, x_n$. We denote the abelianization of F_n by H, and its dual group by $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. Let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced from the abelianization of F_n. In this paper we identifies $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H as a free abelian group induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of ρ is called the IA-automorphism group of F_n. It is well known due to Nielsen [21] that IA_2 coincides with the inner automorphsim group $\mathrm{Inn}\, F_2$ of F_2. Namely, IA_2 is a free group of rank 2. However, IA_n for $n \geq 3$ is much larger than the inner automorphism group $\mathrm{Inn}\, F_n$ of F_n. Indeed, Magnus [15] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} x_j{}^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i,\ j \in \{1, 2, \ldots, n\}$ and

$$K_{ijk} : x_t \mapsto \begin{cases} x_i[x_j, x_k], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i,\ j,\ k \in \{1, 2, \ldots, n\}$ such that $j < k$. In this paper, for the convenience, we often use automorphisms $L_{ij} := K_{ij}^{-1}$ and $L_{ijk} := K_{ijk}[K_{ij}^{-1}, K_{ik}^{-1}]$. Then we see that

$$L_{ij} : x_t \mapsto \begin{cases} x_j x_i x_j{}^{-1}, & t = i, \\ x_t, & t \neq i, \end{cases} \quad L_{ijk} : x_t \mapsto \begin{cases} [x_j, x_k] x_i, & t = i, \\ x_t, & t \neq i, \end{cases}$$

and that IA_n is also generated by L_{ij} and L_{ijk}. Recently, Cohen-Pakianathan [5, 6], Farb [7] and Kawazumi [13] inedepedently showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module.

2.3. **Johnson filtration.**

In this subsection we briefly recall the definition and some properties of the Johnson filtration of $\mathrm{Aut}\, F_n$. (For details, see [26] for example.)

Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2.$$

For $k \geq 0$, the action of $\mathrm{Aut}\, F_n$ on each nilpotent quotient $F_n/\Gamma_n(k+1)$ induces a homomorphism

$$\rho^k : \mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

The map ρ^0 is trivial, and $\rho^1 = \rho$. We denote the kernel of ρ^k by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, with $\mathcal{A}_n(1) = IA_n$. We call it the Johnson filtration of $\mathrm{Aut}\, F_n$, and denote each of its graded quotient by $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$.

The Johnson filtration of $\mathrm{Aut}\, F_n$ was originally introduced in 1963 with a remarkable pioneer work by Andreadakis [1] who showed that $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, ... is a descending central series of $\mathcal{A}_n(1)$ and $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. The Johnson filtration has been studied with the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. The study of the Johnson homomorphisms was begun in 1980 by D. Johnson [11]. He [12] studied the Johnson homomorphism of a mapping class group of a closed oriented surface, and determined the abelianization of the Torelli group. The Johnson homomorphisms of $\mathrm{Aut}\, F_n$ are also defined in a similar way, and there is a broad range of remarkable results for them. (For surveys and related topics concerning with the Johnson homomorphisms, see [19] and [20] for example.)

Let $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, ... be the lower central series of IA_n. In this paper, we are interested in the difference between $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$. Andreadakis [1] showed that the filtration $\mathcal{A}_2(1)$, $\mathcal{A}_2(2)$, ... coincides with the lower central series of $\mathcal{A}_2(1) = \mathrm{Inn}\, F_2$, and that $\mathcal{A}_3(3) = \mathcal{A}'_3(3)$. Recently, Cohen-Pakianathan [5, 6], Farb [7] and Kawazumi [13] independently showed that $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ for any $n \geq 3$. Pettet [24] showed that $\mathcal{A}'_n(3)$ has a finite index in $\mathcal{A}_n(3)$ at most for any $n \geq 3$. In general, however, it is still open problem whether the Johnson filtration $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, ... coincides with the lower central series of IA_n or not.

2.4. Magnus representations.

In this subsection we recall the Magnus representation of IA_n. (For details, see [4].) For each $1 \leq i \leq n$, let

$$\frac{\partial}{\partial x_i} : \mathbf{Z}[F_n] \to \mathbf{Z}[F_n]$$

be the Fox derivation defined by

$$\frac{\partial}{\partial x_i}(w) = \sum_{j=1}^{r} \epsilon_j \delta_{\mu_j, i} x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_j}^{\frac{1}{2}(\epsilon_j - 1)} \in \mathbf{Z}[F_n]$$

for any reduced word $w = x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_r}^{\epsilon_r} \in F_n$, $\epsilon_j = \pm 1$. (For details for the fox derivation, see [8].) Let $\varphi : F_n \to G$ be any group homomorphism. If there is no confusion, we also denote by φ both the ring homomorphism $\bar{\varphi} : \mathbf{Z}[F_n] \to \mathbf{Z}[G]$ induced from φ and the group homomorphism $\tilde{\varphi} : \mathrm{GL}(n, \mathbf{Z}[F_n]) \to \mathrm{GL}(n, \mathbf{Z}[G])$ induced from $\bar{\varphi}$. For any matrix $C = (c_{ij}) \in \mathrm{GL}(n, \mathbf{Z}[F_n])$, let C^{φ} be the matrix $(c_{ij}^{\varphi}) \in \mathrm{GL}(n, \mathbf{Z}[G])$. Then we obtain a map $\tau_{\varphi} : \mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbf{Z}[G])$ defined by

$$\sigma \mapsto \left(\frac{\partial x_i{}^{\sigma}}{\partial x_j} \right)^{\varphi}.$$

This map is not a homomorphism in general. Let A_{φ} be a subgroup of $\mathrm{Aut}\, F_n$ consisting of automorphisms σ such that $(x^{\sigma})^{\varphi} = x^{\varphi}$. Then, by restricting τ_{φ} to A_{φ}, we obtain a homomorphism

$$\tau_\varphi : A_\varphi \to \mathrm{GL}(n, \mathbf{Z}[G]),$$

which is called the Magnus representation of A_φ.

Here we consider two particular homomorphisms from F_n. The first one is the abelianization $\mathfrak{a} : F_n \to H$ of F_n. It is clear that $\mathrm{IA}_n \subset A_{\mathfrak{a}}$. We call the Magnus representation $\tau_{\mathfrak{a}} : \mathrm{IA}_n \to \mathrm{GL}(n, \mathbf{Z}[H])$ the Gassner representation of IA_n, denoted by τ_G. Let $s_1, \ldots, s_n$ be the coset classes of $x_1, \ldots, x_n$ in H respectively. Then, for example, $\tau_G(L_{ij})$ and $\tau_G(L_{ijk})$ are given by

$$\begin{array}{cc} & \begin{matrix} & \underline{i} & \underline{j} & \end{matrix} \\ \begin{matrix} \\ \underline{i} \\ \underline{j} \\ \\ \end{matrix} & \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & s_j & 1-s_i & \vdots \\ \vdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{matrix} & \underline{i} & \underline{j} & \underline{k} & \end{matrix} \\ \begin{matrix} \\ \underline{i} \\ \underline{j} \\ \underline{k} \\ \\ \end{matrix} & \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1-s_k & s_j-1 & 0 \\ \vdots & 0 & 1 & 0 & 0 \\ \vdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \end{array}$$

respectively. Bachmuth determined the image $\mathrm{Im}(\tau_G)$ of τ_G:

Theorem 2.1 (Bachmuth, [2]). *For $n \geq 2$ and $C = (c_{ij}) \in \mathrm{GL}(n, \mathbf{Z}[H])$, $C \in \mathrm{Im}(\tau_G)$ if and only if C satisfies*

(1) $\det(C) = s_1^{e_1} s_2^{e_2} \cdots s_n^{e_n}, \quad e_i \in \mathbf{Z}$,

(2) *For any $1 \leq i \leq n$,*

$$\sum_{j=1}^{n} c_{ij}(1 - s_j) = 1 - s_i.$$

Let $I := \mathrm{Ker}(\mathbf{Z}[F_n] \to \mathbf{Z})$ be the augmentation ideal of the group ring $\mathbf{Z}[H]$. By a fundumental argument in Fox's free differential calculus, we see that for any $C = (c_{ij}) \in \mathrm{Im}(\tau_G|_{\mathcal{A}_n(k)})$, $c_{ij} - \delta_{ij} \in I^k$ for any $i \neq j$. Here δ_{ij} is the Kronecker's delta.

Let $\langle s \rangle$ be the infinite cyclic group generated by s. The other homomorphism is $\mathfrak{b} : F_n \to \langle s \rangle$ defined by $x_i \mapsto s$. The group ring $\mathbf{Z}[\langle s \rangle]$ is naturaly considered as the Laurent polynomial ring $\mathbf{Z}[s^{\pm 1}]$ of one indetrminates over the integers. In this paper we identify them. Then we call the Magnus representation

$$\tau_B := \tau_{\mathfrak{b}} : \mathrm{IA}_n \to \mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}]),$$

the Burau representation of IA_n. For a homomorpshim $\mathfrak{c} : H \to \langle s \rangle$ defined by $s_i \mapsto s$, $\tau_B = \mathfrak{c} \circ \tau_G$. By Theorem 2.1, we have:

Lemma 2.1. *For $n \geq 2$, any element $C = (c_{ij}) \in \mathrm{Im}(\tau_B)$ satisfies*

(1) $\det(C) = s^e, \quad e \in \mathbf{Z}$,

(2) *For any $1 \leq i \leq n$,*

$$\sum_{j=1}^{n} c_{ij} = 1.$$

Let $\mathcal{B}_n(k)$ and $\mathcal{B}'_n(k)$ be the images of $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$ by the Burau representation τ_B respectively. Let $J := \mathrm{Ker}(\mathbf{Z}[s^{\pm 1}] \to \mathbf{Z})$ be the augmentation ideal of the group ring $\mathbf{Z}[s^{\pm 1}]$. For any $k \geq 1$, an ideal J^k is a principal ideal generated by $(1-s)^k$. For any $C = (c_{ij}) \in \mathcal{B}_n(k)$, we see $c_{ij} - \delta_{ij} \in J^k$.

3. Homomorphisms $\psi_{k,l}$

In this section we study homomorphisms from subgroups of $\mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}])$ to certain free abelian groups. The results, obtained in this section, are applied to determine the structure of the graded quotients $\mathrm{gr}^k(\mathcal{B}'_n) := \mathcal{B}'_n(k)/\mathcal{B}'_n(k+1)$ for $k = 1$ and 2 in the next section.

For any $n \geq 2$ and $k \geq 1$, let $\Gamma(n,k)$ be the kernel of a homomorphism $\mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}]) \to \mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}]/J^k)$ induced from a natural projection $\mathbf{Z}[s^{\pm 1}] \to \mathbf{Z}[s^{\pm 1}]/J^k$. From the argument above, we see $\mathcal{B}_n(k) \subset \Gamma(n,k)$. We denote by $M(n,R)$ the abelian group of $(n \times n)$-matrices over a ring R. For any $k \geq 1$ and $1 \leq l \leq k$, we consider a map $\xi_{k,l} : \Gamma(n,k) \to M(n, \mathbf{Z}[s^{\pm 1}]/J^l)$ defined by

$$\xi_{k,l}(C) = C' \mod J^l$$

where $C = E + (1-s)^k C'$, and E denotes the identity matrix. The map $\xi_{k,l}$ is a homomorphism since

$$(E + (1-s)^k C')(E + (1-s)^k D') = E + (1-s)^k (C' + D' + (1-s)^k C'D')$$

for any $C = E + (1-s)^k C'$, $D = E + (1-s)^k D' \in \Gamma(n,k)$. Set

$$\psi_{k,l} := \xi_{k,l} \circ \tau_B|_{\mathcal{A}_n(k)} : \mathcal{A}_n(k) \to M(n, \mathbf{Z}[s^{\pm 1}]/J^l).$$

In the following, we completely determine the image of $\psi_{k,l}$. First, we consider the case where $k = l = 1$.

Proposition 3.1. *For $n \geq 2$, $\mathrm{Im}(\psi_{1,1}) \cong \mathbf{Z}^{\oplus n(n-1)}$.*

Now, for any $l \geq 1$, the quotient ring $\mathbf{Z}[s^{\pm 1}]/J^l$ is a free abelian group of rank l with a basis $\{(1-s)^m \,|\, 0 \leq m \leq l-1\}$. We fix this basis in the following. To study $\mathrm{Im}(\psi_{k,l})$ for $k \geq 2$, we consider some elements in $\mathcal{A}_n(k)$. For $k \geq 2$, $1 \leq l \leq k$ and $0 \leq m \leq l-1$, and distinct i, j and u, set

$$\sigma_m(i,j,u) := [L_{iju}, L_{ij}, L_{ij}, \ldots, L_{ij}] \in \mathcal{A}'_n(m+k) \subset \mathcal{A}_n(k)$$

where L_{ij} appears $m+k-1$ times among the component. Then we see

$$\sigma_m(i,j,u) : x_t \mapsto \begin{cases} [x_j, x_u, x_j, x_j, \ldots, x_j]\, x_i, & t = i \\ x_t, & t \neq i \end{cases}$$

and

$$\psi_{k,l}(\sigma_m(i,j,u)) = \begin{array}{c} \\ \underline{i} \\ \underline{j} \\ \underline{u} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} & \underline{i} & \underline{j} & \underline{u} & & \end{array} \\ \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & (1-s)^m & -(1-s)^m & \vdots \\ \vdots & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \end{array}.$$

For $0 \le m \le l-1$, and distinct i and j, set

$$w_m(i,j) := [K_{ij}, K_{ji}, K_{ij}, K_{ij}, \dots, K_{ij}]^{-1} \in \mathcal{A}'_n(m+k) \subset \mathcal{A}_n(k)$$

where K_{ij} appears $m+k-2$ times among the component. Then we see

$$w_m(i,j) : x_t \mapsto \begin{cases} [x_i, x_j, x_j, \dots, x_j, x_t]\, x_t, & t = i, j, \\ x_t, & t \neq i, j \end{cases}$$

and

$$\psi_{k,l}(w_m(i,j)) = \begin{array}{c} \\ \underline{i} \\ \underline{j} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} & \underline{i} & \underline{j} & \end{array} \\ \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & (1-s)^m & -(1-s)^m & \vdots \\ \vdots & (1-s)^m & -(1-s)^m & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \end{array}.$$

Set

$$\begin{aligned} \mathfrak{E} := & \{\psi_{k,l}(\sigma_m(i,j,n)) \mid 1 \le j < i \le n-1,\ 0 \le m \le l-1\} \\ & \cup \{\psi_{k,l}(\sigma_m(n,n-1,u)) \mid 1 \le u \le n-2,\ 0 \le m \le l-1\} \\ & \cup \{\psi_{k,l}(w_m(i,j)) \mid 1 \le i < j \le n,\ 0 \le m \le l-1\} \subset \mathrm{Im}(\psi_{k,l}). \end{aligned}$$

Then we see:

Proposition 3.2. *For $n \ge 2$, $k \ge 2$ and $1 \le l \le k$, $\mathrm{Im}(\psi_{k,l})$ is a free abelian group with basis $\mathfrak{E}$. In particular, $\mathrm{Im}(\psi_{k,l}) \cong \mathbf{Z}^{\oplus l(n^2-n-1)}$.*

From the proof of the Propositions above, we see:

Corollary 3.1. *For $n \ge 2$, $k \ge 2$ and $1 \le l \le k$, $\psi_{k,l}(\mathcal{A}_n(k)) = \psi_{k,l}(\mathcal{A}'_n(k))$.*

This shows that the difference between $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$ is characterized by the kernel of $\psi_{k,l}$. Furthermore, observing the image of $\psi_{k,k}$, we have:

Corollary 3.2. *For $n \ge 2$ and $k \ge 2$, $H_1(\mathcal{A}_n(k), \mathbf{Z})$ contains a free abelian group of rank $k(n^2-n-1)$.*

4. Filtration $\mathcal{B}'_n(k)$

In this section, we consider the lower central series $\mathcal{B}'_n(k)$ of $\mathcal{B}'_n(1) := \tau_B(\mathrm{IA}_n)$. In particular, we determine the structure of the graded quotients $\mathrm{gr}^k(\mathcal{B}'_n) := \mathcal{B}'_n(k)/\mathcal{B}'_n(k+1)$ for $k = 1$ and 2, using the homomorphisms $\xi_{1,1}$ and $\xi_{2,1}$. We also show that $\mathcal{B}_n(3) = \mathcal{B}'_n(3)$. First, we consider the case where $k = 1$, namely, the abelianization of $\tau_B(\mathrm{IA}_n)$.

Theorem 4.1. *For any $n \geq 2$, $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \cong \mathbf{Z}^{\oplus n(n-1)}$.*

To study the graded quotients $\mathrm{gr}^k(\mathcal{B}'_n)$ for $k \geq 2$, we consider a certain normal subgroup $\mathcal{Y}_n$ of $\tau_B(\mathrm{IA}_n)$. Let $\mathcal{Y}_n$ be a subgroup of $\tau_B(\mathrm{IA}_n)$ generated by $\overline{L}_{in}$ and $\overline{L}_{nj}$ for $i, j \neq n$. In particular, we show that the lower central series $\mathcal{Y}'_n(k)$ of $\mathcal{Y}_n$ coincides with $\mathcal{B}'_n(k)$ for any $k \geq 2$. In the following, we use $\overline{L}_{ij}$ for $\tau_B(L_{ij})$ for simplicity.

Lemma 4.1. *For any $n \geq 2$, $\mathcal{Y}_n$ is a normal subgroup of $\tau_B(\mathrm{IA}_n)$.*

From this lemma, we see that the natural action of $\tau_B(\mathrm{IA}_n)$ on $H_1(\mathcal{Y}_n, \mathbf{Z})$ by conjugation is trivial. Next, in order to show that $\mathcal{Y}_n$ contains the commutator subgroup of $\tau_B(\mathrm{IA}_n)$, we prepare some lemmas.

Lemma 4.2. *For $1 \leq i \neq j \leq n$, $[\overline{L}_{ij}, \overline{L}_{ji}] \in \mathcal{Y}_n$.*

Lemma 4.3. *For $1 \leq i \neq j \neq k \leq n$, $[\overline{L}_{ij}, \overline{L}_{ik}]$, $[\overline{L}_{ij}, \overline{L}_{jk}] \in \mathcal{Y}_n$.*

Then we have:

Lemma 4.4. *For any $n \geq 2$, $\mathcal{B}'_n(2) \subset \mathcal{Y}_n$.*

Here we remark that the quotient group of $\tau_B(\mathrm{IA}_n)$ by $\mathcal{Y}_n$ is given by

Proposition 4.1. *For $n \geq 2$, $\tau_B(\mathrm{IA}_n)/\mathcal{Y}_n \cong H_1(\tau_B(\mathrm{IA}_{n-1}), \mathbf{Z})$.*

Next we show that $\mathcal{Y}'_n(k)$ coincides with $\mathcal{B}'_n(k)$ for any $k \geq 2$.

Theorem 4.2. *For any $n \geq 2$ and $k \geq 2$, $\mathcal{Y}'_n(k) = \mathcal{B}'_n(k)$.*

Next we determine $\mathrm{gr}^2(\mathcal{B}'_n)$ using the homomorphism $\xi_{2,1}$.

Theorem 4.3. *For $n \geq 2$, $\mathrm{gr}^2(\mathcal{B}'_n) \cong \mathbf{Z}^{\oplus(n^2-n-1)}$.*

As a corollary, we obtain

Corollary 4.1. *For $n \geq 2$, $\mathcal{B}_n(3) = \mathcal{B}'_n(3)$.*

By Pettet [24], $\mathcal{A}'_n(3)$ has a finite index in $\mathcal{A}_n(3)$. From Corollary 4.1, we see that if $\mathcal{A}'_n(3) \neq \mathcal{A}_n(3)$, the defference between them is contained in the kernel of τ_B.

5. The cup product

In this section we determine the image of the cup product

$$\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}).$$

First, we consider an interpretation of the second cohomology group $H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$.

Let F be a free group of rank $n(n-1)$ on $\{\overline{L}_{ij} \mid 1 \le i \ne j \le n\}$. Let $\varphi : F \to \tau_B(\mathrm{IA}_n)$ be a natural surjection and R the kernel of φ. Then we have a minimal presentation of $\tau_B(\mathrm{IA}_n)$

$$1 \to R \to F \xrightarrow{\varphi} \tau_B(\mathrm{IA}_n) \to 1. \tag{2}$$

The word "minimal"means that the number of generators is minimal among any presentation of $\tau_B(\mathrm{IA}_n)$. Since the abelianization of $\tau_B(\mathrm{IA}_n)$ is a free abelian group with basis $\{\overline{L}_{ij} \mid 1 \le i \ne j \le n\}$ by Theorem 4.1, the induced homomorphism

$$\varphi^* : H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^1(F, \mathbf{Z})$$

is an isomorphism. Hence considering the cohomological five-term exact sequence

$$\begin{aligned} 0 \to H^1(\tau_B(\mathrm{IA}_n),\mathbf{Z}) \to H^1(F, \mathbf{Z}) \to H^1(R, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)} \\ \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(F, \mathbf{Z}) = 0. \end{aligned}$$

of (2), we obtain an isomorphism

$$H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \cong H^1(R, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)}.$$

To study the abelian group $H^1(R, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)}$, we consider a descending filtration of R. Let $\Gamma_F(k)$ be the lower central series of F and $\mathcal{L}_F(k) := \Gamma_F(k)/\Gamma_F(k+1)$ for $k \ge 1$. Set $R_k := R \cap \Gamma_F(k)$ and $\overline{R}_k := R/R_k$ for $k \ge 1$. Then $R_k = R$ for $1 \le k \le 2$. The natural projection $R \to \overline{R}_{k+1}$ induces an injective homomorphism

$$\psi^k : H^1(\overline{R}_{k+1}, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)} \to H^1(R, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)}.$$

Hence we can consider $H^1(\overline{R}_{k+1}, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)}$ as a subgroup of $H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$. In the following, we study the case where $k = 2$. In this case, we remark that $H^1(\overline{R}_3, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)} = H^1(\overline{R}_3, \mathbf{Z})$ since $\tau_B(\mathrm{IA}_n)$ acts on $\overline{R}_3$ trivially. Then we have:

Proposition 5.1. *The image of the cup product*

$$\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$$

is $H^1(\overline{R}_3, \mathbf{Z})$.

Since $\mathcal{L}_F(2)$ is a free abelian group of rank $n(n-1)(n^2-n-1)/2$, as a corollary, we obtain:

Theorem 5.1. *For* $n \ge 2$, $\mathrm{Im}(\cup) \cong \mathbf{Z}^{\oplus(n-2)(n+1)(n^2-n-1)/2}$

6. The case $n = 2$

In this section, we completely determine the structures of $\mathrm{gr}^k(\mathcal{B}_2')$ and $\mathrm{gr}^k(\mathcal{B}_2)$ for any $k \ge 1$. Recall that $\mathrm{IA}_2 = \mathrm{Inn}\, F_2$ is generated by K_{21} and K_{12}. For the convenience, set $\iota_1 := K_{21}$ and $\iota_2 := K_{12}$. We remark that from Theorem 4.1, the abelianization of $\tau_B(\mathrm{IA}_2)$ is a free abelian group of rank 2 generated by ι_1 and ι_2.

Theorem 6.1. *For any* $k \ge 2$, $\mathrm{gr}^k(\mathcal{B}_2') \cong \mathbf{Z}$.

Since $\mathcal{A}_2(k) = \mathcal{A}'_2(k)$ for any $k \geq 1$ due to Andreadakis [1], we obtain

Corollary 6.1. *For any $k \geq 2$, $\mathrm{gr}^k(\mathcal{B}_2) \cong \mathbf{Z}$.*

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