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G-ISOVARIANCE AND THE DIAGRAM OBSTRUCTION

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Section 1. Introduction

In 1987, W. Browder [Br] claimed a fundamental theorem relating equivariant vs. isovariant homotopy equivalences, under the Gap Hypothesis. Twenty years have passed since then, but the claim is still "folklore", despite the fact that many people (cf. [We 1]) have developed theories under the assumption that Browder's claim is true. The current author's earlier works [N 2], [N 3] also relied on it.

In 2006, R. Schultz [Sch] published a proof of Browder's theorem for semi-free actions. He used homotopy theoretic methods, and built a new obstruction theory in order to construct an isovariant homotopy equivalence from an equivariant homotopy equivalence in the semi-free situation. However, for general (non-semi-free) cases, the situation is not settled yet. If one wants to generalize Schultz' proof for non-semi-free cases, one would have to construct even more complicated obstruction theories, which do not look so straightforward.

In this note, we would like to generalize the homotopy theoretic methods done by Schultz and other people, to investigate a possible proof of Browder's theorem in a more general case, rather than the very restricted case done by Schultz. In order to do that, we generalize the diagram cohomoloogy obstruction theory developed by Dula and Schultz [DS] to more general group actions. We have not succeeded in proving the theorem yet, but we will give some construction that we hope to be able to be applied in the general situation, which we would like to work on elsewhere.

Section 2. Definition and the Basic Example

Let G be a finite group. Let M be a closed, connected, G-oriented smooth G-manifold. For any subgroup H of G, let M^H be the fixed-point set, which may consist of submanifolds of various dimension. A G-manifold M is said to satisfy the Gap Hypothesis if the following holds:

The Gap Hypothesis. *For any pair of subgroups $K \lneqq H$ of G, and for any pair of connected components $B \subset M^H$ and $C \subset M^K$ such that $B \subsetneqq C$, the inequality $2\dim B + 2 \leq \dim C$, in other words, $\dim B < [\frac{1}{2}\dim C]$, holds.*

The Gap Hypothesis provides general position arguments and transversality between each isotropy type pieces, thus making it possible to provide various geometric constructions in the equivariant settings. Madsen and Rothenberg ([MR 2]) constructed a beautiful surgery exact sequence in an equivariant category, and used it to classify spherical space forms.

Browder's insight told us to use this condition to construct isovariant homotopy equivalences from equivariant homotopy equivalences. And that is what we would like to consider here.

Definition. *A map $f : X \to Y$ between G-sapces X and Y is called equivariant if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. In other words, the isotropy subgroup G_x is included in the isotropy subgroup $G_{f(x)}$ for all $x \in X$. The map f is called isovariant if G_x is equal to $G_{f(x)}$ for all $x \in X$.*

Browder [Br] claimed the folowing:

Theorem (Browder). *Let M and N be closed, connected, G-oriented smooth G-manifolds. Assume that M satisfies the Gap Hypothesis. Then, any G-homotopy equivalence $f : M \to N$ is G-equivariantly homotopic to a G-isovariant homotopy equivalence f'. Moreover, if $M \times I$ satisfies the Gap Hypothesis, then the f' is unique up to G-homotopy.*

Here is an example, given by Browder, that illustrates the principal obstruction in deforming an equivariant map into an isovariant map:

Let G be a cyclic group of prime order, and let it act on the sphere S^q by rotation, with 2 fixed points 0 and ∞. Let $Y = S^k \times S^q$ where G acts trivially on the first coordinate S^k, thus the fixed point set is $Y^G = \left(S^k \times 0\right) \cup \left(S^k \times \infty\right)$. Let $X = \left(S^k \times S^q\right) \sharp_G G\left(S^k \times S^q\right)$, the equivariant connected sum of $Y = S^k \times S^q$ and $|G|$ copies of G-trivial $\left(S^k \times S^q\right)$ with G freely acting by circulating the $|G|$ copies, and the equivariant connected sum is made on a free orbit.

Define $f : X \to Y$ to be the identity on the first component $S^k \times S^q$, and via the composition of the projection $G\left(S^k \times S^q\right) \to GS^q$ and the canonical G-map $GS^q \to S^q$ on the second component of the equivariant connected sum.

By construction, f is a degree 1 equivariant map. But it is not an isovariant map, because the fixed point set X^G is just the "central" $(S^k \times 0)$ on the first component, thus $f^G : X^G \to Y^G$ is just the identity, but the free part of X is $X - X^G = S^k \times (S^{q-1} \times \mathbb{R}) \sharp_G G(S^k \times S^q)$, which contains all the S^q-cycles on the $|G|$ copies of $(S^k \times S^q)$. When mapped onto Y, this free part must intersect with the fixed-point set Y^G in Y, thus f could not be deformed in any way to an isovariant map.

Note that both X and Y satisfy the Gap Hypothesis if $q \geq k+2$, thus it is a serious obstruction in considering Browder's deformation of equivariant things into isovariant things. The Gap Hypothesis and degree 1 maps are not enough; being an equivariant homotopy equivalence is an essential condition, and so this is really a deep geometrical problem.

SECTION 3. THE METHODS OF SCHULTZ

Schultz [Sch] gave a proof of Browder's theorem under the additional assumption that the G-action is semi-free (that is, $M - M^G$ is G-free) everywhere. In the semi-free case, the only possible isotropy types are G-free and trivial types, so one can do the construction considering only those two distinct types. Thus, Schultz (and Dula and Schultz [DS]) constructed an obstruction theory in a form of equivariant cohomology, which they called "diagram cohomology", of triads of the form (manifold; regular neighborhood of the fixed-point set, and the free-part).

Since the fixed point sets $N^G = \coprod_\alpha N_\alpha$ and $M^G = \coprod_\alpha M_\alpha$ with $M_\alpha = f^{-1}(N_\alpha) \cap M^G$ is in one-to-one correspondence component-wise, one can first deform f inside the regular neighborhood of each of the components M_α of the fixed-point set. The normal bundles of M_α and N_α are stably fiber homotopy equivalent, but thanks to the Gap Hypothesis, it is unstably fiber homotopy equivalent. Therefore, it is possible to deform f to be isovariant in the regular neighborhood of M_α for each α, by using standard construction.

Next one pushes down the non-isovariant points into the system of tubular neighborhoods of M_α. That is, deform the map f so that any non-isovariant point is contained in a closed tubular neighborhood W_α of M_α for some α. (See Proposition 4.2 of [Sch].) Here, the deformation is done via the "diagram cohomology" obstruction theory. One notes that the map $f : X \to Y$ in the example of the previous section cannot be deformed this way, since the "diagram cohomology" detects its non-trivial obstruction.

Finally, one deforms the result map into a G-isovariant map. Again, one uses the "diagram cohomology" to detect the deformation obstruction. First, one uses G-transversality (due to the Gap Hypothesis) to construct appropriate "diagram maps" that have necessary local isovariancy properties (which they call "almost isovariant maps,") and then apply the "diagram cohomology" obstruction theory to see that the obstruction vanishes, producing the desired deformation, to get a global G-isovariant map. (See Proposition 5.3 of [Sch].)

Schultz has successfully built an appropriate obstruction theory just enough for proving the theorem in the semi-free case. As he remarks in the last section in his paper, he seems to be interested in applying the obstruction theory to situations where the Gap Hypothesis fails, and to build a new framework of applications of equivariant homotopy theory into equivariant surgery. However, in non-semi-free cases, the "diagram cohomology" obstruction theory (of [DS]) does not seem to be directly applicable, and things seem to be much complicated if one pursues to reduce them into algebraic topology methods. So, here we try to consider a different direction, that is, to look into more naive geometric methods, to reduce things into the deep theories of equivariant surgery.

However, a more generalized version of obstruction theory is still needed, and so we first work out a new form of "diagram cohomology" in the style of Dula and Schultz [DS].

Claim. *The diagram cohomology obstruction theory of Dula and Schultz can be directly generalized to non-semi-free actions of metacyclic groups. In particular, Theorem 4.5 of* [DS] *still holds for an arbitrary action of any metacyclic group.*

In order to prove this, we go back to Serre-type spectral sequence of Bredon cohomology with twisted coefficients, as developed by J. M. Møller [Mo] and I. Moerduk and J.-A. Svensson [MoS]. Working parrarel to Dula and Schultz for such group actions using Bredon cohomology with twisted coefficients, Dula and Schultz' arguments can be directly generalized to our cases, too, and Theorem 4.5 of [DS] can be proved in such cases, providing recognition principle for a diagram map to produce an isovariant map. We will discuss further details elsewhere.

SECTION 3. EQUIVARIANT SURGERY

First we make use of the following theorem of W. Lück:

Theorem (Lück). *Let M and N be smooth G-manifolds with codimension ≥ 3 gaps, $f : M \to N$ a G-homotopy equivalence, and $x \in M^G$. Then, the tangent representation at $x \in M$ is G-homotopy equivalent to that of $f(x) \in N$.*

Therefore, under our Gap Hypothesis, the equivariant normal bundles of the fixed-point sets are G-homotopy invariant between M and N. We would like to construct an equivariant unstable fiber homotopy equivalence between the regular neighborhoods of the fixed-points sets, and so we rely on the following classic theorem of C. T. C. Wall ([W], §11 and §12) :

Codimension 1 Embedding Theorem (Wall). *Let M and N be smooth G-manifolds with the Gap Hypothesis, and $f : M \to N$ a G-homotopy equivalence. Assume that N is divided into G-submanifolds $N = N_1 \cup N_2$ such that $N_0 = N_1 \cap N_2 = \partial N_1 = \partial N_2$ and $\pi_1 N_0 \cong \pi_1 N_1$. Assume further that N_0 is in the G-free part $N - \mathcal{S}N$, where $\mathcal{S}N = \bigcup_{H \neq e} N^H$. Then, f is G-homotopic to a map f' such that $M_i = f'^{-1}(N_i)$ is G-homotopy equivalent to N_i, respectively for $i = 0, 1, 2$, via the map f'.*

Making use of it, we can deform the G-homotopy equivalence between the normal bundles of the fixed-point sets into an (unstable) fiber homotopy equivalence between the regular neighborhoods. Thus far, the argument is similar to the one explained in Schultz' paper [Sch].

In order to approach toward the proof of Browder's theorem, we proceed inductively on the system of isotropy types. For now, we start by assuming that the theorem is true over $\mathcal{S}M$.

So, we assume that $f : M \to N$ a G-homotopy equivalence such that $f|_{\partial M}$ is already an isovariant homotopy equivalence. We need to deform f (by G-homotopy) relative to ∂M into a G-isovariant map.

Let U be a regular neighborhood of $\mathcal{S}N$ in N. $N - \partial N$ is G-free, and $f^{-1}(N - U) \subset M = \partial M$ by assumption. Now let $N_1 = \overline{U}$ and $N_2 = \overline{N - U}$, which readily satisfies the assumptions in the Codimension 1 Embedding Theorem because $f|_{\partial M}$ is assumed to be an isovariant homotopy equivalence.

Now apply the Codimension 1 Embedding Theorem to deform the map to get a thickening (in the line of the argument of §11 of Wall's book [W])

$$M = V \cup M_2 \longrightarrow U \cup N_2 = N$$

where $V \longrightarrow U$ is a G-homotopy equivalence, and V is G-h-cobordant to the regular neighborhood W of $\mathcal{S}M$.

We have now "divided" the manifolds into the "interior" and the "exterior" of the regular neighborhoods of $\mathcal{S}M$ and $\mathcal{S}N$ respectively.

Note that the argument is still similar to Schultz' paper [Sch]. He has also divided things to "interior" (good neighborhood of the singular set) and "exterior" (free part on the target manifold, where the map may go non-isovariant). From here, Schultz goes ahead to construct an obstruction theory to handle the deformation obstruction of the "exterior" relative to the "interior". We would like to go from here toward the equivariant surgery methods, to avoid a much complicated algebraic system in the non-semi-free case.

Since the regular neighborhoods are (unstably) G-fiber homotopy equivalent to each other, the proof could be completed once we could perform an equivariant surgery process to deform the G-homotopy equivalence $f|_{\partial W}$ into a G-homotopy equivalence $f|\partial V$.

That last process could be reduced to the π-π Theorem in the equivariant surgery. We now rely on the arguments of §13.2 of Weinberger's book [We 1]. Assuming some variant of the Gap Hypothesis, Weinberger has established a form of the equivariant surgery exact sequence. (See §13.2 of [We 1], p.225):

Equivariant Surgery Exact Sequence. *Suppose that G is a finite group acting orientation preservingly an a (topological) manifold M with small gaps and with all fixed point sets locally flat submanifolds. Suppose also that all fixed sets have dimension at least five. Then we have a long exact surgery sequence for isovariant structure sets.*

We could follow Weinberger's techniques, to perform equivariant surgery to deform the G-homotopy equivalence $f|_{\partial W}$ into a G-homotopy equivalence $f|_{\partial V}$. However, in the non-semi-free situation, the deformation must be done relative to the system of pieces of neighborhoods of the isotropy sets that are already deformed to be isovariant. So, we need to rely on some kind of "stratification" of such pieces of isotropy set neighborhoods.

Since we have assumed the Gap Hypothesis, those pieces can be assumed to be in the general position, and thus the stratified surgery can be applied. We use the following form of the π-π Theorem. (See Section 7.1 of [We 1]):

Stratified π-π Theorem. *Suppose (Y, X) is a strongly stratified pair, $X = \partial Y$, and each pure stratum of Y touches exactly one stratum of X for which the inclusion is a 1-equivalence. If all strata of X are of dimension ≥ 5, then any normal invariant of $(W, V) \to (Y, X)$ can be surged into a simple homotopy equivalence.*

Since our Gap Hypothesis is stronger than the condition needed here, our general position situation is enough to apply the Stratified π-π Theorem to our stratified data, we can surger the data to construct a K-homotopy equivalence. However, in order to get an equivariant homotopy equivalence map in the global level, we still need a destabilization obstruction, as explained in Section 6.2 of [We 1]:

$$\mathcal{S}(X) \longrightarrow \mathcal{S}^{-\infty}(X) \longrightarrow \hat{H}\left(\mathbb{Z}/2 : \mathrm{Wh}^{\mathrm{Top}}(X)\right)$$

where the latter term is 2-torsion only. Thus, the surgery can be done up to 2-torsion. This provides the desired deformation, at least up to 2-torsion.

In order to handle the 2-torsion obstruction, we probably need to make use of the Nil arguments of Cappell and Weinberger (see §14.2 of [We 1]), which was originally invented by Cappell in order to deepen Wall's submanifold embedding theorems.

In the case of actions of metacyclic groups, those obstructions can be reduced to certain explicit construction built upon the diagram cohomology obstructions discussed at the end of Section 2, and can be used to show that the desired deformation is possible.

The L-group term in the equivariant surgery exact sequence consists of the hierarchical strata-wise L-group classes, each of which is interpreted (by the original realization theorem of C. T. C. Wall ([W], Section3)) as appropriate classes of equivariant normal maps. They were computed by various people in various situation, including Madsen-Rothenberg ([MR 2]), Cappell-Weinberger-Yan ([CWY]) and Weinberger-Yan ([WY 2]). In our case, since we have started with a G-homotopy equivalence, we could be successful in reducing the surgery obstruction into the π-π Theorem situation, at least up to 2-torsion, as above.

In this way, reducing the deformation construction into the stratified π-π Theorem seems to work in the general non-semi-free case. Unlike Schultz's methods, it really depends on the deep geometric results of equivariant surgery theories, but on the other hand, it may open up a deeper geometric understanding on the properties of isovariant homotopy equivalences, so we hope to work further in this direction. We hope to provide more details to this generality in a future work.

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