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JENSEN INEQUALITY に関わる作用素不等式について

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ABSTRACT. Jensen's inequality について Mond と Pečarić は，逆不等式を考察した．彼らの定理の一般化を本稿では概説する．特に，Hölder-McCarthy inequality の逆についても触れる．

また，Jensen 型のノルム不等式 — Araki-Cordes inequality — は，凹または凸関数を用いて拡張される．そこで本稿では，submultiplicative な増加凹関数に対してその不等式の逆が与えられることを先の一般化を用いて概説する．得られた結果の応用として，Bourin が得たスペクトル半径による作用素ノルムの評価不等式の一般化について触れる．

1. はじめに

本稿では，[17], [23] で得られた Mond-Pečarić method による Jensen inequality の逆評価の拡張と，その拡張にまつわる Araki-Cordes 型不等式について，概要を報告する．

本稿で，作用素 (operator) は，ヒルベルト空間 H 上の有界線形作用素 (bounded linear operator) を意味し，正作用素 (positive operator) A を $A \geq 0$ で表す．

古典的 Jensen's inequality (cf. [15]) は，凸関数に関する最も重要な不等式の一つである: Let $f(t)$ be a convex continuous function on an interval $[m, M]$ and $w = (w_1, \ldots, w_n)$ a weight, i.e., $\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0$. Then for $t_1, \ldots, t_n \in [m, M]$

$$f\left(\sum_{i=1}^{n} w_i t_i\right) \leq \sum_{i=1}^{n} w_i f(t_i).$$

上記 Jensen inequality は，次のようにも表現される:

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{1.1}$$

for a selfadjoint operator A on H with $m \leq A \leq M$ and a unit vector $x \in H$. 特に，もし $f(t) = t^p$ であるとき, 次の Hölder-McCarthy inequality が得られる: For all $1 \leq p$ (resp $0 \leq p \leq 1$)

$$\langle A^p x, x\rangle \geq \langle Ax, x\rangle^p \quad (\text{resp. } \langle A^p x, x\rangle \leq \langle Ax, x\rangle^p). \tag{1.2}$$

第 2 章では，連続関数 g に対して $g(\langle Ax, x\rangle) - \lambda\langle f(A)x, x\rangle$ の評価を調べる．そのために用いる Mond-Pečarić Method [14] は有用な手法であり，ある曲線と割線との関係を曲線と接線との関係変化させている．また，この評価は (1.1) の逆不等式を含むことになる．応用として，この評価は，興味深い定数 $K(h, p)$ (see (2.10)) を用いて Hölder-McCarthy inequality (1.2) の逆不等式を得る．

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第 3 章では，Cordes と Araki による次のノルム不等式の一般化と逆について触れる．作用素ノルムに関する Cordes inequality [5] は，H 上の正作用素 A, B に関する次の不等式である:

$$\| A^p B^p \| \leq \| AB \|^p \quad \text{for all} \quad 0 \leq p \leq 1. \tag{1.3}$$

[1] において，Araki は，次の不等式を導く trace inequality を与えた:

$$\| B^p A^p B^p \| \leq \| BAB \|^p \quad \text{for all} \quad 0 \leq p \leq 1. \tag{1.4}$$

上記二つのノルム不等式 (1.3) と (1.4) とは，同値であり ([3], [9])，Hölder-McCarthy inequality (1.2) の一般化である．更に，Furuta [11] は，Cordes's inequality (1.3) が次の Löwner-Heinz inequality (e.g. [20]) に同値であることを示した:

$$A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for all } 0 \leq p \leq 1. \tag{1.5}$$

応用として，次の Bourin's reverse inequality [4] を一般化する: For a positive definite matrix A with $0 < m \leq A \leq M$ and a positive semidefinite matrix B

$$\| AB \| \leq \frac{M+m}{2\sqrt{Mm}} r(AB) \tag{1.6}$$

where r(·) is the spectral radius.

2. Mond-Pečarić Method による reverse Jensen's inequality

$m < M$ を満たす実数 m, M をとる．区間 $I(\supset [m, M])$ 上の実数値連続関数 f に対して，次のように定数 α_f と β_f を定める:

$$\alpha_f = \alpha_f(m, M) := \frac{f(M) - f(m)}{M - m}, \qquad \beta_f = \beta_f(m, M) := \frac{Mf(m) - mf(M)}{M - m}. \tag{2.1}$$

Mond と Pečarić は，実数値凸関数を用いた正作用素に関する次の不等式を示した (cf. [18, Theorem 4]):

Theorem M-P. *Let A be a positive operator on a Hilbert space H such that $m \leq A \leq M$ where $0 < m < M$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$ and J an interval including $f[m, M]$. If $F[u, v]$ is a real valued function defined on $J \times J$, non-decreasing in u, then*

$$F\left[\langle f(A)x, x\rangle, f(\langle Ax, x\rangle)\right] \leq \max_{m \leq t \leq M} F[\alpha_f t + \beta_f, f(t)]$$

for every unit vector x in H.

更に，Mond と Pečarić [19, Theorems 1,2] は，次の多重正作用素の場合における Jensen 型不等式を示した:

Theorem A. *Let A_j be positive operators on a Hilbert space H satisfying $m \leq A_j \leq M$ $(j = 1, 2, \ldots, k)$ where $0 < m < M$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$. Suppose that $x_1, x_2, \ldots, x_k$ are any finite number of vectors in H such that*

$\sum_{j=1}^{k} \| x_j \|^2 = 1$. *Then the following inequalities hold*

$$f\left(\sum_{j=1}^{k}\langle A_j x_j, x_j\rangle\right) \leq \sum_{j=1}^{k}\langle f(A_j)x_j, x_j\rangle, \tag{2.2}$$

$$\sum_{j=1}^{k}\langle f(A_j)x_j, x_j\rangle \leq \alpha_f \sum_{j=1}^{k}\langle A_j x_j, x_j\rangle + \beta_f. \tag{2.3}$$

次に，Theorem M-P の拡張を示す:

Theorem 2.1. *Let A_j be positive operator on a Hilbert space H satisfying $m \leq A_j \leq M$ $(j = 1, 2, \ldots, k)$ where $0 < m < M$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$ and also let $g(t)$ be a real valued continuous function on $[m, M]$. Suppose that $x_1, x_2, \ldots, x_k$ are any finite number of vectors in H such that $\sum_{j=1}^{k} \| x_j \|^2 = 1$, and U and V are two intervals such that $U \supset f[m, M]$ and $V \supset g[m, M]$. If $F[u, v]$ is a real valued function defined on $U \times V$, non-decreasing in u, then*

$$F\left[\sum_{j=1}^{k}\langle f(A_j)x_j, x_j\rangle, g\left(\sum_{j=1}^{k}\langle A_j x_j, x_j\rangle\right)\right] \leq \max_{m \leq t \leq M} F[\alpha_f t + \beta_f, g(t)]. \tag{2.4}$$

Proof. Let take $t_0 = \sum_{j=1}^{k}\langle A_j x_j, x_j\rangle$ in (2.3). The hypothesis ensures the inequality $m = \sum_{j=1}^{k}\langle m x_j, x_j\rangle \leq \sum_{j=1}^{k}\langle A_j x_j, x_j\rangle \leq \sum_{j=1}^{k}\langle M x_j, x_j\rangle = M$, i.e., $m \leq t_0 \leq M$. Using the non-decreasing character of $F[\cdot, v]$, we have

$$F\left[\sum_{j=1}^{k}\langle f(A_j)x_j, x_j\rangle, g\left(\sum_{j=1}^{k}\langle A_j x_j, x_j\rangle\right)\right] \leq F\left[\alpha_f t + \beta_f, g(t_0)\right]$$

and hence the desired inequality holds. □

Theorem 2.2. *Assume that the conditions of Theorem 2.1 hold except that $F[u, v]$ is non-increasing in u. Then the following inequality holds*

$$F\left[\sum_{j=1}^{k}\langle f(A_j)x_j, x_j\rangle, g\left(\sum_{j=1}^{k}\langle A_j x_j, x_j\rangle\right)\right] \geq \min_{m \leq t \leq M} F[\alpha_f t + \beta_f, g(t)]. \tag{2.5}$$

[18, Theorems 3,4] において，Mond と Pečarić は $g = f$ の場合における Theorems 2.1, 2.2 を示した．Theorem 2.1 の応用として，[21, Theorem 1] の拡張を考察する．更に，等号が成立する条件について考える．

Theorem 2.3. *Assume that the conditions of Theorem 2.1 hold. Then for any real number λ*

$$\sum_{j=1}^{k}\langle f(A_j)x_j, x_j\rangle \leq \lambda g\left(\sum_{j=1}^{k}\langle A_j x_j, x_j\rangle\right) + \mu(\lambda) \tag{2.6}$$

holds for $\mu(\lambda) = \max_{m \leq t \leq M}\{\alpha_f t + \beta_f - \lambda g(t)\}$.

Moreover, suppose that $\mu(\lambda) = \alpha_f \sum_{j=1}^k \langle A_j x_j, x_j\rangle + \beta_f - \lambda g\left(\sum_{j=1}^k \langle A_j x_j, x_j\rangle\right)$ *for some vectors* x_j *in* H *such that* $\sum_{j=1}^k \| x_j \|^2 = 1$. *Then the equality is attained in* (2.6) *if and only if there exist orthogonal vectors* y_j *and* z_j *such that*

$$x_j = y_j + z_j, \quad A_j y_j = m y_j \quad \textit{and} \quad A_j z_j = M z_j. \tag{2.7}$$

Proof. Put $t_0 = \sum_{j=1}^k \langle A_j x_j, x_j\rangle$, then the hypothesis ensures the inequality $m \le t_0 \le M$. Also, put $F[u,v] = u - \lambda v$, $u = \sum_{j=1}^k \langle f(A_j)x_j, x_j\rangle$ and $v = g(t_0)$. Then it follows from Theorem 2.1 that

$$\begin{aligned}\sum_{j=1}^k \langle f(A_j)x_j, x_j\rangle - \lambda g\Big(\sum_{j=1}^k \langle A_j x_j, x_j\rangle\Big) &\le \max_{m\le t\le M} F[\alpha_f t + \beta_f, g(t)] \\ &= \max_{m\le t\le M}\{\alpha_f t + \beta_f - \lambda g(t)\}\end{aligned}$$

which gives the desired inequality.

We next investigate conditions under which the equality holds. Suppose that the equality $\sum_{j=1}^k \langle f(A_j)x_j, x_j\rangle = \lambda g(t_0) + \mu(\lambda)$ holds. By definition of $\mu(\lambda)$, notice that the equality $\sum_{j=1}^k \langle f(A_j)x_j, x_j\rangle = \lambda g(t_0) + \mu(\lambda)$ holds if and only if the equality $\sum_{j=1}^k \langle f(A_j)x_j, x_j\rangle = \alpha_f t_0 + \beta_f$ holds. Let $E_j(t)$ be the spectral resolution of the identity of A_j, that is, $A_j = \int_{m-0}^M t dE_j(t)$. Put $P_j = E_j(M) - E_j(M-0)$, $Q_j = E_j(M-0) - E_j(m)$ and $R_j = E_j(m) - E_j(m-0)$. Then $\langle A_j P_j x_j, x_j\rangle = M\langle P_j x_j, x_j\rangle$ and $\langle A_j R_j x_j, x_j\rangle = m\langle R_j x_j, x_j\rangle$. Note also that

$$\begin{aligned}\langle f(A_j)P_j x_j, x_j\rangle &= \int_{m-0}^M f(t) d\langle E_j(t)P_j x_j, x_j\rangle = f(M)\langle P_j x_j, x_j\rangle \\ &= \langle (f(m) + \alpha_f(M-m))P_j x_j, x_j\rangle\end{aligned}$$

and

$$\begin{aligned}\langle f(A_j)R_j x_j, x_j\rangle &= \int_{m-0}^M f(t) d\langle E_j(t)R_j x_j, x_j\rangle = f(m)\langle R_j x_j, x_j\rangle \\ &= \langle (f(m) + \alpha_f(m-m))R_j x_j, x_j\rangle.\end{aligned}$$

Since $\sum_{j=1}^k \langle f(A_j)x_j, x_j\rangle = \alpha_f t + \beta_f$, it follows that $\sum_{j=1}^k \langle (\alpha_f A_j + \beta_f - f(A_j))Q_j x_j, x_j\rangle = 0$ and hence $Q_j x_j = 0$ for any j because $\alpha_f s + \beta_f - f(s) > 0$ for $s \in (m, M)$. Thus we obtain the desired decomposition of x_j setting $y_j = R_j x_j$ and $z_j = P_j x_j$.

Assume conversely (2.7). Then it follows that

$$\begin{aligned}\alpha_f \sum_{j=1}^k \langle A_j x_j, x_j\rangle + \beta_f &= \alpha_f \sum_{j=1}^k (m \| y_j \|^2 + M \| z_j \|^2) + \beta_f \sum_{j=1}^k (\| y_j \|^2 + \| z_j \|^2) \\ &= f(m)\sum_{j=1}^k \| y_j \|^2 + f(M)\sum_{j=1}^k \| z_j \|^2 \\ &= \sum_{j=1}^k \langle f(A_j)x_j, x_j\rangle\end{aligned}$$

which is the desired equality. □

$g = f$ とし，[21, Theorem 1] の多重作用素版を与える:

Theorem 2.4. *Let A_j be positive operator on a Hilbert space H satisfying $m \le A_j \le M$ $(j = 1, 2, \ldots, k)$ where $0 < m < M$. Let f be a real valued continuous strictly convex differentiable function on $[m, M]$. Suppose that $x_1, x_2, \ldots, x_k$ are any finite number of vectors in H such that $\sum_{j=1}^{k} \| x_j \|^2 = 1$. Then for each $\lambda > 0$*

$$\sum_{j=1}^{k} \langle f(A_j)x_j, x_j \rangle \le \lambda f \left(\sum_{j=1}^{k} \langle A_j x_j, x_j \rangle \right) + \mu(\lambda) \tag{2.8}$$

holds for $\mu(\lambda) = \alpha_f t + \beta_f - \lambda f(t_0)$ and

$$t_0 = \begin{cases} M & \text{if } M \le f'^{-1}(\frac{\alpha_f}{\lambda}) \\ m & \text{if } f'^{-1}(\frac{\alpha_f}{\lambda}) \le m \\ f'^{-1}(\frac{\alpha_f}{\lambda}) & \text{otherwise.} \end{cases}$$

The equality is attained in (2.8) if and only if there exist orthogonal vectors y_j and z_j such that $x_j = y_j + z_j$, $A_j y_j = m y_j$, $A_j z_j = M z_j$ and $t_0 = m \sum_{j=1}^{k} \| y_j \|^2 + M \sum_{j=1}^{k} \| z_j \|^2$.

Proof. By virtue of Theorem 2.3, it is sufficient to see that $\mu(\lambda) = \alpha_f t_0 + \beta_f - \lambda f(t_0)$. Put $h_\lambda(t) = \alpha_f t + \beta_f - \lambda f(t)$. Since $f(t)$ is strictly convex, we put $t_1 = f'^{-1}(\frac{\alpha_f}{\lambda})$. Then we have $h'(t) = 0$ if and only if $t = t_1$. If $m \le t_1 \le M$, then $\mu(\lambda) = \max_{m \le t \le M} h_\lambda(t) = h_\lambda(t_1)$. If $M \le t_1$, then $h_\lambda(t)$ is increasing on $[m, M]$ and hence the maximum value on $[m, M]$ of $h_\lambda(t)$ is attained for $t_0 = M$. Similarly, we have $t_0 = m$ if $t_1 \le m$.

Next, since the graph of $\lambda f(t) + \beta_f$ touches the line of $\alpha_f t + \beta_f$ at the point t_0, it follows that the equality $\sum_{j=1}^{k} \langle f(A_j)x_j, x_j \rangle = \lambda f \left(\sum_{j=1}^{k} \langle A_j x_j, x_j \rangle \right) + \mu(\lambda)$ holds if and only if two equalities $t_1 = \sum_{j=1}^{k} \langle A_j x_j, x_j \rangle$ and $\sum_{j=1}^{k} \langle f(A_j)x_j, x_j \rangle = \alpha_f t + \beta_f$ hold. Therefore we obtain the Theorem 2.4 by the same proof as Theorem 2.3. □

$\lambda > 0$ に対して，方程式 $\mu(\lambda) = 0$ は，唯一解 $\lambda = \lambda_f$ を持つ. Theorem 2.4 において，$f(t) = t^p$, $k = 1$ のとき，次のように Hölder-McCarthy inequality (1.2) の商に関する逆不等式が得られる:

Corollary 2.5. *Let A be a positive operator on a Hilbert space H such that $0 < m \le A \le M$ for some scalars $m < M$ and $h := \frac{M}{m} (> 1)$. Then for $p \ge 1$ (resp. $0 < p \le 1$)*

$$\langle A^p x, x \rangle \le K(h,p) \langle Ax, x \rangle^p \quad (\text{resp. } \langle A^p x, x \rangle \ge K(h,p) \langle Ax, x \rangle^p) \tag{2.9}$$

holds for all unit vectors $x \in H$ where $K(h,p)$ is a generalized Kantorovich constant (cf. [7], [12], [13]) defined by

$$K(h,p) := \frac{1}{h-1} \frac{h^p - h}{p-1} \left(\frac{p-1}{h^p - h} \frac{h^p - 1}{p} \right)^p \tag{2.10}$$

for all $h > 0$ and $p \in \mathbb{R}$.

3. Jensen type norm inequalities

Hölder-McCarthy inequality (1.2) は，Araki(-Cordes) norm inequality (1.4) (, (1.3)) より導かれる．実際，勝手な単位ベクトル $x \in H$ をとると任意のベクトル $y \in H$ に対して次が成り立つ:

$$(x \otimes \bar{x})A(x \otimes \bar{x})y = \langle y, x\rangle (x \otimes \bar{x})Ax = \langle y, x\rangle\langle Ax, x\rangle x = \langle Ax, x\rangle\langle y, x\rangle x = \langle Ax, x\rangle (x \otimes \bar{x})y.$$

よって

$$(x \otimes \bar{x})A(x \otimes \bar{x}) = \langle Ax, x\rangle (x \otimes \bar{x}).$$

同様に計算することにより

$$(x \otimes \bar{x})A^p(x \otimes \bar{x}) = \langle A^p x, x\rangle (x \otimes \bar{x}).$$

これゆえに，Araki(-Cordes) norm inequality (1.4) において，$B = x \otimes \bar{x}$ とおくと Hölder-McCarthy inequality (1.2) が得られる．上記関係の観点から，(2.8) (, (3.4)) に関するある種の拡張として Jensen 型ノルム不等式について考察する．

ここで，後の議論のために幾つかの定義を行う．f を $[0, \infty)$ 上の実数値連続関数とする．このとき，$A \geq B \geq 0$ に対し $f(A^{\frac{1}{2}})^2 \geq f(B^{\frac{1}{2}})^2$ ならば，f は semi-operator monotone と呼ばれる．また，f が submultiplicative (resp. supermultiplicative) であるとは，任意の $a, b \geq 0$ に対して $f(ab) \leq f(a)f(b)$ (resp. $f(ab) \geq f(a)f(b)$) であることを意味する．f の adjoint f^* は次のように定義される: $t > 0$ に対して $f^*(t) := f(t^{-1})^{-1}$ ([16]).

J.I. Fujii と M. Fujii は，(1.3) の拡張を与えた ([6], cf. [2, Theorem 2.6]):

Theorem B. *If a nonnegative semi-operator monotone function f on $(0, \infty)$ is submultiplicative, then*

$$\| f(A)f^*(B) \| \leq f(\| AB \|) \tag{3.1}$$

for all positive operators A and B.

次に，(1.4) の一般化であり，(3.1) に同値な不等式として次の定理を記述する．尚，得られた結果は，[2, Theorem 2.9] のある種の改良である．

Theorem 3.1. *If a nonnegative operator monotone function f on $(0, \infty)$ is submultiplicative, then*

$$\| f^*(B^2)^{\frac{1}{2}} f(A^2)^{\frac{1}{2}} f^*(B^2)^{\frac{1}{2}} \| \leq \| f^*(B^2)^{\frac{1}{2}} f(A) f^*(B^2)^{\frac{1}{2}} \| \leq f(\| BAB \|) \tag{3.2}$$

for all positive operators A and B.

(3.2) の第 2 不等式に関する逆不等式を導くため，次の区間と定数を定義する．$[m, M]$ 上で増加狭義凹 (resp. 狭義凸) 微分可能関数 f に対して，次の区間 I_f を定義する:

$$I_f = I_{f,m,M} := \left[\frac{f'(M)}{\alpha_f}, \frac{f'(m)}{\alpha_f}\right] \quad \left(\text{resp. } I_f = I_{f,m,M} := \left[\frac{f'(m)}{\alpha_f}, \frac{f'(M)}{\alpha_f}\right]\right).$$

ところで 任意の $\lambda \in I_f$ に対して，方程式 $f'(\mu) = \lambda\alpha_f$ は，唯一解 $\mu = \mu_\lambda \in [m, M]$ を持つ．この唯一解を用いて，次の定数 $F(m, M, f; \lambda)$ を定める:

$$F(m, M, f; \lambda) := \begin{cases} (1-\lambda)f(c_1) & \text{if } 0 < \lambda < \frac{f'(c_1)}{\alpha_f} \\ f(\mu_\lambda) - (\mu_\lambda\alpha_f + \beta_f)\lambda & \text{if } \lambda \in I_f \\ (1-\lambda)f(c_2) & \text{if } \lambda > \frac{f'(c_2)}{\alpha_f} \end{cases} \tag{3.3}$$

where $c_1 = M$ and $c_2 = m$ (resp. $c_1 = m$ and $c_2 = M$).

ここで，関数 $F(m,M,p;\lambda)$ は単調減少し，λ に関する方程式 $F(m,M,p;\lambda)=0$ は，唯一解 $\lambda=\lambda_f(\in I_f)$ をもつ ([22]).

次の定理は，Theorem 2.4 により確かめられる:

Theorem 3.2. *Let A be a positive operator on a Hilbert space H such that $m \le A \le M$ for some scalars $0<m<M$. Let f be a real valued continuous strictly concave (resp. strictly convex) differentiable function on $[m,M]$ with $f(m)\ne f(M)$. Then for each $\lambda>0$*

$$\begin{aligned} f(\langle Ax,x\rangle)-\lambda\langle f(A)x,x\rangle \le F(m,M,f;\lambda) \\ \big(\textit{resp. } f(\langle Ax,x\rangle)-\lambda\langle f(A)x,x\rangle \ge F(m,M,f;\lambda)\ \big) \end{aligned} \tag{3.4}$$

holds for all unit vectors $x\in H$.

この定理を用いて，Theorem 3.1 の逆不等式を得る:

Theorem 3.3. *Let A and B be positive operators on a Hilbert space H such that $m_1 \le A \le M_1$ and $m_2 \le B \le M_2$ for some scalars $0<m_i<M_i$ $(i=1,2)$. Let f and g be nonnegative real valued differentiable functions on $(0,\infty)$. Then the following assertions* (i) *and* (ii) *hold and they are equivalent:*

(i) *Suppose that f is increasing strictly concave submultiplicative and λ_f is a unique solution of $F(m_1,M_1,f;\lambda)=0$. Then for each $\lambda\in(0,\lambda_f]$*

$$\begin{aligned} f(\| BAB \|) \ \le\ & \lambda \sup_{t\in[m_2,M_2]} f(t^2)f\left(\frac{1}{t^2}\right) \| f^*(B^2)^{\frac12} f(A) f^*(B^2)^{\frac12} \| \\ &+\ F(m_1,M_1,f;\lambda)\, f(M_2^2). \end{aligned} \tag{3.5}$$

(ii) *Suppose that g is increasing strictly convex supermultiplicative and λ_g is a unique solution of $F(g(m_1),g(M_1),g^{-1};\lambda)=0$. Then for each $\lambda\in(0,\lambda_g]$*

$$\begin{aligned} g^{-1}\left(\| g^*(B^2)^{\frac12} g(A) g^*(B^2)^{\frac12} \|\right) \ \le\ & \lambda \sup_{t\in[m_2,M_2]} g^{-1}(g^*(t^2))t^{-2} \| BAB \| \\ &+\ F\left(g(m_1),g(M_1),g^{-1};\lambda\right)\ g^{-1}(g^*(M_2^2)). \end{aligned} \tag{3.6}$$

Proof. Firstly we prove the case (i). For each $\lambda>0$ and unit vector $x\in H$

$$\begin{aligned} f(\langle BABx,x\rangle) &= f\left(\left\langle A\frac{Bx}{\| Bx\|},\frac{Bx}{\| Bx\|}\right\rangle \| Bx \|^2\right) \\ &\le f\left(\left\langle A\frac{Bx}{\| Bx\|},\frac{Bx}{\| Bx\|}\right\rangle\right) f(\| Bx\|^2) \\ &\le \left\{\lambda\left\langle f(A)\frac{Bx}{\| Bx\|},\frac{Bx}{\| Bx\|}\right\rangle + F(m_1,M_1,f;\lambda)\right\} f(\| Bx\|^2) \qquad \text{(by (3.4))} \\ &= \lambda\left\langle f(B^{-2})^{-\frac12} f(A) f(B^{-2})^{-\frac12}\cdot\frac{f(B^{-2})^{\frac12}Bx}{\| f(B^{-2})^{\frac12}Bx\|},\frac{f(B^{-2})^{\frac12}Bx}{\| f(B^{-2})^{\frac12}Bx\|}\right\rangle \\ &\qquad\times\frac{f(\| Bx\|^2)\,\| f(B^{-2})^{\frac12}Bx\|^2}{\| Bx\|^2} + F(m_1,M_1,f;\lambda) f(\| Bx\|^2) \\ &\le \lambda \| f^*(B^2)^{\frac12} f(A) f^*(B^2)^{\frac12} \| \cdot\frac{f(\| Bx\|^2)\,\| f(B^{-2})^{\frac12}Bx\|^2}{\| Bx\|^2} + F(m_1,M_1,f;\lambda) f(\| Bx\|^2). \end{aligned}$$

Here, we have

$$
\begin{aligned}
f(\| Bx \|^2) \, \| f(B^{-2})^{\frac{1}{2}} \frac{Bx}{\| Bx \|} \|^2 &= f(\| Bx \|^2) \left\langle f(B^{-2}) \frac{Bx}{\| Bx \|}, \frac{Bx}{\| Bx \|} \right\rangle \\
&\leq f(\| Bx \|^2) f\left(\left\langle \frac{x}{\| Bx \|}, \frac{x}{\| Bx \|} \right\rangle \right) \\
&= f(\| Bx \|^2) f\left(\frac{1}{\| Bx \|^2} \right) \\
&\leq \sup_{t \in [m_2, M_2]} f(t^2) f\left(\frac{1}{t^2} \right).
\end{aligned}
\tag{3.7}
$$

Moreover since $0 < f(\| Bx \|^2) \leq f(M_2^2)$ and $F(m_1, M_1, f; \lambda) \geq 0$ for $\lambda \in (0, \lambda_f]$, we have $0 < F(m_1, M_1, f; \lambda) f(\| Bx \|^2) \leq F(m_1, M_1, f; \lambda) f(M_2^2)$. So the desired inequality (3.5) holds.

Next we show (3.5) $\Longrightarrow$ (3.6). We replace A, B and f by $g(A)$, $g^*(B^2)^{\frac{1}{2}}$ and g^{-1}, respectively in (3.5). Since $(g^{-1})^*(g^*(X)) = X$ for all positive operator X and g^* is also increasing, the inequality (3.5) ensures the inequality (3.6). Similarly we can show (3.6) $\Longrightarrow$ (3.5). □

$f_0(t) := f(t^{\frac{1}{2}})^2$ を増加狭義凹で submultiplicative な関数とし，λ_f を λ に関する方程式 $F(m_1^2, M_1^2, f_0; \lambda) = 0$ の唯一解とする．このとき，(3.5) において，A, f をそれぞれ A^2, f_0 とおくと，任意の $\lambda \in (0, \lambda_f]$ に対して

$$
f(\| AB \|)^2 \leq \lambda \sup_{t \in [m_2, M_2]} f(t)^2 f\left(\frac{1}{t} \right)^2 \| f(A) f^*(B) \|^2 + F(m_1^2, M_1^2, f_0; \lambda) \, f(M_2)^2.
$$

尚，これは Theorem B の逆不等式である．

また，Theorem 3.3 において $f(t) = t^p$ $(p \geq 0)$ とおくと，次の系が得られる (see [10]):

Corollary 3.4. *Let A and B be positive operators on a Hilbert space H such that $m_1 \leq A \leq M_1$ and $m_2 \leq B \leq M_2$ and $h_i = \frac{M_i}{m_i}$ for some scalars $0 < m_i < M_i$ $(i = 1, 2)$. Then the following assertions* (i) *and* (ii) *hold and they are equivalent:*

(i) *Suppose that $0 \leq p \leq 1$. Then for each $\lambda \in (0, K(h, p)^{-1}]$*

$$
\| BAB \|^p \leq \lambda \| B^p A^p B^p \| + F(m_1, M_1, (\cdot)^p; \lambda) \, M_2^{2p}. \tag{3.8}
$$

(ii) *Suppose that $p \geq 1$. Then for each $\lambda \in (0, K(h, p)]$*

$$
\| B^p A^p B^p \|^{\frac{1}{p}} \leq \lambda \| BAB \| + F\left(m_1^p, M_1^p, (\cdot)^{\frac{1}{p}}; \lambda \right) M_2^2. \tag{3.9}
$$

Remark 3.5. (3.5) により次の商に関する不等式が得られる：

$$
f(\| BAB \|) \leq \lambda_f \sup_{t \in [m_2, M_2]} f(t^2) f\left(\frac{1}{t^2} \right) \| f^*(B^2)^{\frac{1}{2}} f(A) f^*(B^2)^{\frac{1}{2}} \|.
$$

一方 *Theorem 3.3* において，もし $\lambda_f < \lambda$ ならば，類似した幾つかの不等式を得ることが出来る．例えば，$f(\| Bx \|^2) \geq f(m_2^2) > 0$ かつ $F(m_1, M_1, f; \lambda) < 0$ なので，(3.5) に

関連して次が得られる：

$$f(\| BAB \|) \leq \lambda \sup_{t\in[m_2,M_2]} f(t^2)f\left(\frac{1}{t^2}\right) \| f^*(B^2)^{\frac{1}{2}}f(A)f^*(B^2)^{\frac{1}{2}} \| + F(m_1,M_1,f;\lambda)\, f(m_2^2)$$

同様の手法により次が得られる:

Theorem 3.6. *Let A and B be positive operators on a Hilbert space H such that $m_1 \leq A \leq M_1$ and $m_2 \leq B \leq M_2$ for some scalars $0 < m_i < M_i$ $(i = 1, 2)$. Let f and g be nonnegative real valued differentiable functions on $(0, \infty)$. Then the following assertions* (i) *and* (ii) *hold and they are equivalent:*

(i) *If f is increasing strictly convex submultiplicative, then for each $\lambda > 0$*

$$f(\| BAB \|) \leq \lambda \sup_{t\in[m_2,M_2]} f(t^2)f\left(\frac{1}{t^2}\right) \| f^*(B^2)^{\frac{1}{2}}f(A)f^*(B^2)^{\frac{1}{2}} \| - \lambda F(m_1m_2^2, M_1M_2^2, f; \frac{1}{\lambda}). \tag{3.10}$$

(ii) *If g is increasing strictly concave supermultiplicative, then for each $\lambda > 0$*

$$g^{-1}\left(\| g^*(B^2)^{\frac{1}{2}}g(A)g^*(B^2)^{\frac{1}{2}} \|\right) \leq \lambda \sup_{t\in[m_2,M_2]} g^{-1}(g^*(t^2))t^{-2} \| BAB \| - \lambda F(g(m_1)g^*(m_2^2), g(M_1)g^*(M_2^2), g^{-1}; \frac{1}{\lambda}). \tag{3.11}$$

[4] において，Bourin は，スペクトル半径 $r(\cdot)$ に関する既知な不等式 $r(A) \leq \| A \|$ の逆不等式をして (1.6) を示した．この (1.6) の一般化として [8] において著者らは次の定理を導いた:

Theorem C. *If A and B are positive operators such that $m_1 \leq A \leq M_1$ for some scalars $0 < m_1 < M_1$, then for each $\lambda > 0$*

$$\| (BA^pB)^{\frac{1}{p}} \| \leq \lambda r(AB^{\frac{2}{p}}) + F(m_1^p, M_1^p, (\cdot)^{\frac{1}{p}}; \lambda) \| B \|^{\frac{2}{p}} \quad \text{for } p > 1. \tag{3.12}$$

Theorems 3.3 と 3.6 により，Theorem C の更なる一般化を与える．

Corollary 3.7. *Let A and B be positive operators such that $m_1 \leq A \leq M_1$ and $m_2 \leq B \leq M_2$ for some scalars $0 < m_i < M_i$ $(i = 1, 2)$. Let f be a nonnegative real valued increasing differentiable function on $(0, \infty)$. Then the following assertions hold:*

(i) *Suppose that f is strictly convex supermultiplicative and λ_f is a unique solution of $F(f(m_1), f(M_1), f^{-1}; \lambda) = 0$. Then for each $\lambda \in (0, \lambda_f]$*

$$\| f^{-1}(Bf(A)B) \| \leq \lambda \sup_{t\in[m_2,M_2]} f^{-1}(t^2)f^{-1}\left(\frac{1}{t^2}\right) r(A\cdot(f^{-1})^*(B^2)) + F(f(m_1), f(M_1), f^{-1}; \lambda)\, f^{-1}(M_2^2). \tag{3.13}$$

(ii) *Suppose that f is strictly concave supermultiplicative. Then for each $\lambda > 0$*

$$\| f^{-1}(Bf(A)B) \| \leq \lambda \sup_{t\in[m_2,M_2]} f^{-1}(t^2)f^{-1}\left(\frac{1}{t^2}\right) r(A\cdot(f^{-1})^*(B^2)) - \lambda F(f(m_1)m_2^2, f(M_1)M_2^2, f^{-1}; \frac{1}{\lambda}). \tag{3.14}$$

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