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A generalization of Hardy spaces on spaces of homogeneous type

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1. Introduction

This is an announcement of my recent work [10].

Let $X = (X, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss [1, 2] (see the next section for the definition). Using atoms, Coifman and Weiss [2] introduced the Hardy space $H^p(X)$. The purpose of this report is to generalize the definition of Hardy space $H^p(X)$ and prove that the generalized Hardy spaces have the same property as $H^p(X)$. Our definition includes a kind of Hardy spaces with variable exponent. The results are new even for the $\mathbb{R}^n$ case.

First we state definitions of Campanato and Hölder spaces. Let $1 \le p < \infty$ and $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$. For a function $f \in L^1_{\mathrm{loc}}(X)$ and for a ball B, let $f_B = \mu(B)^{-1} \int_B f(x)\, d\mu(x)$. Then the Campanato spaces $\mathcal{L}_{p,\phi}(X)$ and the Hölder spaces $\Lambda_\phi(X)$ are defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_\phi} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \right)^{1/p},$$
$$\|f\|_{\Lambda_\phi} = \sup_{x,y \in X,\, x \ne y} \frac{2|f(x) - f(y)|}{\phi(x, d(x,y)) + \phi(y, d(y,x))}.$$

Let $\mathcal{C}$ be the space of all constant functions. Then $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$ and $\Lambda_\phi(X)/\mathcal{C}$ are Banach spaces with the norm $\|f\|_{\mathcal{L}_{p,\phi}}$ and $\|f\|_{\Lambda_\phi}$, respectively. Campanato spaces of these type were studied in [11, 7, 8, 12, 9]. See [9] for relations among these spaces. When $p = 1$, we denote $\mathcal{L}_{1,\phi}(X)$ by $\mathrm{BMO}_\phi(X)$. If $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(X) = \mathrm{BMO}(X)$.

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For $\phi(x,r)=r^{\alpha(x)}$, $\alpha(x)>0$, we denote $\Lambda_\phi(X)$ by $\mathrm{Lip}_{\alpha(\cdot)}(X)$. Then

$$\|f\|_{\mathrm{Lip}_{\alpha(\cdot)}}=\sup_{x,y\in X,\,x\neq y}\frac{2|f(x)-f(y)|}{d(x,y)^{\alpha(x)}+d(y,x)^{\alpha(y)}}.$$

If $\alpha(\cdot)$ satisfies a certain condition, then $\mathrm{Lip}_{\alpha(\cdot)}(X)=\mathcal{L}_{p,\phi}(X)$ for all $p\in[1,\infty)$.

Using atoms, Coifman and Weiss [2] defined the Hardy space $H^p(X)$ as a subspace of the dual of $\mathrm{Lip}_\alpha(X)$ and they proved that $\mathrm{Lip}_\alpha(X)$ is the dual of $H^p(X)$. Their results are generalization of the case $X=\mathbb{R}^n$. In [2] $\mathrm{Lip}_\alpha(X)$ was regarded as the space of functions modulo constants. Therefore, we denote by $(H^p(X))^*=\mathrm{Lip}_\alpha(X)/C$ the fact above.

In this report, using $[\phi,q]$-atoms, we define a generalized Hardy space $H_U^{[\phi,q]}(X)$ as a subspace of the dual of $\mathcal{L}_{q',\phi}(X)/C$ and prove that $\mathcal{L}_{q',\phi}(X)/C$ is the dual of $H_U^{[\phi,q]}(X)$, i.e. $\left(H_U^{[\phi,q]}(X)\right)^*=\mathcal{L}_{q',\phi}(X)/C$, where $1<q\le\infty$, $1/q+1/q'=1$, U is a concave strictly increasing function from $[0,\infty)$ to itself and $U(0)=0$ (see the third section for the precise definition of $H_U^{[\phi,q]}(X)$). The definition of $H^p(X)$ in [2], $0<p\le1$, is a special case of ours, since $\mathrm{Lip}_\alpha(X)$ is a special case of $\mathcal{L}_{q',\phi}(X)$.

Coifman and Weiss [2] first defined $H^{p,q}(X)$, and then proved $H^{p,q}(X)=H^{p,\infty}(X)$, which was denoted by $H^p(X)$. We will prove that $H_U^{[\phi,q]}(X)=H_U^{[\phi,\infty]}(X)$ under a certain condition. In particular, for Hardy spaces with variable exponent $p(x)$, we use the condition that $p(x)$ is log-Hölder continuous (see Corollary 4.2).

The log-Hölder continuity was used to prove boundedness of the Hardy-Littlewood maximal operator on $L^{p(x)}$, Lebesgue spaces with variable exponent, as follows.

Let $G\subset\mathbb{R}^n$ be bounded. For a function $p:G\to[1,\infty)$, let

$$L^{p(x)}(G)=\left\{f\in L^1(G):\int_G(c\,|f(x)|)^{p(x)}\,dx<\infty\ \text{ for some } c>0\right\}.$$

For $f\in L^{p(x)}(G)$, let

$$\|f\|_{p(x)}=\inf\left\{\lambda>0:\int_G\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx\le1\right\}.$$

Then $\|\cdot\|_{p(x)}$ is a norm and thereby $L^{p(x)}(G)$ is a Banach space. For a function f on G, the Hardy-Littlewood maximal function of f is defined by

$$Mf(x)=\sup_{B\ni x}\frac{1}{|B|}\int_{B\cap G}|f(y)|dy,$$

where the supremum is taken over all balls B containing x. By the definition we have

$$\|Mf\|_\infty\le\|f\|_\infty.$$

We say that $p(x)$ is log-Hölder continuous if

$$|p(x) - p(y)| \leq \frac{c}{|\log|x-y||} \quad \text{for } |x-y| \leq \frac{1}{2}.$$

Theorem 1.1 (Diening [3]). *If $p(x)$ is log-Hölder continuous, then the operator M is bounded on $L^{p(x)}(G)$.*

Remark 1.1. Let

$$p(x) = \begin{cases} 4 & (-1 < x \leq 0) \\ 2 & (0 < x < 1). \end{cases}$$

If $f(x) = \begin{cases} 0 & (-1 < x \leq 0) \\ x^{-1/3} & (0 < x < 1), \end{cases}$ then $Mf(x) \geq c|x|^{-1/3}$. In this case $f \in L^{p(x)}(-1,1)$ and $Mf \notin L^{p(x)}(-1,1)$.

2. Space of homogeneous type

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a nonnegative measure μ such that

$$d(x,y) \geq 0 \quad \text{and} \quad d(x,y) = 0 \text{ if and only if } x = y,$$

$$d(x,y) = d(y,x),$$

$$d(x,y) \leq K_1\,(d(x,z) + d(z,y)), \tag{2.1}$$

the balls (d-balls) $B(x,r) = B^d(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x,2r)) \leq K_2\,\mu(B(x,r)) < \infty, \tag{2.2}$$

If there are constants θ $(0 < \theta \leq 1)$ and $K_3 \geq 1$ such that

$$|d(x,z) - d(y,z)| \leq K_3\,(d(x,z) + d(y,z))^{1-\theta} d(x,y)^{\theta}, \qquad x, y, z \in X, \tag{2.3}$$

then the balls are open sets. Note that (2.1) for some $K_1 \geq 1$ follows from (2.3) (Lemarié [4]). Conversely, from (2.1) it follows that there exist $\theta > 0$, $K_3 \geq 1$ and a quasi-distance which is equivalent to the original d such that (2.3) holds (Macías and Segovia [5]). Therefore We always assume (2.3) in this report.

It is known that, if $\mu(X) < +\infty$, then there is a constant $R_0 > 0$ such that

$$X = B(x, R_0) \quad \text{for all } x \in X \tag{2.4}$$

(see [12, Lemma 5.1]).

3. Definitions

Definition 3.1 ($[\phi, q]$-atom (resp. $(p(\cdot), q)$-atom)). Let $\phi : X \times (0, \infty) \to (0, \infty)$ and $1 < q \le \infty$. A function a on X is called a $[\phi, q]$-atom (resp. $(p(\cdot), q)$-atom) if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$,

(ii) $\|a\|_q \le \dfrac{1}{\mu(B)^{1/q'}\phi(B)}$

$\big($resp. $\|a\|_q \le \mu(B)^{1/q-1/p(x)}$, where x is the center of $B\big)$,

(iii) $\displaystyle\int_X a(x)\, d\mu(x) = 0$,

where $\|a\|_q$ is the L^q norm of a and $1/q + 1/q' = 1$. We denote by $A[\phi, q]$ the set of all $[\phi, q]$-atoms. (We denote by $A(p(\cdot), q)$ the set of all $(p(\cdot), q)$-atoms.)

We note that $(p(\cdot), q)$-atoms are special cases of $[\phi, q]$-atoms. If $p(x) \equiv p$, then the $(p(\cdot), q)$-atom is the usual (p, q)-atom. Let $p_- = \inf p(x)$ and $p_+ = \sup p(x)$.

Remark 3.1. Assume that $\mu(B(x, r)) \sim r^Q$ $(Q > 0)$ for $x \in X$ and $0 < r < \infty$ $(0 < r < R_0$ if $\mu(X) < \infty)$. Let $\alpha(x) = Q(1/p(x) - 1)$. If $Q/(\theta + Q) \le p_- \le p_+ < 1$, then $0 < \alpha_- \le \alpha_+ \le \theta$ and $\mathrm{Lip}_{\alpha(\cdot)}(X) = \mathcal{L}_{q',\phi}(X)$ for all $q' \in [1, \infty)$.

If a is a $[\phi, q]$-atom and a ball B satisfies (i)–(iii), then

$$\begin{aligned} \left|\int_X a(x)g(x)\, d\mu(x)\right| &= \left|\int_B a(x)(g(x) - g_B)\, d\mu(x)\right| \\ &\le \|a\|_q \left(\int_B |g(x) - g_B|^{q'}\, d\mu(x)\right)^{1/q'} \\ &\le \frac{1}{\phi(B)}\left(\frac{1}{\mu(B)}\int_B |g(x) - g_B|^{q'}\, d\mu(x)\right)^{1/q'} \\ &\le \|g\|_{\mathcal{L}_{q',\phi}}. \end{aligned} \tag{3.1}$$

That is, the mapping $g \mapsto \int_X ag\, d\mu$ is a bounded linear functional on $\mathcal{L}_{q',\phi}(X)/\mathcal{C}$ with norm not exceeding 1.

Definition 3.2 ($H_U^{[\phi,q]}(X)$). Let $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$, $1 < q \le \infty$ and $1/q + 1/q' = 1$. Let U be a continuous, concave, increasing and bijective function from $[0, +\infty)$ to itself. Assume that $\mathcal{L}_{q',\phi}(X)/\mathcal{C} \ne \{0\}$. We define the space $H_U^{[\phi,q]}(X) \subset (\mathcal{L}_{q',\phi}(X)/\mathcal{C})^*$ as follows:

$f \in H_U^{[\phi,q]}(X)$ if and only if there exist sequences $\{a_j\} \subset A[\phi, q]$ and positive numbers $\{\lambda_j\}$ such that

$$f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{q',\phi}(X)/\mathcal{C})^* \quad \text{and} \quad \sum_j U(\lambda_j) < \infty. \tag{3.2}$$

From $U(0) = 0$ and the concavity of U it follows that

$$U(Cr) \le CU(r), \quad 1 \le C < \infty,\ 0 \le r < \infty, \tag{3.3}$$

$$U(r+s) \le U(r) + U(s), \quad 0 \le r, s < \infty. \tag{3.4}$$

Then $H_U^{[\phi,q]}(X)$ is a linear space. (3.4) implies

$$\sum_j \lambda_j \le U^{-1}\left(\sum_j U(\lambda_j)\right). \tag{3.5}$$

Therefore, if $\sum_j U(\lambda_j) < \infty$, then $\sum_j \lambda_j < \infty$ and $\sum_j \lambda_j a_j$ converges in $(\mathcal{L}_{q',\phi}(X)/\mathcal{C})^*$.

In general, the expression (3.2) is not unique. We define

$$\|f\|_{H_U^{[\phi,q]}} = \inf\left\{U^{-1}\left(\sum_j U(\lambda_j)\right)\right\},$$

where the infimum is taken over all expressions as in (3.2). We note that $\|f\|_{H_U^{[\phi,q]}}$ is not a norm in general. Let $d(f,g) = U(\|f-g\|_{H_U^{[\phi,q]}})$ for $f, g \in H_U^{[\phi,q]}(X)$. Then $d(f,g)$ is a metric and $H_U^{[\phi,q]}(X)$ is complete with respect to this metric. If $I(r) = r$, then $\|f\|_{H_I^{[\phi,q]}}$ is a norm and $H_I^{[\phi,q]}(X)$ is a Banach space.

In the case of $(p(\cdot), q)$-atoms instead of $[\phi, q]$-atoms, we denote $H_U^{[\phi,q]}(X)$ by $H_U^{p(\cdot),q}(X)$.

4. Results

Theorem 4.1. *If there exists a constant $C_* > 0$ such that*

$$U(rs) \le C_* U(r) U(s) \quad \text{for} \quad 0 < r, s \le 1, \tag{4.1}$$

$$U\left(\frac{\mu(B_1)\phi(B_1)}{\mu(B_2)\phi(B_2)}\right) \le C_* \frac{\mu(B_1)}{\mu(B_2)} \quad \text{for all balls } B_1 \text{ and } B_2 \text{ with } B_1 \subset B_2, \tag{4.2}$$

then

$$H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X),$$

with equivalent topologies.

Corollary 4.2. *Let $Q > 0$. Assume that $\mu(X) < \infty$ and that $\mu(B(x,r)) \sim r^Q$ for all $x \in X$ and $0 < r < R_0$, where R_0 is the constant in (2.4). Let $U(r) = r^{p_+}$ with $0 < p_- \le p_+ \le 1$, where $p_- = \inf p(x)$ and $p_+ = \sup p(x)$. If there exists a constant $C_0 > 0$ such that*

$$|p(x) - p(y)| \le \frac{C_0}{\log(1/d(x,y))} \quad \text{for} \quad d(x,y) < 1/2, \tag{4.3}$$

then

$$H_U^{p(\cdot),q}(X) = H_U^{p(\cdot),\infty}(X),$$

with equivalent topologies.

In this case we denote $H_U^{p(\cdot),q}(X)$ by $H^{p(\cdot)}(X)$ simply, which is a kind of Hardy spaces with variable exponent.

Proof of Corollary 4.2. The inequality (4.1) holds clearly. We show (4.2).

For $B(x,r) \subset B(y,s)$,

$$\frac{U\left(\frac{\phi(x,r)\mu(B(x,r))}{\phi(y,s)\mu(B(y,s))}\right)}{\frac{\mu(B(x,r))}{\mu(B(y,s))}} \sim \left(\frac{r}{s}\right)^{Qp_+(1/p(x)-1/p_+)} s^{Qp_+(1/p(x)-1/p(y))} \le s^{Qp_+(1/p(x)-1/p(y))},$$

since $r/s \le 1$. If $1/2 < s < R_0$, then

$$s^{Qp_+(1/p(x)-1/p(y))} \le R_0{}^{Qp_+/p_-}.$$

If $s \le 1/2$, then $d(x,y) < s$ and

$$\begin{aligned} \log s^{Qp_+(1/p(x)-1/p(y))} &\le Qp_+ \left|\frac{1}{p(y)} - \frac{1}{p(x)}\right| \log(1/s) \\ &\le Qp_+ \left|\frac{p(x)-p(y)}{p(x)p(y)}\right| \log(1/d(x,y)) \le \frac{C_0 Qp_+}{p_-{}^2}. \quad \square \end{aligned}$$

Lemma 4.3. *Let $E = H_U^{[\phi,q]}(X)$. If*

$$\sup_{0<s\le 1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0), \tag{4.4}$$

then

$$\|\ell\|_{E^*} = \sup\{|\ell(f)| : \|f\|_E \le 1\}$$

is finite for all $\ell \in E^*$, and $\|\ell\|_{E^*}$ is a norm.

Remark 4.1. If (4.1) holds, then (4.4) holds. If (4.4) holds, then there exist constants $C > 0$ and $p > 0$ such that $U(r) \le Cr^p$ for $r \in (0,1]$. If $\alpha > 0$ and $U(r) = (\log(1/r))^{-\alpha}$ for small $r > 0$, then U does not satisfy (4.4).

Let $L^q_c(X)$ be the set of all L^q-functions with bounded support, and let

$$L^{q,0}_c(X) = \left\{ f \in L^q_c(X) : \int_X f\,d\mu = 0 \right\}.$$

Then, for $1 < q \le \infty$, $L^{q,0}_c(X)$ is dense in $H^{[\phi,q]}_U(X)$.

Theorem 4.4. *If U satisfies* (4.4), *then*

$$\left(H^{[\phi,q]}_U(X)\right)^* = \mathcal{L}_{q',\phi}(X)/\mathcal{C}.$$

More precisely, if $g \in \mathcal{L}_{q',\phi}(X)/\mathcal{C}$, then the mapping $\ell : f \mapsto \int_X f(g+c)\,d\mu$, for $f \in L^{q,0}_c(X)$, can be extended to a continuous linear functional on $H^{[\phi,q]}_U(X)$. Conversely, if ℓ is a continuous linear functional on $H^{[\phi,q]}_U(X)$, then there exists $g \in \mathcal{L}_{q',\phi}(X)/\mathcal{C}$ such that $\ell(f) = \int_X f(g+c)\,d\mu$ for $f \in L^{q,0}_c(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{\mathcal{L}_{q',\phi}}$.

Corollary 4.5. *Assume the conditions in Remark 3.1 and Corollary 4.2. Then*

$$\left(H^{p(\cdot)}(X)\right)^* = \mathrm{Lip}_{\alpha(\cdot)}(X)/\mathcal{C}.$$

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