数理解析研究所講究録
第 1612 巻 2008 年 158-169

DIFFEOMORPHISM TYPE OF REAL BOTT TOWERS

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INTRODUCTION

A real Bott tower is described as a sequence of $\mathbb{RP}^1$-bundles just as the real restriction to Bott towers [2] . From the viewpoint of group actions, an n-dimensional real Bott tower is viewed as the quotient of the n-dimensional torus $T^n = S^1 \times ... \times S^1$ by the product $(\mathbb{Z}_2)^n$ of cyclic groups of order 2. A *Bott matrix* A of size n is a upper triangular matrix whose diagonal entries are one and the other entries are either one or zero. By the definition, there are $2^{\frac{n^2-n}{2}}$ distinct Bott matrices of size n. The free action of $(\mathbb{Z}_2)^n$ on T^n can be expressed by each row of the Bott matrix A whose orbit space $M(A) = T^n/(\mathbb{Z}_2)^n$ is the real Bott tower. It is easy to see that $M(A)$ is a compact euclidean space form (Riemannian flat manifold). Then we can apply the Bieberbach theorem [5] to classify real Bott towers. Using this theorem, the classification of real Bott towers up to dimension 4 has been obtained [3]. In [2] we have proved that every n-dimensional real Bott tower $M(A)$ admits an injective Seifert fibred structure, that is there exists a k-torus action on $M(A)$ whose quotient space is an $(n-k)$-dimensional real Bott tower orbifold $M(B)$ by some $(\mathbb{Z}_2)^s$-action $(1 \leq s \leq k)$. Moreover we have shown the smooth rigidity which states that real Bott towers $M(A_i)$ $i = 1, 2$ are diffeomorphic if and only if the corresponding actions $((\mathbb{Z}_2)^{s_i}, M(B_i))$ are equivariantly diffeomorphic. When the low dimensional real Bott towers with $(\mathbb{Z}_2)^s$-actions are determined, we can distinguish the diffeomorphism classes of higher dimensional ones by the rigidity.

The main purpose of this paper is to determine the diffeomorphism classes of 4 dimensional real Bott towers from the classifications of 2, 3 dimensional real Bott towers with $(\mathbb{Z}_2)^s$-actions $(s = 1, 2)$. This method also works for dimension 5 but the classification of low dimensional real Bott towers with $(\mathbb{Z}_2)^s$-actions are a bit complicated. The classification of 5 dimensional real Bott towers will be appeared elsewhere. (cf. [4])

1. REVIEW OF [2]

Each i-th row of a Bott matrix A defines a $\mathbb{Z}_2$-action on T^n by

$$g_i(z_1, z_2, \ldots, z_n) = (z_1, \ldots, z_{i-1}, -z_i, \tilde{z}_{i+1}, \ldots, \tilde{z}_n),\ \ (i = 1, ..., n)$$

where (i, i)-(diagonal) entry 1 acts as $z_i \rightarrow -z_i$ while $\tilde{z}_j$ is either z_j or $\bar{z}_j$ depending on whether (i, j)-entry $(i < j)$ is 0 or 1 respectively. Note that $\bar{z}$ is the conjugate of the complex number $z \in S^1$. It always trivial; $z_j \rightarrow z_j$ whenever $j < i$. Here $(z_1, ..., z_n)$ are the standard coordinates of the n-dimensional torus T^n. Those $\langle g_1, \ldots, g_n\rangle$ constitute the

2000 *Mathematics Subject Classification.* 53C24, 57S25.
Key words and phrases. Bott tower, Crystallographic group, Bieberbach group, Flat Riemannian Manifold, Seifert fibration, Diffeomorphism.

generators of $(\mathbb{Z}_2)^n$. It is easy to see that $(\mathbb{Z}_2)^n$ acts freely on T^n such that the orbit space $M(A) = T^n/(\mathbb{Z}_2)^n$ is a smooth compact manifold. In this way, given a Bott matrix A of size n, we obtain a free action of $(\mathbb{Z}_2)^n$ on T^n.

Now let us recall operations I, II, III and IV [2] to a Bott matrix A of size n under which the diffeomorphism class of $M(A)$ does not change.

I. Interchange the coordinates z_i, z_j in T^n, $(z_j \to z_i', z_i \to z_j')$.

II. Interchange the generators g_i, g_j $(i < j)$, $(g_j \to g_i', g_i \to g_j')$.

Performing the operations I and II iteratively, we get a Bott matrix

$$A' = \left(\begin{array}{c|c} I_k & C \\ \hline 0 & B \end{array}\right) \quad B = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \tag{1.1}$$

where I_k is a maximal block of identity matrix of size k, the entries of the $*$ are either 1 or 0, B is the Bott matrix of size $(n-k)$ which presents a real Bott tower $M(B) = T^{n-k}/(\mathbb{Z}_2)^{n-k}$. Since I_k is a maximal block of identity matrix, each $k+j$ $(j = 1, \ldots, n-k)$-*th* column of A' has at least two non zero elements.

Associated to the Bott matrix A', the $(\mathbb{Z}_2)^n$-action splits into $(\mathbb{Z}_2)^k \times (\mathbb{Z}_2)^{n-k}$ and T^n splits into $T^k \times T^{n-k}$. Hence

$$M(A) = T^n/(\mathbb{Z}_2)^n \cong \frac{T^k \times T^{n-k}}{(\mathbb{Z}_2)^k \times (\mathbb{Z}_2)^{n-k}} = T^k \underset{(\mathbb{Z}_2)^k}{\times} M(B) = M(A'). \tag{1.2}$$

Note that above $(\mathbb{Z}_2)^k$-action of (1.2) is not necessarily effective on $M(B)$ but we can reduce it to the effective $(\mathbb{Z}_2)^s$-action on $M(B)$ for some s $(1 \le s \le k)$. In order to do so, we have two more operations.

III. If there is an m-th row $(1 \le m \le k)$ whose entries in C are all zero, then divide $T^k \times M(B)$ by the corresponding $\mathbb{Z}_2$-action.

IV. If the p-th row and ℓ-th row $(1 \le p < \ell \le k)$ have the common entries in C, then compose the $\mathbb{Z}_2$-action of p-th row with l-th row and divide $T^k \times M(B)$ by this $\mathbb{Z}_2$-action.

By an iteration of **III**, **IV**, the quotient is again diffeomorphic to $T^k \times M(B)$ but eventually the $(\mathbb{Z}_2)^k$-action is reduced to the effective $(\mathbb{Z}_2)^s$-action on $T^k \times M(B)$. Therefore the Bott matrix A' reduces to

$$A'' = \left(\begin{array}{c|c|c} I_{k-s} & 0 & 0 \\ \hline 0 & I_s & * \\ \hline 0 & 0 & B \end{array}\right) \tag{1.3}$$

in which

$$\begin{aligned} M(A') &= T^k \underset{(\mathbb{Z}_2)^k}{\times} M(B) \\ &= \frac{T^{k-s} \times T^s \times M(B)}{(\mathbb{Z}_2)^{k-s} \times (\mathbb{Z}_2)^s} = M(A''). \end{aligned}$$

Since $(\mathbb{Z}_2)^{k-s}$ acts trivially on $T^s \times M(B)$ then we have

$$M(A'') \cong T^k \underset{(\mathbb{Z}_2)^s}{\times} M(B).$$

From now on, we write $M(A)$ instead of $M(A'')$.

Remark 1.1. *From the submatrix* $*$ *of* (1.3), *the group* $(\mathbb{Z}_2)^s = \langle g_{k-s+1}, \ldots, g_k \rangle$ *acts on* $T^k \times M(B)$ *by*

$$\begin{aligned} &g_i(z_1, \ldots, z_{k-s+1}, \ldots, z_k, [z_{k+1}, \ldots, z_n]) \\ &= (z_1, \ldots, z_{k-s+1}, \ldots, -z_i, \ldots, z_k, [\tilde{z}_{k+1}, \ldots, \tilde{z}_n]) \end{aligned} \tag{1.4}$$

where $\tilde{z} = \bar{z}$ *or* z. *So there induces an action of* $(\mathbb{Z}_2)^s$ *on* $M(B)$ *by*

$$g_i([z_{k+1}, \ldots, z_n]) = [\tilde{z}_{k+1}, \ldots, \tilde{z}_n]. \tag{1.5}$$

Moreover in [2], we have shown that

Theorem 1.2 (Structure). *Given a real Bott tower* $M(A)$, *there exists a maximal* T^k-*action* ($k \geq 1$) *such that*

$$M(A) = T^k \underset{(\mathbb{Z}_2)^s}{\times} M(B)$$

is an injective Seifert fiber space over the $(n-k)$-*dimensional real Bott orbifold* $M(B)/(\mathbb{Z}_2)^s$;

$$T^k \longrightarrow M(A) \longrightarrow M(B)/(\mathbb{Z}_2)^s. \tag{1.6}$$

There is a central extension of the fundamental group $\pi(A)$ *of* $M(A)$:

$$1 \longrightarrow \mathbb{Z}^k \longrightarrow \pi(A) \longrightarrow Q_B \longrightarrow 1 \tag{1.7}$$

such that

(i) $\mathbb{Z}^k$ *is the maximal central free abelian subgroup*

(ii) *The induced group* Q_B *is the semidirect product* $\pi(B) \rtimes (\mathbb{Z}_2)^s$ *for which* $\mathbb{R}^{n-k}/\pi(B) = M(B)$.

See [2] for the proof.

By this theorem, a real Bott tower $M(A)$ which admits a maximal T^k-action ($k \geq 1$) can be created from an $(n-k)$-dimensional real Bott tower $M(B)$ by a $(\mathbb{Z}_2)^s$-action, and the corresponding Bott matrix A has the form as in (1.3) above.

Next, we can apply the following theorem to check whether two real Bott towers are diffeomorphic.

Theorem 1.3 (Rigidity). *Let* $M(A_1)$, $M(A_2)$ *be* n-*dimensional real Bott towers and* $1 \longrightarrow \mathbb{Z}^{k_i} \longrightarrow \pi(A_i) \longrightarrow Q_{B_i} \longrightarrow 1$ *be the associated group extensions* ($i = 1, 2$). *Then the following are equivalent:*

(i) $\pi(A_1)$ *is isomorphic to* $\pi(A_2)$.

(ii) *There exists an isomorphism of* $Q_{B_1} = \pi(B_1) \rtimes (\mathbb{Z}_2)^{s_1}$ *onto* $Q_{B_2} = \pi(B_2) \rtimes (\mathbb{Z}_2)^{s_2}$ *preserving* $\pi(B_1)$ *and* $\pi(B_2)$.

(iii) *The action* $((\mathbb{Z}_2)^{s_1}, M(B_1))$ *is equivariantly diffeomorphic to* $((\mathbb{Z}_2)^{s_2}, M(B_2))$.

See [2] for the proof.

Note that two real Bott towers $M(A_1)$ and $M(A_2)$ are diffeomorphic if and only if $\pi(A_1)$ is isomorphic to $\pi(A_2)$ by the Bieberbach theorem [5]. Moreover Theorem 1.3 implies that if $M(B_1)$ and $M(B_2)$ are not diffeomorphic then $M(A_1)$ is not diffeomorphic to $M(A_2)$. Therefore two real Bott towers which admit different maximal T^k-action are not diffeomorphic. If they have the same maximal T^k-action, then the quotients $((\mathbb{Z}_2)^{s_i}, M(B_i))$ are compared. If $M(B_1)$ is not diffeomorphic to $M(B_2)$ or $s_1 \neq s_2$, then $M(A_1)$ and $M(A_2)$ are not diffeomorphic. So our task is to distinguish the $(\mathbb{Z}_2)^s$-actions on $M(B_i)$ when it is the case that $s_1 = s_2 = s$ and $M(B_1)$ is diffeomorphic to $M(B_2)$.

Proposition 1.4. *The $(\mathbb{Z}_2)^s$-action on $M(B)$ is distinguished by the number of components and types of each positive dimensional fixed point subsets.*

See [2] for the proof.

Note that from (1.5), the fixed point set of $(\mathbb{Z}_2)^s$ acting on $M(B)$ is characterized by the equation:

$$(\tilde{z}_{k+1}, ..., \tilde{z}_n) = g(z_{k+1}, ..., z_n)$$

for some $g \in (\mathbb{Z}_2)^{n-k}$.

Definition. We say that two Bott matrices A and A' are *equivalent* (denoted by $A \sim A'$) if $M(A)$ and $M(A')$ are diffeomorphic.

In order to understand easily, we shall give the explicit calculations in the following examples how to create and distinguish the diffeomorphism type of real Bott towers.

Example 1.1.

We create Bott matrices of size 4 where the corresponding real Bott towers admit the maximal T^2-actions. By Theorem 1.2, such Bott matrices can be created from a Bott matrix B of size 2. In this example we choose $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. There are 12 Bott matrices of size 4 created from B with the $(\mathbb{Z}_2)^s$-actions where $s = 1, 2$ (see subsection 3.2 below). Now we choose four of them as follows

$$A_3 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right), A_4 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

$$A_5 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right), A_6 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

a). *Let us consider the Bott matrices A_3 and A_4. As mentioned in the previous paragraph, by the operation IV, the $(\mathbb{Z}_2)^2$-action on $T^2 \times M(B)$ corresponding to Bott matrix A_3 reduces to the $\mathbb{Z}_2$-action. ($M(B)$ = a Klein bottle). Therefore Bott matrix A_3 is equivalent to A_4.*

b). *Now the induced action of $(\mathbb{Z}_2)^2$ on $M(B)$ corresponding to Bott matrices A_5 and A_6 are*

(i). $g_1([z_3, z_4]) = [\bar{z}_3, \bar{z}_4]$, $g_2([z_3, z_4]) = [z_3, \bar{z}_4]$ *and*
(ii). $h_1([z_3, z_4]) = [\bar{z}_3, z_4]$, $h_2([z_3, z_4]) = [z_3, \bar{z}_4]$
respectively. We change the generator g_1 *into* $g_1'([z_3, z_4]) = g_1g_2\,([z_3, z_4]) = [\bar{z}_3, z_4]$. *Then define an equivariant diffeomorphism* $\varphi : ((\mathbb{Z}_2)^2, M(B)) \to ((\mathbb{Z}_2)^2, M(B))$ *by* $\varphi([z_3, z_4]) = ([z_3, z_4])$ *such that* $\varphi g_1' = h_1\varphi$ *and* $\varphi g_2 = h_2\varphi$. *Hence* $M(A_5)$ *is diffeomorphic to* $M(A_6)$ *by Theorem* 1.3.

c). *To show that* $M(A_4)$ *is not diffeomorphic to* $M(A_6)$, *we use the following argument. Since the* $(\mathbb{Z}_2)^2$*-action on* $M(B)$ *corresponding to* A_4 *reduces to the* $\mathbb{Z}_2$*-action then* $M(A_4) = T^2 \times_{\mathbb{Z}_2} M(B)$, *but* $M(A_6) = T^2 \times_{(\mathbb{Z}_2)^2} M(B)$.

Example 1.2.
We shall create 5*-dimensional real Bott towers which admit maximal* S^1*-actions. Therefore the corresponding Bott matrices can be created from the Bott matrices of size* 4. *In this example we create the Bott matrix* A *from* A_4 *(see Example* 1.1.*). We introduce* 3 *of* 4 *Bott matrices as follows*

$$A_7 = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), A_8 = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), A_9 = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

a). *The induced action of* $\mathbb{Z}_2$ *on* $M(A_4)$ *corresponding to Bott matrices* A_7 *and* A_8 *are*

$$\begin{aligned} g_1([z_2, z_3, z_4, z_5]) &= [\bar{z}_2, \bar{z}_3, z_4, z_5] \text{ and} \\ h_1([z_2, z_3, z_4, z_5]) &= [\bar{z}_2, \bar{z}_3, \bar{z}_4, z_5] = [h_3(\bar{z}_2, \bar{z}_3, \bar{z}_4, z_5)] \\ &= [\bar{z}_2, -\bar{z}_3, z_4, z_5] \end{aligned}$$

respectively. We define an equivariant diffeomorphism

$$\varphi : (\mathbb{Z}_2, M(A_4)) \to (\mathbb{Z}_2, M(A_4))$$

by $\varphi([z_2, z_3, z_4, z_5]) = ([z_2, \mathrm{i}z_3, z_4, z_5])$, *such that* $\varphi g_1 = h_1\varphi$. *Hence* $M(A_7)$ *is diffeomorphic to* $M(A_8)$.

b). *Real Bott tower* $M(A_7)$ *is not diffeomorphic to* $M(A_9)$, *because they are distinguished by the positive dimensional fixed point sets of* $\mathbb{Z}_2$*-actions on* $M(A_4)$ *where the fixed point sets corresponding to* A_7 *and* A_9 *are (i)* 2*-components* T^2, 2*-components* S^1, 4 *points, (ii)* 6*-components* S^1, 4 *points, respectively.*

2. Two and Three Dimensional Real Bott Towers

2.1. **Two dimensional real Bott towers.** We shall classify the diffeomorphism classes of 2-dimensional real Bott towers.

Theorem 2.1. *The diffeomorphism classes of* 2*-dimensional real Bott towers consist of two.*

We shall explain Theorem 2.1. The Bott matrices are

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } A_2 = \left(\begin{array}{c|c} 1 & 1 \\ \hline 0 & 1 \end{array}\right). \tag{2.1}$$

Then the corresponding real Bott towers $M(A_1)$, $M(A_2)$ are not diffeomorphic because $M(A_1)$ is a torus T^2 and $M(A_2)$ is a Klein bottle.

2.2. **Three dimensional real Bott towers.** Using our Theorem 1.2, 3-dimensional real Bott towers are obtained from the 1, 2-dimensional real Bott towers with $(\mathbb{Z}_2)^s$-actions.

Theorem 2.2. *The diffeomorphism classes of* 3*-dimensional real Bott towers consist of four.*

2.2.1. S^1*-actions with two dimensional quotients.*

The Bott matrices of $M(A)$ admitting S^1-actions are the following forms

$$\left(\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \left(\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right), \left(\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right). \tag{2.2}$$

By the $\mathbb{Z}_2$-actions on two dimensional real Bott towers $M(B)$, the first row of each matrix is determined as above. However the second and third Bott matrices are equivalent

$$\left(\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right). \tag{2.3}$$

In fact, let $M(B)$ be the Klein bottle where $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then there is an equivariant diffeomorphism $\varphi : (\mathbb{Z}_2, M(B)) \to (\mathbb{Z}_2, M(B))$ which is defined by $\varphi([z_2, z_3]) = [\mathbf{i}z_2, z_3]$ such that $\varphi g_1 = h_1 \varphi$ where $g_1([z_2, z_3]) = [\bar{z}_2, z_3]$ and $h_1([z_2, z_3]) = [\bar{z}_2, \bar{z}_3] = [h_2(\bar{z}_2, \bar{z}_3)] = [-\bar{z}_2, z_3]$.

So the diffeomorphism classes of this case consist of two.

2.2.2. T^2*-actions with one dimensional quotients.*

The Bott matrices of $M(A)$ admitting T^2-actions have the form

$$\left(\begin{array}{cc|c} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ \hline 0 & 0 & 1 \end{array}\right),$$

where $a_{13}, a_{23} = \{0, 1\}$. In this case $M(B) = M(1) = S^1$ with $\mathbb{Z}_2$-action.

The following shows all the possibilities of the above form.

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right). \tag{2.4}$$

However they are equivalent to each other by operations I, II, III and IV. In this case it consists of just one diffeomorphism class.

Obviously a 3-dimensional real Bott tower admitting T^3-action is T^3 whose Bott matrix is the identity matrix of rank 3. Combined with the cases of S^1, T^2-actions we get four distinct diffeomorphism classes.

Remark 2.3. *We have already introduced seven of eight Bott matrices of size* 3*. The remaining is the following Bott matrix*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

However it is equivalent to the first Bott matrix in (2.4) *by using operations I and II* (*i.e., interchange the coordinates* z_2, z_3 *and the generators* g_2, g_3).

3. Four Dimensional Real Bott Towers

In this section we shall classify the diffeomorphism classes of 4-dimensional real Bott towers where by using Theorem 1.2, such real Bott towers are obtained from 1, 2 or 3-dimensional real Bott towers with the $(\mathbb{Z}_2)^s$-actions.

Theorem 3.1. *The diffeomorphism classes of* 4*-dimensional real Bott towers consist of twelve.*

We shall explain Theorem 3.1.

3.1. S^1-actions with three dimensional quotients.

The Bott matrices of $M(A)$ admitting S^1-actions have the following form

$$\left(\begin{array}{c|ccc} 1 & 1 & a_{13} & a_{14} \\ \hline 0 & & & \\ 0 & & \mathrm{B} & \\ 0 & & & \end{array}\right) \tag{3.1}$$

where $a_{13}, a_{14} = \{0,1\}$. In this case $M(B)$ corresponds to Bott matrices (2.2), (2.4) and I_3 with the $\mathbb{Z}_2$-actions.

The following shows all the possibilities of the form (3.1).

(i).

$$\left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{3.2}$$

(ii).

$$\left(\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{c|ccc} 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{3.3}$$

The two Bott matrices in (3.2) (resp. (3.3)) are equivalent by the equivariant diffeomorphism $\varphi : (\mathbb{Z}_2, M(B)) \to (\mathbb{Z}_2, M(B))$ defined by $\varphi([z_2, z_3, z_4]) = [\mathrm{i}z_2, z_3, z_4]$, where $M(B)$ corresponds to the first Bott matrix in (2.2). Then the fixed point sets of the $\mathbb{Z}_2$-actions on $M(B)$ corresponding to the Bott matrices (3.2) and (3.3) are (a) T^2, 4 points, (b) 4-components S^1, respectively.

(iii).

$$\left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{*} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{**} \left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{*} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{3.4}$$

(iv).

$$(3.5)\qquad \left(\begin{array}{c|ccc} 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{*} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{**} \left(\begin{array}{c|ccc} 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{*} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

The Bott matrices in (3.4) as well as (3.5) are equivalent to each other by the following equivariant diffeomorphisms $\varphi : (\mathbb{Z}_2, M(B)) \to (\mathbb{Z}_2, M(B))$ which is defined by $\varphi([z_2, z_3, z_4]) = [\mathbf{i}z_2, z_3, z_4]$ for "$\sim^{*}$" and by $\varphi([z_2, z_3, z_4]) = [\mathbf{i}z_2, \mathbf{i}z_3, z_4]$ for "$\sim^{**}$". Here $M(B)$ corresponds to the second or third Bott matrix in (2.2). On the other hand the fixed point sets of the $\mathbb{Z}_2$-actions on $M(B)$ corresponding to the Bott matrices (3.4) and (3.5) are (a) T^2, S^1, 2 points, (b) 3-components S^1, 2 points, respectively.

(v).

$$(3.6)\qquad \begin{array}{l} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{a)} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{b)} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{a)} \\ \left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{d)} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim^{c)} \left(\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right). \end{array}$$

The Bott matrices in (3.6) are equivalent to each other by the following equivariant diffeomorphisms

$$\begin{aligned} &\varphi : (\mathbb{Z}_2, M(B)) \to (\mathbb{Z}_2, M(B)) \\ &\varphi([z_2, z_3, z_4]) = [\mathbf{i}z_2, z_3, z_4] \ \ (for \text{ " } \sim^{a)} \text{ "}) \\ &\varphi([z_2, z_3, z_4]) = [z_2z_3, z_3, z_4] \ \ (for \text{ " } \sim^{b)} \text{ "}) \\ &\varphi([z_2, z_3, z_4]) = [z_2, \mathbf{i}z_3, z_4] \ \ (for \text{ " } \sim^{c)} \text{ "}), \end{aligned}$$

and for " $\sim^{d)}$ " we interchange the coordinates z_2, z_3 and the generators g_2, g_3 (by operations I, II). Here $M(B)$ corresponds to the Bott matrices in (2.4).

Remark 3.2. *We have two more Bott matrices which are equivalent to the Bott matrices in* (3.6), *namely*

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These Bott matrices are obtained from

$$\left(\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \text{ and } \left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

respectively, by interchanging the coordinates z_3, z_4 *and the generators* g_3, g_4.

(vi). Obviously it consists of just one Bott matrix of $M(A)$ obtained from $M(B)$ with $\mathbb{Z}_2$-action, where $B = I_3$, namely

$$\left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{3.7}$$

So the diffeomorphism classes of the case of S^1-actions with three dimensional quotients consist of six.

3.2. **T^2-actions with two dimensional quotients.**

The Bott matrices of $M(A)$ admitting T^2-actions have the following form

$$\left(\begin{array}{c|c} I_2 & * \\ \hline 0 & B \end{array}\right). \tag{3.8}$$

In this case $M(B)$ corresponds to Bott matrix I_2 or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with the $(\mathbb{Z}_2)^s$-actions where $s = 1, 2$.

The following shows all the possibilities of the form (3.8).

(i).

$$\begin{aligned} &\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \\ &\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \end{aligned} \tag{3.9}$$

(ii).

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{3.10}$$

Similar to Example 1.1, one can check that the Bott matrices in (3.9) (resp. (3.10)) are equivalent to each other. We can check that the $(\mathbb{Z}_2)^2$-actions on 2-dimensional real Bott towers $M(B)$ to each B in (3.10) can be reduced to a $\mathbb{Z}_2$-action on it. Moreover the class of real Bott tower in (3.9) is not equivalent to that of (3.10).

Remark 3.3. *There are two more Bott matrices which are equivalent to the Bott matrices in* (3.9), *namely*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These Bott matrices are obtained from

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \textit{ and } \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

respectively, by interchanging the coordinates z_2, z_3 and the generators g_2, g_3.

Next from the Bott matrix

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{3.11}$$

in (3.10) *we obtain the Bott matrices*

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \textit{ and } \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the first Bott matrix is obtained by interchanging the coordinates z_2, z_3 and the generators g_2, g_3, while the second Bott matrix is obtained by interchanging the coordinates z_2, z_4 and the generators g_2, g_4 of the Bott matrix (3.11).

(iii).

$$\begin{gathered} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \\ \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right). \end{gathered} \tag{3.12}$$

(iv).

$$\begin{gathered} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \\ \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right). \end{gathered} \tag{3.13}$$

We have already checked the equivalence of (iii) and (iv) respectively. Compare Example 1.1.

Remark 3.4. *There are four more Bott matrices which are equivalent to the Bott matrices in* (3.12), *namely*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.14}$$

The first Bott matrix (resp. the second Bott matrix) in (3.14) *is obtained from*

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

by interchanging the coordinates z_2, z_3 and the generators g_2, g_3 (resp. the coordinates z_2, z_3, z_4 and the generators g_2, g_3, g_4), while the third Bott matrix (resp. the fourth Bott matrix) in (3.14) *is obtained from*

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

by the same operations with above.

Associated to the class (3.13)*, there are two more Bott matrices which are equivalent to this class, namely*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where these Bott matrices are obtained from

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \text{ and } \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

respectively by interchanging the coordinates z_2, z_3 and the generators g_2, g_3.

So the diffeomorphism classes of the case of T^2-actions with two dimensional quotients consist of four.

3.3. T^3-actions with one dimensional quotients.

The Bott matrices of $M(A)$ admitting T^3-actions have the following form

$$\left(\begin{array}{ccc|c} & I_3 & & * \\ \hline 0 & 0 & 0 & 1 \end{array}\right). \tag{3.15}$$

In this case $M(B) = M(1) = S^1$ with $\mathbb{Z}_2$-action.

The following shows all the possibilities of the form (3.15).

$$\begin{aligned} &\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \sim \\ &\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right). \end{aligned} \tag{3.16}$$

However they are equivalent to each other by the operations I, II, III and IV. So in this case it consists of just one diffeomorphism class.

Remark 3.5. *There are four more Bott matrices which are equivalent to the Bott matrices in* (3.16), *namely*

$$(3.17)\qquad \begin{pmatrix} 1&0&0&0\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}.$$

The first and fourth Bott matrices in (3.17) *are obtained from*

$$\left(\begin{array}{ccc|c} 1&0&0&0\\ 0&1&0&1\\ 0&0&1&0\\ \hline 0&0&0&1 \end{array}\right) \text{ and } \left(\begin{array}{ccc|c} 1&0&0&1\\ 0&1&0&1\\ 0&0&1&0\\ \hline 0&0&0&1 \end{array}\right)$$

respectively, by interchanging the coordinates z_3, z_4 and the generators g_3, g_4. The second Bott matrix (*resp. the third Bott matrix*) *in* (3.17) *is obtained from*

$$\left(\begin{array}{ccc|c} 1&0&0&1\\ 0&1&0&0\\ 0&0&1&0\\ \hline 0&0&0&1 \end{array}\right)$$

by interchanging the coordinates z_2, z_4 and the generators g_2, g_4 (*resp. the coordinates z_3, z_4 and the generators g_3, g_4*).

Obviously the corresponding Bott matrix of size 4 of a real Bott tower admitting T^4-action is the identity matrix of rank 4. Combined with the cases of S^1, T^2, T^3-actions we get 12 distinct diffeomorphism classes of 4-dimensional real Bott towers.

Acknowledgment. I would like to thank Professor M. Masuda for his useful suggestions.

REFERENCES

[1] M. Grossberg and Y. Karshon, *Bott towers, complete integrability, and the extended character of representations,* Duke Math. J 76 (1994) 23-58.
[2] Y. Kamishima and A. Nazra, *Seifert fibred structure and rigidity on real Bott towers,* 2008. preprint.
[3] A. Nazra, *Real Bott Tower,* Tokyo Metropolitan University, Master Thesis 2008.
[4] A. Nazra, *Determination of Real Bott Towers,* 2008, in preparation.
[5] J.A. Wolf, *Spaces of Constant Curvature,* McGraw-Hill Book Company, 1967.

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