数理解析研究所講究録
第 1612 巻 2008 年 111-122

Borsuk's antipodal theorem for set-valued mappings

明治大学 四反田義美 (Shitanda Yoshimi)
Meiji University Izumi Campus

1 Introduction

When a non empty closed set $\varphi(x)$ in a topological space Y is assigned for each x of a topological space X, we call the correspondence a set-valued mapping and write $\varphi : X \to Y$ by the Greek alphabet. For single-valued mapping, we write $f : X \to Y$ etc. by the Roman alphabet. In this paper, we assume that set-valued mappings are upper semi-continuous.

In this paper, we shall prove Borsuk's antipodal theorem for an admissible mapping $\varphi : \partial\overline{U} \to \mathbf{R}^n$ where U is a bounded symmetric open neighborhood of the origin of $\mathbf{R}^{n+k}$ ($k \geqq 1$) and generalize to the case of an admissible mapping $\varphi : \partial\overline{U} \to \mathbf{E}$ where U is a bounded symmetric open neighborhood of the origin of the normed space $\mathbf{E}$.

In the second section, we review various cohomology theories and summerize some definitions and result. In this paper, we shall mainly use Alexander-Spanier cohomology theory $\bar{H}^*(X;\mathbf{F})$ with coefficient in a field $\mathbf{F}$.

In the third section, we define an equivariant mapping in the class of set-valued mappings (cf. Definition 3.4) and discuss about Borsuk's antipodal theorem for admissible mappings. Y.S.Chang proved a generalization of Borsuk's antipodal theorem (cf. Theorem 4 in [1]) for closed convex valued mappings by using the method of general topology and analysis. We shall prove the following theorem which is a generalization of his theorem by using the method of algebraic topology (cf. Theorem 3.6).

Main Theorem 1. *Let U be a bounded open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 1$ which is symmetric with respect to the involution $T(x) = -x$. Assume that $\varphi : \partial\overline{U} \to \mathbf{R}^m$ is an equivariant admissible mapping. Then there exists point $x_0 \in \partial\overline{U}$ such that $\varphi(x_0) \ni 0$.*

We shall prove the following theorem (cf. Theorem 3.7) which is a generalization of Theorem 6 in [1] and also a generalization of Theorem 9.1, 9.2 of §10 in [6] for set-valued mappings.

Main Theorem 2. *Let U be a bounded open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 0$ which is symmetric with respect to the involution $T(x) = -x$. Assume that $\varphi : \overline{U} \to \mathbf{R}^m$ is an admissible mapping which is equivariant on the boundary $\partial\overline{U}$ of $\overline{U}$. Then there exists a point $x_0 \in \overline{U}$ such that $\varphi(x_0) \ni 0$ and a point $x_1 \in \overline{U}$ such that $\varphi(x_1) \ni x_1$.*

In the last section, we discuss a generalization of results of §3 to the infinite dimensional normed space. We obtain the following theorem (cf. Theorem 4.2) which is a generalization of Theorem 7 in [1] in the case of the normed space.

Main Theorem 3. *Let U be a symmetric bounded open neighborhood of the origin in a normed space* $\mathbf{E}$. *Assume that* $\varphi : \overline{U} \to \mathbf{E}$ *is upper semi-continuous, compact convex valued mapping and is equivariant on* $\partial\overline{U}$. *Then there exist a fixed point* $z_0 \in \overline{U}$ *such that* $\varphi(z_0) \ni z_0$.

In the above theorem, we can not deduce the existence of the zero value of φ. We shall generalize Borsuk-Ulam theorem to the case of infinite dimensional spaces.

Main Theorem 4. *Let* $\mathbf{E}_k$ *be a closed subspace of codimension* $k \geqq 1$ *of* $\mathbf{E}$ *and* U *be a symmetric bounded open neighborhood of the origin of* $\mathbf{E}$. *If* $\Phi : \partial\overline{U} \to \mathbf{E}_k$ *is a compact admissible mapping, there is a point* $x_0 \in \partial\overline{U}$ *such that* $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$ *where* $\varphi(x) = x - \Phi(x)$.

2 Various cohomology theories

To begin with, we give some remarks about several cohomology theories. For the detail, see Y.Shitanda [12]. The Alexander-Spanier cohomology theory $\bar{H}^*(-;G)$ is isomorphic to the singular cohomology theory $H^*(-;G)$, that is,

$$\mu : \bar{H}^*(X;G) \cong H^*(X;G)$$

if the singular cohomology theory satisfies the continuity condition (cf. Theorem 6.9.1 in [13]). For a paracompact Hausdorff space X, it holds also the isomorphism between Čech cohomology theory $\check{H}^*(-;G)$ with coefficient in a constant sheaf and the Alexander-Spanier cohomology theory $\bar{H}^*(-;G)$ (cf. Theorem 6.8.8 in [13])

$$\check{H}^*(X;G) \cong \bar{H}^*(X;G).$$

An ANR space is an r-image of some open set of a normed space (cf. Proposition 1.8 in [5]). For an ANR space X, it holds also the isomorphism:

$$\check{H}^*(X;G) \cong \bar{H}^*(X;G) \cong H^*(X;G)$$

by Theorem 6.1.10 of [13]. The remarkable feature of the Alexander-Spanier cohomology theory is that it satisfies the continuity property (cf. Theorem 6.6.2 in [13]). Hereafter we mainly use the Alexander-Spanier (co)homology theory with coefficient field $\mathbf{F}$.

Definition 2.1. *Let X and Y be paracompact Hausdorff spaces. A mapping $f : X \to Y$ is called a Vietoris mapping, if it satisfies the following conditions:*

1. *f is proper and onto continuous mapping.*
2. *$f^{-1}(y)$ is an acyclic space for any $y \in Y$, that is,* $\bar{H}^*(f^{-1}(y);\mathbf{F}) = 0$ *for positive dimension.*

When f is closed and onto continuous mapping and satisfies the condition (2), *we call it weak Vietoris mapping.*

If $f^{-1}(K)$ is compact set for any compact subset $K \subset Y$, f is called a proper mapping. Note that a proper mapping is closed. A mapping $f : X \to Y$ is called a compact mapping, if $f(X)$ is contained in a compact set of Y, or equivalently its closure $\overline{f(Y)}$ is compact.

The following theorem is called Vietoris's theorem and is essentially important for our purpose (cf. Theorem 6.9.15 in [13]).

Theorem 2.2. *Let $f : X \to Y$ be a weak Vietoris mapping between paracompact Hausdorff spaces X and Y. Then,*

$$f^* : \bar{H}^m(Y; \mathbf{F}) \to \bar{H}^m(X; \mathbf{F}) \tag{1}$$

is an isomorphism for all $m \geqq 0$.

The graph of set-valued mapping $\varphi : X \to Y$ is defined by $\Gamma_\varphi = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$. If φ is upper semi-continuous, Γ_φ is closed, but the converse is not true. If the image $\varphi(X)$ is contained in a compact set, the converse is true (cf. §14 in [5]).

Definition 2.3. *An upper semi-continuous mapping $\varphi : X \to Y$ is admissible, if there exists a paracompact Hausdorff space Γ satisfying the following conditions:*

1. *there exist a Vietoris mapping $p : \Gamma \to X$ and a continuous mapping $q : \Gamma \to Y$,*
2. *$\varphi(x) \supset q(p^{-1}(x))$ for each $x \in X$.*

A pair of mappings (p, q) is called a selected pair of φ.

Define $\varphi^* : \bar{H}^*(Y) \to \bar{H}^*(X)$ by the set $\{(p^*)^{-1}q^*\}$ where (p, q) is a selected pair of admissible mapping $\varphi : X \to Y$. And φ_* is similarly defined.

Let N be a paracompact Hausdorff space with a free involution T and $p : \Gamma \to N$ a Vietoris mapping. Consider the following diagram:

$$\begin{array}{ccc} \hat{\Gamma} & \xrightarrow{\hat{\Delta}} & \Gamma \times \Gamma \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle p \times p} \\ N & \xrightarrow{\Delta} & N \times N \end{array} \tag{2}$$

where Δ is given by $\Delta(x) = (x, T(x))$. $\hat{\Gamma}$ is defined by the pull-back square and $\hat{p}$ and $\hat{\Delta}$ are induced mappings in the pull-back square, i.e. $\hat{p}(y, y') = p(y)$. Involutions on N^2, Γ^2 are given by switching mappings $T(x, x') = (x', x)$. All mappings are equivariant with respect to their involutions. Clearly $\hat{\Gamma}$ has free involution $\hat{T}$. The following lemma is proved in Lemma 4.6 of [12].

Lemma 2.1. *Let N be a paracompact Hausdorff space with a free involution T and $p : \Gamma \to N$ be a Vietoris mapping. Then $\hat{p} : \hat{\Gamma} \to N$ is a π-equivariant Vietoris mapping and $\hat{\Gamma}$ is a paracompact Hausdorff space. $\hat{p}_\pi : \hat{\Gamma}_\pi \to N_\pi$ is a Vietoris mapping and $\hat{\Gamma}_\pi$ is a paracompact Hausdorff space. Moreover if N is a metric space and A is a π-invariant closed subspace of N, then $\bar{H}^*(\hat{\Gamma} - \hat{p}^{-1}(A); \mathbf{F}_2)$ and $\bar{H}^*(\hat{\Gamma}_\pi - \hat{p}_\pi^{-1}(A_\pi); \mathbf{F}_2)$ are isomorphic to $\bar{H}^*(N - A; \mathbf{F}_2)$ and $\bar{H}^*(N_\pi - A_\pi; \mathbf{F}_2)$ respectively.*

3 Borsuk's antipodal theorem

The classical Borsuk's antipodal theorem says that an equivariant mapping $f : S^m \to \mathbf{R}^m$ has the zero value, that is, there exists a point $x_0 \in S^m$ such that $f(x_0) = 0$ (cf. Theorem 5.2 of §5 in [6]). A generalized Borsuk's antipodal theorem is also stated as follows (cf. Theorem 9.2 of §10 in [6]).

Theorem 3.1. *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^m$. Assume that the closure $\overline{U}$ of U is a finite polyhedron and $f : \overline{U} \to \mathbf{R}^m$ be a continuous mapping which is equivariant on the boundary $\partial\overline{U}$ of $\overline{U}$. Then f has the zero value , that is, there exists a point $x_0 \in \overline{U}$ such that $f(x_0) = 0$.*

S.Y.Chang proved the following Borsuk antipodal theorems for upper semi-continuous mappings which are closed convex set valued (cf. Theorem 4 in [1]). A set valued mapping $F : X \to Y$ is called antipodal mapping in his paper, if F satisfies $F(x) \cap (-F(-x)) \neq \emptyset$ for all $x \in X$.

Theorem 3.2. *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^{m+1}$, and $F : \partial\overline{U} \to \mathbf{R}^m$ be upper semi-continuous, closed convex-valued, and antipodal preserving. Then F has the zero value , that is, there exists a point $x_0 \in \overline{U}$ such that $F(x_0) \ni 0$.*

We prepare a theorem for later applications.

Theorem 3.3. *Let N be a paracompact Hausdorff space with a free involution T and M an m-dimensional closed manifold with a free involution T'. Assume that $c^m \neq 0$ for $c = c(N,T) \in \bar{H}^1(N_\pi; \mathbf{F}_2)$ and f is an equivariant mapping. Then $f^* : \bar{H}^*(M; \mathbf{F}_2) \to \bar{H}^*(N; \mathbf{F}_2)$ is not trivial for a positive dimension.*

Proof. Let $h : M \to S^\infty$ be an equivariant mapping such that $h_\pi^*(\omega) = c(M, T')$. Here ω is the generator of $\bar{H}^1(RP^\infty; \mathbf{F}_2)$. $hf : N \to S^\infty$ is also an equivariant mapping such that $(hf)_\pi^*(\omega) = c(N,T)$. From $c(N,T)^m \neq 0$, it holds $c(M,T')^m \neq 0$. By Gysin-Smith exact sequence, we see $\phi^*(c_M) = c(M,T')^m$ where c_M is the dual cocycle of the m-dimensional fundamental cycle $[M]$. By

$$\phi^* f^*(c_M) = f_\pi^* \phi^*(c_M) = f_\pi^*(c(M,T')^m) = c(N,T)^m \neq 0,$$

we obtain the result. □

In this paper we adopt a new definition of an equivariant mapping for set valued mappings. Our definition is a generalization of S. Y. Chang's definition.

Definition 3.4. *Let X and Y be paracompact Hausdorff spaces with involutions T and T' respectively. An admissible mapping $\varphi : X \to Y$ is said to be equivariant, if there exist a paracompact Hausdorff space Γ with a free involution and an equivariant Vietoris mapping $p : \Gamma \to X$ and an equivariant continuous mapping $q : \Gamma \to Y$ such that $qp^{-1}(x) \subset \varphi(x)$ for $x \in X$. An admissible mapping $\varphi : X \to Y$ is said to be equivariant on a closed* *subspace X_0 of X, if there exists an equivariant Vietoris mapping $p_0 : \Gamma_0 \to X_0$ and equivariant mapping $q_0 : \Gamma_0 \to Y$ and satisfies the following commutativity:*

$$\begin{array}{ccccc} X_0 & \xleftarrow{p_0} & \Gamma_0 & \xrightarrow{q_0} & Y \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle id} \\ X & \xleftarrow{p} & \Gamma & \xrightarrow{q} & Y \end{array}$$

where (p, q) is a selected pair of φ and i is a closed inclusion.

For an equivariant mapping $\varphi : X \to Y$, it holds $qp^{-1}(x) \subset \varphi_0(x)$ for $x \in X$ where $\varphi_0(x) = \varphi(x) \cap T'\varphi(T(x))$. For an admissible mapping $\varphi : X \to Y$ which is equivariant on X_0, it holds $q_0p_0^{-1}(x) \subset \varphi_0(x)$ for $x \in X_0$.

We shall generalize Theorem 3.1 and 3.2 in what follows. Let $\partial\overline{U}$ be the boundary of $\overline{U}$, that is, $\partial\overline{U} = \overline{U} - Int\overline{U}$.

Proposition 3.5. *Let U be a bounded open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 1$ which is symmetric with respect to the involution $T(x) = -x$. Assume that the boundary $\partial\overline{U}$ is an $(m + k - 1)$-dimensional manifold and $\varphi : \partial\overline{U} \to \mathbf{R}^m$ is an admissible mapping and is equivariant on $\partial\overline{U}$. Then there exists a point $x_0 \in \overline{U}$ such that $\varphi(x_0) \ni 0$.*

Proof. Set $M = \overline{U - D}$ where D is an open disk centered at 0 with a small radius $r > 0$. M is a topological manifold with boundary which has the free involution T. We have $i^*(c(M, T)) = c(\partial\overline{U}, T)$ for the inclusion $i : \partial\overline{U} \to M$ and $j^*(c(M, T)) = c(\partial\overline{D}, T)$ for the inclusion $j : \partial\overline{D} \to M$. We can prove the following formula:

$$c^{m+k-1}(\partial\overline{U}, T)[(\partial\overline{U})_\pi] = c^{m+k-1}(S^{m+k-1}, T)[S_\pi^{m+k-1}]$$

by the method of Theorem 4.9 in J.Milnor [7]. Since $c^{m+k-1}(S^{m+k-1}, T)$ is not zero, we obtain

$$c^{m+k-1}(\partial\overline{U}, T) \neq 0. \tag{3}$$

By our assumption, there exists an equivariant Vietoris mapping $p_0 : \Gamma_0 \to \partial\overline{U}$ and an equivariant mapping $q_0 : \Gamma_0 \to \mathbf{R}^m$ such that $q_0p_0^{-1}(x) \subset \varphi(x)$ for $x \in \partial\overline{U}$. We have a formula:

$$c(\Gamma_0, T') = p_{0\pi}^*(c(\partial\overline{U}), T) \neq 0. \tag{4}$$

Assume that $\varphi(x)$ does not contain zero. q_0 is considered as $q_0 : \Gamma_0 \to \mathbf{R}^m - \{0\}$. Since q_0 is equivariant, we have a formula:

$$q_{0\pi}^*(c) = c(\Gamma_0, T') \tag{5}$$

where c is the first Stiefel-Whitney class of $\mathbf{R}^m - \{0\}$. From the results (4), (5), we have

$$(q_{0\pi})^*(c^{m+k-1}) = c(\Gamma_0, T')^{m+k-1} = (p_{0\pi})^*(c(\partial\overline{U}, T)^{m+k-1}). \tag{6}$$

The left side of the equation is zero by $c^m = 0$ and the right side is not zero by the results (3) and (4) and the bijectivity of $(p_{0\pi})^*$. From the contradiction, we obtain the conclusion. □

We must remark that φ is defined on $\partial\overline{U}$, not on $\overline{U}$. We can also generalize Proposition 3.5 for the case that $\partial\overline{U}$ is not an $(m+k-1)$-dimensional closed manifold. The following theorem is a generalization of Theorem 4 of S.Y.Chang [1].

Theorem 3.6. *Let U be a bounded open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 1$ which is symmetric with respect to the involution $T(x) = -x$. Assume that $\varphi : \partial\overline{U} \to \mathbf{R}^m$ is an equivariant admissible mapping. Then there exists point $x_0 \in \partial\overline{U}$ such that $\varphi(x_0) \ni 0$.*

Proof. We symmetrically cover $\overline{U}$ by finitely many open disks $\{V_\alpha\}_{\alpha\in A}$ with a small radius below $r > 0$ such that $\overline{U} \subset \cup_{\alpha\in A} V_\alpha$. We may assume that $W = \cup_{\alpha\in A}\overline{V_\alpha}$ is a manifold with boundary. Moreover we may assume that the boundary ∂W is a manifold. If ∂W is not a manifold, it happened at a point x where two closed disks $\overline{V}_1$ and $\overline{V}_2$ are tangent each other. Since the point x is clearly outside of $\overline{U}$, it is sufficient to add two small disks symmetrically at x and $T(x)$. Therefore we have

$$c^{m+k-1}(\partial W, T) \neq 0. \tag{7}$$

as in the proof of Proposition 3.5.

Set $\overline{U}_r = \{x \in \overline{U} \mid d(x, \partial\overline{U}) \geqq 2r\}$ where $d(x,\partial\overline{U})$ is the distance between x and $\partial\overline{U}$. We symmetrically cover $\overline{U}_r$ by finitely many open disks $\{V'_\beta\}_{\beta\in B}$ with a small radius below $r > 0$ such that $\overline{U}_r \subset \cup_{\beta\in B} V'_\beta \subset \overline{U}$. Set $W' = \cup_{\beta\in B}\overline{V'_\beta}$. We may assume that W' is a manifold with boundary and satisfies $W' \subset Int\overline{U}$. By Proposition 3.5 and $\partial(W - IntW') = \partial W \cup \partial W'$, we obtain

$$c^{m+k-1}(\partial W', T) \neq 0, \quad c^{m+k-1}(W - IntW', T) \neq 0.$$

Since families $\{IntW - W'\}$ and $\{W - IntW'\}$ are cofinal coverings of $\partial\overline{U}$, we have the isomorphism

$$\bar{H}^*(\partial\overline{U}) \cong \varinjlim \bar{H}^*(IntW - W') \cong \varinjlim \bar{H}^*(W - IntW') \tag{8}$$

by the continuity of the Alexander-Spanier cohomology theory. By the naturality of Stiefel-Whitney class with respect to $\{W - IntW'\}$, we see

$$c^{m+k-1}(\partial\overline{U}, T) \neq 0. \tag{9}$$

Therefore we obtain the result by the similar method as the proof of Proposition 3.5. □

Let ∂U be the boundary of U. Note that $\partial U = \overline{U} - U$. Generally ∂U and $\partial\overline{U}$ are different and $\partial\overline{U} \subset \partial U$. For an open set U of a normed space $\mathbf{E}$, it is said to be balanced if satisfies $sU \subset U$ for all s, $(0 \leqq s \leqq 1)$. Since a bounded open symmetric balanced space U satisfies the condition of the following theorem, we obtain easily Theorem 6 in [1].

Theorem 3.7. *Let U be a bounded open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 0$ which is symmetric with respect to the involution $T(x) = -x$. Assume that $\varphi : \overline{U} \to \mathbf{R}^m$ is an admissible mapping which is equivariant on the boundary $\partial\overline{U}$ of $\overline{U}$. Then there exists a point $x_0 \in \overline{U}$ such that $\varphi(x_0) \ni 0$ and a point $x_1 \in \overline{U}$ such that $\varphi(x_1) \ni x_1$.*

Proof. We define a new open neighborhood V of the origin in $\mathbf{R}^{m+k+1}$:

$$V = \{(x,s) \in \mathbf{R}^{m+k+1} \mid x \in Int\overline{U},\ |s| < d(x, \partial\overline{U})\}. \tag{10}$$

Clearly V is an open neighborhood of the origin in $\mathbf{R}^{m+k+1}$ and bounded symmetric with respect to the antipodal involution in $\mathbf{R}^{m+k+1}$. We easily see :

$$\overline{V} = \{(x,s) \in \mathbf{R}^{m+k+1} \mid x \in \overline{U},\ |s| \leqq d(x, \partial\overline{U})\}. \tag{11}$$

The boundary $\partial\overline{V}$ of $\overline{V}$ is

$$\partial\overline{V} = \{(x,s) \in \mathbf{R}^{m+k+1} \mid x \in \overline{U},\ |s| = d(x, \partial\overline{U})\}. \tag{12}$$

Define a mapping $J : \overline{U} \to \mathbf{R}^{m+k+1}$ by

$$J(x) = x + d(x, \partial\overline{U})e_{m+k+1} \tag{13}$$

where $x \in \mathbf{R}^{m+k}$ and e_{m+k+1} is the $(m+k+1)$-th unit vector in $\mathbf{R}^{m+k+1}$. Clearly we see $\partial\overline{V} = J(\overline{U}) \cup \{TJ(\overline{U})\}$. As Theorem 3.6, we have $c(\partial\overline{V}, T)^{m+k} \neq 0$ and $c(\partial\overline{U}, T)^{m+k-1} \neq 0$.

For the case $k > 0$ the theorem is proved by the similar method as Theorem 3.6. We shall prove for the case $k = 0$. Let $\hat{\varphi} : \overline{U} \to \mathbf{R}^m$ be defined as follows:

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in Int\overline{U} \\ \varphi(x) \cup \{T\varphi(Tx)\} & \text{if } x \in \partial\overline{U}. \end{cases} \tag{14}$$

Since φ is upper semi-continuous, we can easily verify that $\hat{\varphi}$ is upper semi-continuous. Since φ is an equivariant admissible mapping on $\partial\overline{U}$, we can easily verify that $\hat{\varphi}$ is equivariant admissible on $\partial\overline{U}$. Note $\hat{\varphi}(Tx) = T\hat{\varphi}(x)$ for $x \in \partial\overline{U}$.

Define $\Psi : \partial\overline{V} \to \mathbf{R}^m$ by

$$\Psi(z) = \begin{cases} \hat{\varphi}(J^{-1}(z)) & \text{if } z \in J(\overline{U}) \\ T\hat{\varphi}(J^{-1}(Tz)) & \text{if } z \in TJ(\overline{U}). \end{cases} \tag{15}$$

Ψ is well-defined and an upper semi-continuous mapping defined on $\partial\overline{V}$.

Let $p : \Gamma \to \overline{U}$ and $q : \Gamma \to \mathbf{R}^m$ be a selected pair of φ. We shall show that Ψ is equivariant on $\partial\overline{V}$. Let $\hat{\Gamma}$ be the space obtained by the pushout $\Gamma \xleftarrow{i} \Gamma_0 \xrightarrow{iT} \Gamma$. Here we note $i_1 : \Gamma_0 \to \Gamma_1$ in the place of $i : \Gamma_0 \to \Gamma$ and $i_2 : \Gamma_0 \to \Gamma_2$ in the place of $iT : \Gamma_0 \to \Gamma$. $\hat{\Gamma}$ has the involution $\hat{T}$ induced by the following diagram:

$$\begin{tikzcd} \Gamma_1 \arrow[d, "h"] & \Gamma_0 \arrow[l, "i_1"'] \arrow[r, "i_2"] \arrow[d, "T"] & \Gamma_2 \arrow[d, "k"] \\ \Gamma_2 & \Gamma_0 \arrow[l, "i_2"'] \arrow[r, "i_1"] & \Gamma_1 \end{tikzcd}$$

where $h : \Gamma_1 \longrightarrow \Gamma_2$, $k : \Gamma_2 \longrightarrow \Gamma_1$ are defined by the identity $\Gamma \longrightarrow \Gamma$.

$\hat{p} : \hat{\Gamma} \longrightarrow \partial\overline{V}$ is defined by

$$\hat{p}(x) = \begin{cases} J(p(x)) & \text{if } x \in \Gamma_1 \\ TJ(p(\hat{T}x)) & \text{if } x \in \Gamma_2. \end{cases}$$

We easily see $\hat{p} : \hat{\Gamma} \longrightarrow \partial\overline{V}$ is a Vietoris mapping.

$\hat{q} : \hat{\Gamma} \longrightarrow \mathbf{R}^m$ is defined by

$$\hat{q}(x) = \begin{cases} q(x) & \text{if } x \in \Gamma_1 \\ Tq(\hat{T}x) & \text{if } x \in \Gamma_2. \end{cases}$$

By Theorem 3.6, we obtain a point $x_0 \in \partial\overline{V}$ such that $\Psi(x_0) \ni 0$. This means $\varphi(y_0) \ni 0$ for a point $y_0 \in \overline{U}$.

For the second part, define $\varphi_1 : \overline{U} \longrightarrow \mathbf{R}^{m+k}$ by $\varphi_1(x) = x - j\varphi(x)$ for $x \in \overline{U}$ where $j : \mathbf{R}^m \longrightarrow \mathbf{R}^{m+k}$. $p : \Gamma \longrightarrow \overline{U}$ and $p - jq : \Gamma \longrightarrow \mathbf{R}^{m+k}$ are the selected pair of φ_1. We easily verify that φ_1 is equivariant on $\partial\overline{U}$ by our hypothesis on φ. By apply the former part of this theorem to the case, there exists an element $x_1 \in \overline{U}$ such that $\varphi_1(x_1) \ni 0$, i.e. $\varphi(x_1) \ni x_1$. □

4 Generalization to normed spaces

For a normed space $\mathbf{E}$, D is defined by $\{x \in \mathbf{E} \mid \|x\| \leqq 1\}$ and S its boundary. We easily see that S is acyclic for an infinite dimensional normed space. Let S_π be the orbit space of S by the antipodal involution. The cohomology ring of S_π is the polynomial ring or truncated polynomial ring according to the infinite or finite dimensional normed spaces. This is easily proved by using the Gysin-Smith exact sequence of a double covering space.

We shall give a generalization of Theorem 3.7 to the normed space. We prepare the Schauder approximation theorem for our application (cf. Theorem 12.9 in [5]).

Theorem 4.1. *Let X be a Hausdorff space and U an open set of a normed space $\mathbf{E}$ and $f : X \longrightarrow U$ a continuous compact mapping. Then, for any $\epsilon > 0$, there exists a continuous compact mapping $f_\epsilon : X \longrightarrow U$ satisfying the following condition:*

1. *$f_\epsilon(X) \subset \mathbf{E}^{n(\epsilon)}$ for a finite dimensional subspace $\mathbf{E}^{n(\epsilon)}$ of $\mathbf{E}$*
2. *$\|f_\epsilon(x) - f(x)\| < \epsilon$ for any $x \in X$*
3. *$f_\epsilon(x)$, $f(x) : X \longrightarrow U$ are homotopic, noted by $f_\epsilon \simeq f$.*

In what follows, we assume that Γ is a metric space. The following theorem 4.2 is called Borsuk's fixed point theorem (cf. Theorem 3.3 in §6 in [6], Theorem 3.7). Y.S.Chang proved Theorem 4.2 for the case of a bounded symmetric balanced neighborhood of the origin in a locally convex topological space (cf. Theorem 7 in [1]). We shall extend his theorem to the case of spaces which is not necessarily contractible.

Theorem 4.2. *Let U be a symmetric bounded open neighborhood of the origin in a normed space $\mathbf{E}$. Assume that $\varphi : \overline{U} \to \mathbf{E}$ is upper semi-continuous, compact convex valued mapping and is equivariant on $\partial\overline{U}$. Then there exist a fixed point $z_0 \in \overline{U}$ such that $\varphi(z_0) \ni z_0$.*

Proof. The normed space $\mathbf{E}$ has the involution T defined by $T(x) = -x$. Let $p : \Gamma \to \overline{U}$ and $q : \Gamma \to \mathbf{E}$ be a selected pair of φ. Let $p_0 : \Gamma_0 \to \overline{U}$ and $q_0 : \Gamma_0 \to \mathbf{E}$ be a selected pair of φ_0 which are equivariant mappings and $\varphi_0(x) = \varphi(x) \cap (T\varphi(T(x)))$ for $x \in \partial\overline{U}$.

For any natural number n, we find finite dimensional vector subspaces $\{\mathbf{V}_n\}$ in $\mathbf{E}$ and $\{q_n : \Gamma \to \mathbf{V}_n\}$ such that

$$\|q(y) - q_n(y)\| < \frac{1}{n} \quad (y \in \Gamma) \tag{16}$$

by the approximation theorem of Schauder. Note that we can choose vector spaces $\{\mathbf{V}_n\}$ such that $\dim \mathbf{V}_n$ increases as n increases and $\mathbf{V}_n \subset \mathbf{V}_{n+1}$ for all n by seeing the construction in the approximation theorem.

Note that Γ_0 has the involution $\tilde{T}$. Define $q_{n,0} : \Gamma_0 \to \mathbf{V}_n$ by

$$q_{n,0}(z) = \frac{1}{2}\{q_n(z) - q_n(\tilde{T}(z))\} \tag{17}$$

which is equivariant. We obtain the following inequality:

$$\|q_{n,0}(z) - q_0(z)\| < \frac{1}{n} \tag{18}$$

for $z \in \Gamma_0$. This is proved by $\|q_n(z) - q_0(z)\| < \frac{1}{n}$ for $z \in \Gamma_0$ and $\|q_n(\tilde{T}z) + q_0(z)\| = \|q_n(\tilde{T}z) - q_0(\tilde{T}z)\| < \frac{1}{n}$ for $z \in \Gamma_0$. And it holds also

$$\|q_{n,0}(z) - q_n(z)\| < \frac{1}{n} \tag{19}$$

for $z \in \Gamma_{n,0}$. This is proved by $\|q_n(\tilde{T}z)+q_n(z)\| \leqq \|q_n(\tilde{T}z)-q_0(\tilde{T}z)\|+\|q_0(\tilde{T}z)+q_n(z)\| \leqq \|q_n(\tilde{T}z)-q_0(\tilde{T}z)\|+\|-q_0(z)+q_n(z)\| < \frac{2}{n}$ for $z \in \Gamma_0$. Especially q_n and $q_{n,0}$ are homotopic.

Let $\varphi_n : \overline{U} \to \mathbf{V}_n$ be defined by

$$\varphi_n(x) = B_n(\varphi(x)) \cap \mathbf{V}_n. \tag{20}$$

where $B_n(\varphi(x)) = \{z \in \mathbf{E} |\ d(z, \varphi(x)) \leqq \frac{1}{n}\}$. By the inequality (16), it holds $\varphi_n(x) \neq \emptyset$ for $x \in \overline{U}$. The graph of a set valued mapping $\hat{h}(x) = B_n(\varphi(x))$ is clearly a closed set in $\mathbf{E} \times \mathbf{E}$ and also the graph of φ_n is a closed set in $\mathbf{V}_n \times \mathbf{V}_n$. Since the image of $\varphi_n(\overline{U})$ is contained in a compact set by the condition of φ and the definition of φ_n, φ_n is upper semi-continuous and compact mapping. $p_n = p$ and q_n is a selected pair of φ_n. Since $q_{n,0}$ is equivariant, φ_n is equivariant on $\partial\overline{U}$ by the inequality (19).

Set $K_n = \overline{U \cap \mathbf{V}_n}$ in $\mathbf{V}_n$ and $K_{n,0} = \partial(\overline{U \cap \mathbf{V}_n})$. Set $\Gamma_n = p^{-1}(K_n)$, $\Gamma_{n,0} = (p_0)^{-1}(K_{n,0})$. $p_n : \Gamma_n \to K_n$ and $p_{n,0} : \Gamma_{n,0} \to K_{n,0}$ are the restrictions of p to Γ_n and $\Gamma_{n,0}$ respectively. The restriction of q_n to Γ_n is also written by $q_n : \Gamma_n \to \mathbf{V}_n \subset \mathbf{E}$.

Let $\psi_n : K_n \to \mathbf{V}_n$ be the restriction of φ_n to K_n. We see $\psi_n(x) \neq \emptyset$ for $x \in K_n$ by (18). Let $\psi_{n,0} : K_{n,0} \to \mathbf{V}_n$ be defined by $\psi_{n,0}(x) = \psi_n(x) \cap T\psi_n(T(x))$ for $x \in K_{n,0}$. We see $\psi_{n,0}(x) \neq \emptyset$ for $x \in K_{n,0}$ by (19).

We apply Theorem 3.7 to the case $\psi_n : K_n \to \mathbf{V}_n$. We have a point $z_n \in K_n$ such that $z_n \in \varphi_n(z_n)$. From $z_n \in \psi_n(z_n)$, i.e. $z_n \in \varphi_n(z_n)$, we have a sequence $\{w_n\}$ satisfying $\|z_n - w_n\| < \frac{1}{n}$ and $w_n \in \varphi(z_n)$. Since φ is a compact mapping, a subsequence of $\{w_n\}$ converges to w_0. Therefore we may assume that $\{z_n\}$ converge to a point w_0. Since φ is upper semi-continuous, we have $z_0 \in \varphi(z_0)$.

□

In the above theorem, we can not prove the existence of the zero value of φ as the finite dimensional case. Now we shall give some examples. Let D be the unit disk in a Hilbert space $\mathbf{H}$. Let $f : D \to D$ be defined by

$$f(\{z_n\}) = (\sqrt{1 - \|z\|^2}, \{z_n\}). \tag{21}$$

Clearly f is a continuous mapping on D and equivariant on the boundary S and not a compact mapping. If f has a zero value, it holds the equations $\sqrt{1 - \|z\|^2} = 0, \quad z_n = 0$ for all n. We obtain easily the contradiction from the equations. Therefore f has not a zero value. We see also easily that f has not a fixed point.

Let $g : D \to D$ be defined by

$$g(\{z_n\}) = (\sqrt{1 - \|z\|^2}, \{\frac{z_n}{n}\}). \tag{22}$$

Clearly g is a continuous mapping on D and equivariant on the boundary S and a compact mapping. If g has a zero value, it holds the equations $\sqrt{1 - \|z\|^2} = 0, \quad \frac{z_n}{n} = 0$ for all n. We obtain easily the contradiction from the equations. Therefore g has not the zero value. Of course g has a fixed point (cf. §12 in [5], §3 in [12]).

Definition 4.3. *Let X be a subset of a vector space $\mathbf{V}$ and $\Phi : X \to \mathbf{V}$ a compact admissible mapping. A set-valued mapping $\varphi : X \to \mathbf{V}$ is called an admissible compact field, if φ is defined by $\varphi(x) = x - \Phi(x)$.*

Let $\mathbf{E}_k$ be a closed subspace of codimension k of a normed space $\mathbf{E}$. K.Geba and L. Górniewicz [3] proved the following theorem for the case of the unit sphere of a normed space. Our method is different from their method.

Theorem 4.4. *Let $\mathbf{E}_k$ be a closed subspace of codimension $k \geqq 1$ of $\mathbf{E}$ and U be a symmetric bounded open neighborhood of the origin of $\mathbf{E}$. If $\Phi : \partial\overline{U} \to \mathbf{E}_k$ is a compact admissible mapping, there is a point $x_0 \in \partial\overline{U}$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$ where $\varphi(x) = x - \Phi(x)$.*

Proof. Let (p, q) a selected pair of Φ where $p : \Gamma \to \partial\overline{U}$ is a Vietoris mapping and $q : \Gamma \to \mathbf{E}_k$ continuous mapping. There is a k-dimensional subspace $\mathbf{L}_k$ such that $\mathbf{E} = \mathbf{E}_k \oplus \mathbf{L}_k$.

By the approximation theorem of Schauder, there are finite dimensional vector subspace $\mathbf{V}_n \subset \mathbf{E}_k$ and $q_n : \Gamma \to \mathbf{V}_n$ such that

$$\|q(y) - q_n(y)\| < \frac{1}{n}$$

for $y \in \Gamma$. We may assume that $\dim \mathbf{V}_n$ increases and $\mathbf{V}_n \subset \mathbf{V}_{n+1}$. Let $\Phi_n : \partial\overline{U} \to \mathbf{V}_n$ be a set-valued mapping defined by

$$\Phi_n(x) = B_n(\Phi(x)) \cap \mathbf{V}_n$$

where $B_n(\Phi(x)) = \{y \in \mathbf{E} \mid d(\Phi(x), y) \leqq \frac{1}{n}\}$. Since the graph of Φ_n is closed and $\Phi_n(\partial\overline{U})$ is compact, Φ_n is upper semi-continuous. Clearly Φ_n has a selected pair $p : \Gamma \to \partial\overline{U}$ and $q_n : \Gamma \to \mathbf{V}_n$. Therefore Φ_n is a compact admissible mapping.

Set $\varphi_n(x) = x - \Phi_n(x)$. Consider $\Psi_n : W_n \to \mathbf{V}_n$ defined by the restriction of Φ_n to W_n where $W_n = \partial\overline{U} \cap (\mathbf{V}_n \oplus \mathbf{L}_k)$. Note that $c(W_n, T)^{i_n+k-1} \neq 0$ by Proposition 3.5 where $\dim W_n = i_n$.

By applying Theorem 6.3 of Y.Shitanda [12] to $\psi_n(x) = x - \Psi_n(x)$, we have a point $x_n \in W_n$ such that $\psi_n(x_n) \cap \psi_n(T(x_n)) \neq \emptyset$. This means $x_n - y_n = -x_n - z_n$ for some $y_n \in \Psi_n(x_n)$ and $z_n \in \Psi_n(T(x_n))$. Since Φ is compact mapping, there are convergent points y_0 and z_0 of $\{y_n\}$ and $\{z_n\}$ respectively. Therefore there is a convergent point x_0 where $x_n \to x_0$ and $x_0 = \dfrac{y_0 - z_0}{2}$. We see easily $y_0 \in \Phi(x_0)$ and $z_0 \in \Phi(T(x_0))$. By $x_0 - y_0 = -x_0 - z_0$, we have $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$, i.e. $A(\varphi) \neq \emptyset$ where $A(\varphi) = \{x \in \partial\overline{U} \mid \varphi(x) \cap \varphi(T(x)) \neq \emptyset\}$. $\square$

Let X be a space with a free involution T and S^k a k-dimensional sphere with the antipodal involution. Define $\gamma(X)$ and $\mathrm{Ind}(X)$ by

$$\begin{aligned}\gamma(X) &= \inf\{k \mid f : X \to S^k \text{ equivariant mapping}\}\\ \mathrm{Ind}(X) &= \sup\{k \mid c^k \neq 0\}\end{aligned}$$

respectively, where $c \in \bar{H}^1(X_\pi; \mathbf{F}_2)$ is the class $c = f_\pi^*(\omega)$ for an equivariant mapping $f : X \to S^\infty$. If X is a compact space with a free involution, it holds the following formula (cf. §3 in [2]):

$$\mathrm{Ind}(X) \leqq \gamma(X) \leqq \dim X. \tag{23}$$

K. Gęba and L. Górniewicz proved $\mathrm{Ind}A(\varphi) \geqq k - 1$ (cf. Theorem 2.5 in [2]). We shall generalize their result.

Corollary 4.5. *Under the hypothesis of Theorem 4.4, it holds*

$$\mathrm{Ind}A(\varphi) \geqq k - 1.$$

Proof. We use the notation of Theorem 4.4. Consider $\varphi_n : W_n \to \mathbf{V}_n$ where $\tilde{\varphi}_n = \varphi(x) \cap \mathbf{V}_n$ for $x \in W_n$. Clearly it holds $A(\tilde{\varphi}_n) \subset A(\tilde{\varphi}_{n+1})$. By Theorem 6.3 of [12], we have $\mathrm{Ind}(A(\tilde{\varphi}_n)) \geqq k - 1$. Therefore we obtain the result. $\square$

References

[1] Chang S.Y. Borsuk's Antipodal and Fixed Point Theorems for Set-Valued Maps, Proc. of Amer. Math. Soc. vol.121,937-941, (1994).

[2] Gęba K. and Górniewicz L. On the Bourgin-Yang Theorem for Multi-valued Maps I, Bull. Polish Acad. Sci. Math. Vol.34 No.5-6. (1986), 315-322.

[3] Gęba K. and Górniewicz L. On the Bourgin-Yang Theorem for Multi-valued Maps II, Bull. Polish Acad. Sci. Math. Vol.34 No.5-6. (1986), 323-327.

[4] Górniewicz,L. A Lefschetz-type fixed point theorem, Fundamenta Mathematicae 88, (1975),103-115.

[5] Górniewicz,L. Topological Fixed Point Theory of Multivalued Mappings, Kluwer Academic Publishers, (1999).

[6] Granas A. and Dugundji J. Fixed Point Theory, Springer Monographs in Mathematics, Springer Verlag, New York Inc (2003)

[7] Milnor J. Characteristic classes, Annals of Mathematics Studies, Princeton University Press

[8] Nakaoka M. Generalizations of Borsuk-Ulam Theorem, Osaka J. Math. 7, (1970),423-441.

[9] Nakaoka M. Continuous map of manifolds with involution I, Osaka J. Math. 11, (1974),129-145.

[10] Nakaoka M. Continuous map of manifolds with involution II, Osaka J. Math. 11, (1974),147-162.

[11] Nakaoka M. Equivariant point theorems for involution, Japan J. Math. Vol.4,No.2, (1978),263-298.

[12] Shitanda Y. Fixed point theorems and equivarint points for set-valued mappings, submitted.

[13] Spanier E. H. Algebraic Topology, Springer-Verlag, New York (1966)