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Fixed Point Theorems and Duality Theorems for Nonlinear Operators in Banach Spaces

高橋 渉 (Wataru Takahashi)
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology

Abstract

In this article, we first define nonlinear operators which are connected with resolvents of maximal monotone operators in Banach spaces and then prove fixed point theorems for the nonlinear operators in smooth strictly convex and reflexive Banach spaces. Further, we prove duality theorems for two nonlinear mappings in Banach spaces, i.e., a relatively nonexpansive mapping and a generalized nonexpansive mapping. Finally, motivated by such duality theorems, we define nonlinear operators in Banach spaces which are connected with the conditional expectations in the probability theory. Then, we obtain orthgonal properties for the nonlinear operators.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\|Tx-Ty\|\leq\|x-y\|$ for all $x, y\in C$. We denote by $F(T)$ the set of fixed points of T. Let $g: H\rightarrow(-\infty, \infty]$ be a proper convex lower semicontinuous function and consider the convex minimization problem:

$$\min\{g(x):x\in H\}. \tag{1.1}$$

For such g, we can define a multivalued operator ∂g on H by

$$\partial g(x)=\{x^{*}\in H:g(y)\geq g(x)+\langle x^{*}, y-x\rangle, y\in H\}$$

for all $x\in H$. Such ∂g is said to be the subdifferential of g. A multivalued operator $A\subset H\times H$ is called monotone if for $(x_{1}, y_{1}), (x_{2}, y_{2})\in A$,

$$\langle x_{1}-x_{2}, y_{1}-y_{2}\rangle\geq 0.$$

A monotone operator $A\subset H\times H$ is called maximal if its graph

$$G(A)=\{(x, y):y\in Ax\}$$

is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $R(I+\lambda A) = H$ for all $\lambda > 0$. A monotone operator A is also called m-accretive if $R(I+\lambda A) = H$ for all $\lambda > 0$. So, we can define, for each $\lambda > 0$, the resolvent $J_\lambda : R(I+\lambda A) \to D(A)$ by $J_\lambda = (I+\lambda A)^{-1}$. We know that J_λ is a nonexpansive mapping and for any $\lambda > 0$, $F(J_\lambda) = A^{-1}0$, where $A^{-1}0 = \{z \in H : 0 \in Az\}$.

Let E be a smooth Banach space and let E^* be the dual space of E. The function $\phi : E \times E \to \mathbf{R}$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$, where J is the normalized duality mapping from E into E^*. Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [36] if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. Further, a point p in C is said to be a generalized asymptotic fixed point of T [13] if C contains a sequence $\{x_n\}$ such that $\{Jx_n\}$ converges to Jp in the weak* topology and $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. The set of generalized asymptotic fixed points of T is denoted by $\check{F}(T)$. A mapping $T : C \to C$ is called relatively nonexpansive [29] if $\hat{F}(T) = F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x)$$

for all $x \in C$ and $p \in F(T)$. Further, a mapping $T : C \to C$ is called generalized nonexpansive [9, 10] if $F(T) \neq \emptyset$ and

$$\phi(Tx, p) \leq \phi(x, p)$$

for all $x \in C$ and $p \in F(T)$. The class of relatively nonexpansive mappings and the class of generalized nonexpansive mappings contain the class of nonexpansive mappings T in Hilbert spaces with $F(T) \neq \emptyset$.

In this article, motivated by two nonlinear operators of a relatively nonexpansive mapping and a generalized nonexpansive mapping, we first define nonlinear operators which are connected with a relatively nonexpansive mapping and a generalized nonexpansive mapping. Then, we prove fixed point theorems for the nonlinear operators in smooth strictly convex and reflexive Banach spaces. Further, we prove duality theorems for two nonlinear mappings in Banach spaces, i.e., a relatively nonexpansive mapping and a generalized nonexpansive mapping. Finally, motivated by such duality theorems, we define nonlinear operators in Banach spaces which are connected with the conditional expectations in the probability theory. Then, we obtain orthgonal properties for the nonlinear operators.

2 Preliminaries

Throughout this paper, we assume that a Banach space E with the dual space E^* is real. We denote by $\mathbf{N}$ and $\mathbf{R}$ the sets of all positive integers and all real numbers, respectively. We also denote by $\langle x, x^*\rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be strictly convex if $\|x+y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A Banach space E is said to be uniformly convex if for any sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$, $\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Moreover, E is said to have a Fréchet differentiable norm if for each $x \in E$ with $\|x\| = 1$, this limit is attained uniformly for $y \in E$ with $\|y\| = 1$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in E$ with $\|y\| = 1$, this limit is attained uniformly for $x \in E$ with $\|x\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The multivalued operator $J : E \to E^*$ is called the normalized duality mapping of E. From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued. If E is strictly convex, then J is one-to-one, i.e., $x \neq y \Rightarrow J(x) \cap J(y) = \emptyset$. If E is reflexive, then J is a mapping of E onto E^*. So, if E is reflexive, strictly convex and smooth, then J is single-valued, one-to-one and onto. In this case, the normalized duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$. If E has a Fréchet differentiable norm, then J is norm to norm continuous. If E has a uniformly Gâteaux differentiable norm, then J is norm to weak* uniformly continuous on each bounded subset of E; see [43] for more details. Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbf{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. We also define the function $\phi_* : E^* \times E^* \to \mathbf{R}$ by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J^{-1}y^* \rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$. It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \geq 0$ for all $x, y \in E$. We also know the following:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.1}$$

for all $x, y, z \in E$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.2}$$

for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y. \tag{2.3}$$

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\}$$

is always nonempty and a singletone. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e.,

$$\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x)$$

for every $x \in E$. Such Π_C is called the generalized projection of E onto C; see Alber [1]. The following lemma is due to Alber [1] and Kamimura and Takahashi [20].

Lemma 2.1 ([1, 20]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, the following hold:*

(a) $z=\Pi_{C}x$ *if and only if* $\langle y-z, Jx-Jz\rangle \leq 0$ *for all* $y\in C$;
(b) $\phi(z, \Pi_{C}x)+\phi(\Pi_{C}x, x)\leq\phi(z, x)$.

From this lemma, we can prove the following lemma.

Lemma 2.2. *Let M be a nonempty closed linear subspace of a smooth, strictly convex and reflexive Banach space E and let $(x, z)\in E\times M$ Then, $z=\Pi_{M}x$ if and only if*

$$\langle J(x)-J(z), m\rangle=0 \text{ for all } m\in M.$$

Let C be a nonempty subset of E and let R be a mapping from E onto C. Then R is said to be a retraction if $R^{2}=R$. It is known that if R is a retraction from E onto C, then $F(R)=C$. The mapping R is also said to be sunny if $R(Rx+t(x-Rx))=Rx$ whenever $x\in E$ and $t\geq 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas were proved by Ibaraki and Takahashi [10].

Lemma 2.3 ([10]). *Let C be a nonempty closed subset of of a smooth and strictly convex Banach space E and let R be a retraction from E onto C. Then, the following are equivalent:*

(a) R is sunny and generalized nonexpansive;
(b) $\langle x-Rx, Jy-JRx\rangle\leq 0$ *for all* $(x, y)\in E\times C$.

Lemma 2.4 ([10]). *Let C be a nonempty closed sunny and generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([10]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z)\in E\times C$. Then, the following hold:*

(a) $z=Rx$ *if and only if* $\langle x-z, Jy-Jz\rangle\leq 0$ *for all* $y\in C$;
(b) $\phi(Rx, z)+\phi(x, Rx)\leq\phi(x, z)$.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z\in C:\|z-x\|=\min_{y\in C}\|y-x\|\}$$

is always nonempty and a singletone. Let us define the mapping P_{C} of E onto C by $z=P_{C}x$ for every $x\in E$, i.e.,

$$\|P_{C}x-x\|=\min_{y\in C}\|y-x\|$$

for every $x\in E$. Such P_{C} is called the metric projection of E onto C; see [43]. The following lemma is in [43].

Lemma 2.6 ([43]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z)\in E\times C$. Then, $z=P_{C}x$ if and only if $\langle y-z, J(x-z)\rangle\leq 0$ for all $y\in C$.*

An operator $A\subset E\times E^{*}$ with domain $D(A)=\{x\in E:Ax\neq\emptyset\}$ and range $R(A)=\cup\{Ax:x\in D(A)\}$ is said to be monotone if $\langle x-y, x^{*}-y^{*}\rangle\geq 0$ for any $(x, x^{*}), (y, y^{*})\in A$. An operator A is said to be strictly monotone if $\langle x-y, x^{*}-y^{*}\rangle>0$ for any $(x, x^{*}), (y, y^{*})\in A$ $(x\neq y)$.

A monotone operator A is said to be maximal if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the set $A^{-1}0 = \{u \in E : 0 \in Au\}$ is closed and convex (see [44] for more details). Let J be the normalized duality mapping from E into E^*. Then, J is monotone. If E is strictly convex, then J is one to one and strictly monotone. The following theorems are well-known; for instance, see [43].

Theorem 2.7. *Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$. Further, if $R(J + A) = E^*$, then $R(J + rA) = E^*$ for all $r > 0$.*

Theorem 2.8. *Let E be a strictly convex and smooth Banach space and let $x, y \in E$. If $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

3 Nonlinear Mappings and Fixed Point Theorems

Let E be a Banach space and let C be a nonempty closed convex subset of E. Then, C has normal structure if for each bounded closed convex subset of K of C which contains at least two points, there exists an element x of K which is not a diametral point of K, i.e.,

$$\sup\{\|x - y\| : y \in K\} < \delta(K),$$

where $\delta(K)$ is the diameter of K. The following Kirk fixed point theorem [22] for nonexpansive mappings in a Banach space is well-known; see also Takahashi [43].

Theorem 3.1 (Kirk [22]). *Let E be a reflexive Banach space and Let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then, T has a fixed point in C.*

Recently, Kohsaka and Takahashi [27], and Ibaraki and Takahashi [15] proved fixed point theorems for nonlinear mappings which are connected with resolvents of maximal monotone oprators in Banach spaces. Before stating them, we give two nonlinear mappings in Banach spaces. Let E be a smooth Banach space and let C be a closed convex subset of E. Then, $T : C \to C$ is of firmly nonexpansive type [27] if for all $x, y \in E$,

$$\langle Tx - Ty, JTx - JTy\rangle \leq \langle Tx - Ty, Jx - Jy\rangle.$$

This means that for $x, y \in C$,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) - \phi(Tx, x) - \phi(Ty, y).$$

Let us give two examples of such mappings. Let E be a Banach space and let C be a closed convex subset of E. Let $f : C \times C \to \mathbf{R}$ be a bifunction satisfying the following conditions:

(F1) $f(x, x) = 0$ for all $x \in C$;
(F2) $f(x, y) \leq -f(y, x)$ for all $x, y \in C$;
(F3) $f(x, \cdot)$ is lower semicontinuous and convex for all $x \in C$;
(F4) $\lim_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$ for all $x, y, z \in C$.

Theorem 3.2 (Blum and Oettli [2]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex subset of E. Let $f : C \times C \to \mathbf{R}$ be a bifunction satisfying (F1)–(F4). Then, for $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in C.$$

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a closed convex subset of E. Let $f : C \times C \to \mathbf{R}$ be a bifunction satisfying (F1)–(F4). For any $r > 0$ and $x \in E$, define the mapping $T_r : E \to C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in C \right\}.$$

Then, T_r satisfies the following condition: for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle.$$

That is, the mapping T_r is of firmly nonexpansive type. In more general, let E be a smooth, strictly convex and reflexive Banach space and let $A : E \times E^*$ be a maximal monotone operator. Define the mapping $T : E \to E$ as follows: For any $r > 0$ and $x \in E$,

$$T_r x = (J + rA)^{-1} Jx,$$

where J is the duality mapping of E. Then, T_r satisfies the following:

$$\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \leq \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y).$$

That is, the mapping T_r is of firmly nonexpansive type.

Theorem 3.3 (Kohsaka and Takahashi [27]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let T be a firmly nonexpansive type mapping of C into itself. Then, the following are equivalent:*

(1) There exists $x \in C$ such that $\{T^n x\}$ is bounded;
(2) $F(T)$ is nonempty.

Motivated by the mapping of firmly nonexpansive type, Ibaraki and Takahashi [15] also defined the following mapping: Let E be a smooth Banach space and let C be a closed convex subset of E. Then, $T : C \to C$ is of firmly generalized nonexpansive type [15] if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx) - \phi(x, Tx) - \phi(y, Ty), \ \forall x, y \in C.$$

Let us give an example of such a mapping. If $B \subset E^* \times E$ is a maximal monotone mapping with domain $D(B)$ and range $R(B)$, then for $\lambda > 0$ and $x \in E^*$, we can define the resolvent $J_\lambda x$ of B as follows:

$$J_\lambda x = \{y \in E : x \in y + \lambda BJy\}.$$

We know from Ibarak and Takahashi [10] that $J_\lambda : E \to E$ is a single valued mapping. So, we call J_λ the generalized resolvent of B for $\lambda > 0$. We also denote the resolvent J_λ by

$$J_\lambda = (I + \lambda BJ)^{-1}.$$

We know that $D(J_\lambda) = R(I + \lambda BJ)$ and $R(J_\lambda) = D(BJ)$, and J_λ is of firmly generalized nonexpansive type; see [15].

Theorem 3.4 (Ibaraki and Takahashi [15]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a firmly generalized nonexpansive type mapping of E into itself. Then, the following are equivalent:*

(1) There exists $x \in E$ such that $\{T^n x\}$ is bounded;
(2) $F(T)$ is nonempty.

4 Duality theorems for Nonlinear Mappings

Let E be a Banach space. Let C be a nonempty closed convex subset of E and let C be a mapping of C into itself. Then, a point p in C is said to be an asymptotic fixed point of T [36] if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. Further, a point p in C is said to be a generalized asymptotic fixed point of T [13] if C contains a sequence $\{x_n\}$ such that $\{Jx_n\}$ converges to Jp in the weak* topology and $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. The set of generalized asymptotic fixed points of T is denoted by $\check{F}(T)$. A mapping $T : C \to C$ is called relatively nonexpansive [29] if $\hat{F}(T) = F(T) \neq \emptyset$ and

$$V(p, Tx) \leq V(p, x)$$

for each $x \in C$ and $p \in F(T)$. Further, a mapping $T : C \to C$ is called generalized nonexpansive [9, 10] if $F(T) \neq \emptyset$ and

$$V(Tx, p) \leq V(x, p)$$

for each $x \in C$ and $p \in F(T)$. Let E be a reflexive, smooth and strictly convex Banach space, let J be the duality mapping from E into E^* and let T be a mapping from E into itself. In this section, we study the mapping T^* from E^* into itself defined by

$$T^* x^* := JTJ^{-1}x^* \tag{4.1}$$

for each $x^* \in E^*$. We first prove the following theorem for such mappings in a Banach space.

Theorem 4.1 ([16]). *Let E be a reflexive, smooth and strictly convex Banach space, let J be the duality mapping of E into E^* and let T be a mapping of E into itself. Let T^* be a mapping defined by* (4.1)*. Then the following hold:*

(1) $JF(T) = F(T^*)$;
(2) $J\hat{F}(T) = \check{F}(T^*)$;
(2) $J\check{F}(T) = \hat{F}(T^*)$.

For instance, let us show that $JF(T) = F(T^*)$. In fact, we have that

$$\begin{aligned} x^* \in JF(T) &\Leftrightarrow J^{-1}x^* \in F(T) \\ &\Leftrightarrow TJ^{-1}x^* = J^{-1}x^* \\ &\Leftrightarrow JTJ^{-1}x^* = JJ^{-1}x^* \\ &\Leftrightarrow T^*x^* = x^* \\ &\Leftrightarrow x^* \in F(T^*). \end{aligned}$$

This implies that $JF(T) = F(T^*)$.

Let E be a reflexive, smooth and strictly convex Banach space with its dual E^* and let J be the duality mapping from E into E^*. We consider a mapping $\phi_* : E^* \times E^* \to \mathbf{R}$ defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J_* y^*\rangle + \|y^*\|^2$$

for each $x^*, y^* \in E^*$, where J_* is the duality mapping on E^*. From the properties of J, we know that

$$\phi_*(x^*, y^*) = \phi(J^{-1}y^*, J^{-1}x^*) \tag{4.2}$$

for each $x^*, y^* \in E^*$. In fact, we have that

$$\begin{aligned}\phi_*(x^*, y^*) &= \|x^*\|^2 - 2\langle x^*, J_* y^*\rangle + \|y^*\|^2 \\ &= \|JJ^{-1}x^*\|^2 - 2\langle JJ^{-1}x^*, J^{-1}y^*\rangle + \|JJ^{-1}y^*\|^2 \\ &= \|J^{-1}x^*\|^2 - 2\langle JJ^{-1}x^*, J^{-1}y^*\rangle + \|J^{-1}y^*\|^2 \\ &= \phi(J^{-1}y^*, J^{-1}x^*)\end{aligned}$$

for each $x^*, y^* \in E^*$.

Now, we prove the following two theorems for relatively nonexpansive mappings and generalized nonexpansive mappings in a Banach space.

Theorem 4.2 (**[16]**). *Let E be a reflexive, smooth, and strictly convex Banach space, let J be the duality mapping from E into E^* and let T be a relatively nonexpansive mapping form E into itself. Let T^* be a mapping defined by* (4.1). *Then T^* is generalized nonexpansive and $\check{F}(T^*) = F(T^*)$.*

Proof. Since T is relatively nonexpansive, we have that $\hat{F}(T) = F(T) \neq \emptyset$. By Lemma 4.1, we obtain that

$$\check{F}(T^*) = J\hat{F}(T) = JF(T) = F(T^*) \neq \emptyset.$$

Let $x^* \in E^*$ and let $p^* \in F(T^*)$. Then $J^{-1}p^* \in F(T)$. From (4.2), we have that

$$\begin{aligned}\phi_*(T^*x^*, p^*) &= \phi(J^{-1}p^*, J^{-1}T^*x^*) \\ &= \phi(J^{-1}p^*, J^{-1}JTJ^{-1}x^*) \\ &= \phi(J^{-1}p^*, TJ^{-1}x^*) \\ &\leq \phi(J^{-1}p^*, J^{-1}x^*) = \phi_*(x^*, p^*).\end{aligned}$$

This completes the proof. □

Theorem 4.3 (**[16]**). *Let E be a reflexive, smooth, and strictly convex Banach space, let J be the duality mapping from E into E^* and let T be a generalized nonexpansive mapping form E into itself with $\check{F}(T) = F(T)$. Let T^* be a mapping defined by* (4.1). *Then T^* is relatively nonexpansive.*

Proof. From the assumption of $\check{F}(T) = F(T) \neq \emptyset$ and Lemma 4.1, we obtain that

$$\hat{F}(T^*) = J\check{F}(T) = JF(T) = F(T^*) \neq \emptyset.$$

Let $x^* \in E^*$ and let $p^* \in F(T^*)$. Then $J^{-1}p^* \in F(T)$. From (4.2), we have that

$$\begin{aligned}\phi_*(p^*, T^*x^*) &= \phi(J^{-1}T^*x^*, J^{-1}p^*) \\ &= \phi(J^{-1}JTJ^{-1}x^*, J^{-1}p^*) \\ &= \phi(TJ^{-1}x^*, J^{-1}p^*) \\ &\leq \phi(J^{-1}x^*, J^{-1}p^*) = \phi_*(p^*, x^*).\end{aligned}$$

This completes the proof. □

5 Generalized Conditional Expectations

In this section, we start with two theorems proved by Kohsaka and Takahashi [26] which are connected with generalized nonexpansive mappings in Banach spaces.

Theorem 5.1 ([26]). *Let E be a smooth, strictly convex and reflexive Banach space, let C_* be a nonempty closed convex subset of E^* and let Π_{C_*} be the generalized projection of E^* onto C_*. Then the mapping R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$.*

Theorem 5.2 ([26]). *Let E be a smooth, reflexive and strictly convex Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent.*

(1) D is a sunny generalized nonexpansive retract of E;
(2) D is a generalized nonexpansive retract of E;
(3) JD is closed and convex.

In this case, D is closed.

Motivated by these theorems, we define the following nonlinear operator: Let E be a reflexive, strictly convex and smooth Banach space and let J be the normalized duality mapping from E onto E^*. Let Y^* be a closed linear subspace of the dual space E^* of E. Then, the generalized conditional expectation E_{Y^*} with respect to Y^* is defined as follows:

$$E_{Y^*} := J^{-1}\Pi_{Y^*}J,$$

where Π_{Y^*} is the generalized projection from E^* onto Y^*.

Let E be a normed linear space and let $x, y \in E$. We say that x is orthogonal to y in the sense of Birkhoff-James (or simply, x is BJ-orthogonal to y), denoted by $x \perp y$ if

$$\|x\| \leq \|x + \lambda y\|$$

for all $\lambda \in \mathbf{R}$. We know that for $x, y \in E$, $x \perp y$ if and only if there exists $f \in J(x)$ with $\langle y, f \rangle = 0$; see [43]. In general, $x \perp y$ does not imply $y \perp x$. An operator T of E into itself is called left-orthogonal (resp. right-orthogonal) if for each $x \in E$, $Tx \perp (x - Tx)$ (resp. $(x - Tx) \perp Tx$).

Lemma 5.3. *Let E be a normed linear space and let T be an operator of E into itself such that*

$$T(Tx + \beta(x - Tx)) = Tx \tag{5.1}$$

for any $x \in E$ and $\beta \in \mathbf{R}$. Then, the following conditions are equivalent:

(1) $\|Tx\| \leq \|x\|$ for all $x \in E$;
(2) T is left-orthogonal.

Proof. We prove (1) $\Rightarrow$ (2). Since $T(Tx + \beta(x - Tx)) = Tx$ for all $x \in E$ and $\beta \in \mathbf{R}$, we have

$$\begin{aligned}\|Tx\| &= \|T(Tx + \beta(x - Tx))\| \\ &\leq \|Tx + \beta(x - Tx)\|\end{aligned}$$

for any $x \in E$ and $\beta \in \mathbf{R}$. This implies that for each $x \in E$, $Tx \perp (x - Tx)$. Next, we prove (2) $\Rightarrow$ (1). Since T is left-orthogonal, we have

$$\|Tx\| \leq \|Tx + \lambda(x - Tx)\|$$

for any $x \in E$ and $\lambda \in \mathbf{R}$. When $\lambda = 1$, we obtain $\|Tx\| \leq \|x\|$. This completes the proof. □

Using Lemma 5.3, we prove the following theorem.

Theorem 5.4 **([8]).** *Let E be a reflexive, strictly convex and smooth Banach space. Let Y^* be a closed linear subspace of the dual space E^*. Then, the generalized conditional expectation E_{Y^*} with respect to Y^* is left-orthogonal, i.e., for any $x \in E$,*

$$E_{Y^*}x \perp (x - E_{Y^*}x).$$

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

Theorem 5.5 **([8]).** *Let E be a reflexive, strictly convex and smooth Banach space and let I be the identity operator of E into itself. Let Y^* be a closed linear subspace of the dual space E^* and let E_{Y^*} be the generalized conditional expectation with respect to Y^*. Then, the mapping $I - E_{Y^*}$ is the metric projection of E onto $Y^*_\perp$. Conversely, let Y be a closed linear subspace of E and let P_Y be the metric projection of E onto Y. Then, the mapping $I - P_Y$ is the generalized conditional expectation $E_{Y^\perp}$ with respect to $Y^\perp$, i.e., $I - P_Y = E_{Y^\perp}$.*

Let E be a normed linear space and let $Y_1, Y_2 \subset E$ be closed linear subspaces. If $Y_1 \cap Y_2 = \{0\}$ and for any $x \in E$ there exists a unique pair $y_1 \in Y_1$ and $y_2 \in Y_2$ such that

$$x = y_1 + y_2,$$

and any element of Y_1 is BJ-orthogonal to any element of Y_2, i.e., $y_1 \perp y_2$ for any $y_1 \in Y_1$ and $y_2 \in Y_2$, then we represent the space E as

$$E = Y_1 \oplus Y_2 \text{ and } Y_1 \perp Y_2.$$

For an operator T of E into itself, the kernel of T is denoted by $\ker(T)$, i.e.,

$$\ker(T) = \{x \in E : Tx = 0\}.$$

Using Theorem 5.5, we have the following theorem.

Theorem 5.6 ([8]). *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^* of E such that for any $y_1, y_2 \in J^{-1}Y^*$, $y_1+y_2 \in J^{-1}Y^*$. Then, $J^{-1}Y^*$ is a closed linear subspace of E and the generalized conditional expectation E_{Y^*} with respect to Y^* is a norm one linear projection from E to $J^{-1}Y^*$. Further, the following hold:*

(1) $E = J^{-1}Y^* \oplus \ker(E_{Y^*})$ *and* $J^{-1}Y^* \perp \ker(E_{Y^*})$;
(2) $I - E_{Y^*}$ *is the metric projection of* E *onto* $\ker(E_{Y^*})$.

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