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Fourier expansion of Arakawa lifting

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Abstract

This note is a report based on our talk at the conference on automorphic forms held at RIMS during January 21th-25th, 2008. We announce our recent results about Fourier coefficients of Arakawa lifting, i.e. a theta lifting to a cusp form on the quaternion unitary group $GSp(1,1)$ from a pair consisting of an elliptic cusp form f and an automorphic form f' on a definite quaternion algebra over $\mathbb{Q}$. We provide an explicit formula for the Fourier coefficients in terms of toral integrals of f and f'. As an application, we show the existence of non-vanishing Arakawa lifts.

0 Introduction

To explain the background of our study we start with reviewing Böcherer's conjecture on Fourier coefficients of holomorphic Siegel modular forms of degree two. We let F be a Hecke-eigen holomorphic Siegel cusp form of weight k with respect to $Sp(2,\mathbb{Z})$. Its Fourier expansion is described as

$$F(Z)=\sum_{T\in \mathrm{Sym}_2(\mathbb{Z})^*,T>0} C(T)e^{2\pi\sqrt{-1}\,\mathrm{Tr}\,TZ},$$

where $\mathrm{Sym}_2(\mathbb{Z})^*$ denotes the set of half-integral symmetric matices of degree 2, and $T>0$ means that T is positive definite. Now let $-D$ be a fundamental discrminant with $D>0$. For such D we consider the average $A(D)$ of the Fourier coefficients of F as follows:

$$A(D):=\sum_{S\in\{T\mid \det T=D/4\}/SL_2(\mathbb{Z})}\frac{C(S)}{\epsilon(S)}.$$

Here we put $\epsilon(S)=\#\{\gamma\in SL_2(\mathbb{Z})\mid {}^t\gamma S\gamma=S\}$. We let $L_{\mathrm{spin}}(F,\left(\frac{D}{*}\right),s)$ be the quadratic twist of the spinor L-function for F. Then Böcherer's conjecture [B] is formulated as

$$|A(D)|^2=C_F D^{k-1}L_{\mathrm{spin}}(F,\left(\frac{D}{*}\right),k-1)$$

with a constant C_F depending only on F. There are several evidences of this conjecture (cf. [B], [B-S], [K-K]). We note that, in the conjecture, the spinor L-function is evaluated at its central point $s = k-1$. This conjecture can be regarded as a generalization of the formula by Waldspurger-Kohnen-Zagier (cf. [Wa-1], [K-Z]), which says that the twisted central L-value of an integral weight elliptic cusp form f is proportional to the square of a Fourier coefficient of a half-integral weight elliptic cusp form associated with f by Shimura correspondence. Furusawa and Shalika have made a further expectation that Böcherer conjecture would also hold for an inner form G of $GSp(2)$:

$$\{g \in M_2(B) \mid {}^t\bar{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \nu(g) \in \mathbb{Q}^\times\},$$

where B is a quaternion algebra over $\mathbb{Q}$. This expectation is based on their conjectural relative trace formula for G (cf. [F-S]). In this note, we consider the case where B is definite, i.e. $G = GSp(1,1)$.

Our target is the "Arakawa lifting", which is a theta lifting from a pair of elliptic cusp form f and an automorphic form f' on $B^\times_{\mathbb{A}_\mathbb{Q}}$ to a vector-valued cusp form $\mathcal{L}(f, f')$ on $GSp(1,1)_{\mathbb{A}_\mathbb{Q}}$. Its representation type at the Archimedean place is a quaternionic discrete series representation (for the definition see [G-W]). Our result (Theorem 2.2) says that a certain average of the Fourier coefficients of $\mathcal{L}(f, f')$ (an analogue of $A(D)$) is explicitly written in terms of a product of toral integrals of (f, f').

Our formula leads us to two directions of further research. One is to show the existence of non-vanishing lifts, which is discussed in §3. In fact, if (f, f') are Hecke eigenforms with non-vanishing toral integrals, we have $\mathcal{L}(f, f') \not\equiv 0$ in view of our formula. Another direction is to find an explicit formula for the constant of proportionality relating the square norms of the averages of the Fourier coefficients to central L-values. Indeed, Furusawa-Shalika [F-S] expects that such square norm is proportional to the central value of a "Rankin-Selberg L-function" of the Arakawa lift. Waldspurger [Wa-1, Proposition 7] and our theorem tell us that the square norm of the average is proportional to a product of central L-values for the quadratic base changes of the Jacquet-Langlands lifts of f and f' twisted by a Hecke character. Such a product is expected to be a central L-value of a (twisted) Rankin-Selberg L-function of the Arakawa lift. We leave the study in this direction to our further research.

1 Arakawa lifting

Let B be a definite quaternion algebra over $\mathbb{Q}$ with the discriminant d_B. Let $x \mapsto \bar{x}$ be the main involution of B. We fix a maximal order $\mathcal{O}$ of B. We denote by $G = GSp(1,1)$ the $\mathbb{Q}$-algebraic group defined by

$$\{g \in M_2(B) \mid {}^t\bar{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \nu(g) \in \mathbb{Q}^\times\}.$$

From now on we assume that every automorphic form dealt with in this note has the trivial central character. Let D be a divisor of d_B and $\mathcal{S}_\kappa(\Gamma_0(D))$ the space of elliptic cusp forms of weight κ and level D. We regard each element of $\mathcal{S}_\kappa(\Gamma_0(D))$ as an automorphic form on $GL_2(\mathbb{A}_\mathbb{Q})$. We further denote by $\mathcal{A}_\kappa = \mathcal{A}_\kappa(B_{\mathbb{A}_\mathbb{Q}}^\times)$ the space of automorphic forms on $B_{\mathbb{A}_\mathbb{Q}}^\times$ of weight $(\sigma_\kappa = \mathrm{Sym}_\kappa, V_\kappa)$ and right $\prod_{v<\infty} \mathcal{O}_v^\times$-invariant.

Let r be the metaplectic representation of $G_{\mathbb{A}_\mathbb{Q}} \times (GL_2(\mathbb{A}_\mathbb{Q}) \times B_{\mathbb{A}_\mathbb{Q}}^\times)$ introduced in [M-N-1, §3]. We then define a theta series on $G_{\mathbb{A}_\mathbb{Q}} \times (GL_2(\mathbb{A}_\mathbb{Q}) \times B_{\mathbb{A}_\mathbb{Q}}^\times)$ by

$$\Theta_\kappa(g,h,h') := \sum_{(X,t)\in B^2\times\mathbb{Q}^\times} (r(g,h,h')\Phi)(X,t).$$

Here we put $\Phi := \prod_{v\le\infty} \Phi_v$ with

$$\Phi_v(X,t) := \begin{cases} \mathrm{ch}(\mathcal{O}_v^2 \times \mathbb{Z}_v^\times)(X,t) & (v \not| D^{-1}d_B), \\ \mathrm{ch}((\mathcal{O}_v \oplus \mathfrak{P}_v^{-1}) \times \mathbb{Z}_v^\times)(X,t) & (v | D^{-1}d_B), \\ \mathrm{ch}(t \in \mathbb{R}_+^\times) t^{\frac{\kappa+3}{2}} \sigma_\kappa(X_1 + X_2) e^{-2\pi t {}^t\bar{X}X} & (v = \infty), \end{cases}$$

where $\mathfrak{P}_v$ is a maximal ideal at v and $\mathrm{ch}(S)$ denotes the characteristic function for a set S. Then the Arakawa lifting is defined as follows:

$$\mathcal{S}_\kappa(\Gamma_0(D)) \times \mathcal{A}_\kappa(B_{\mathbb{A}_\mathbb{Q}}^\times) \ni (f,f') \mapsto \mathcal{L}(f,f')(g) := \iint_{(\mathbb{R}_+^\times)^2 (GL_2\times B^\times)_\mathbb{Q} \backslash (GL_2 \times B^\times)_{\mathbb{A}_\mathbb{Q}}} \overline{f(h)} \Theta_\kappa(g,h,h') f'(h') dh dh'.$$

This is a cusp form on $G_{\mathbb{A}_\mathbb{Q}}$ belonging to the minimal K_∞-type of a quaternionic discrete series representation at infinity (cf. [M-N-2, Theorem 3.3.2]), where K_∞ denotes a maximal compact subgroup of the real points of $Sp(1,1)$.

2 Main result

2.1

In general, a cuspidal automorphic form F on $G_{\mathbb{A}_\mathbb{Q}}$ admits a Fourier expansion as follows:

$$F(g) = \sum_{\xi \in B^- \backslash \{0\}} F_\xi(g) = \sum_{\xi \in B^- \backslash \{0\}} \sum_{\chi \in X_\xi} F_\xi^\chi(g).$$

Here

$$F_\xi(g) := \int_{B^- \backslash B^-_{\mathbb{A}_\mathbb{Q}}} F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-\mathrm{tr}(\xi x)) dx, \quad F_\xi^\chi(g) := \int_{\mathbb{R}_+^\times \mathbb{Q}(\xi)^\times \backslash \mathbb{A}_{\mathbb{Q}(\xi)}^\times} F_\xi(s 1_2 \cdot g) \chi(s)^{-1} ds,$$

where $B^- = \{x \in B \mid \bar{x} = -x\}$, ψ is the standard additive character on $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ and X_ξ denotes the set of Hecke characters of $\mathbb{A}_\mathbb{Q}^\times \mathbb{Q}(\xi)^\times \backslash \mathbb{A}_{\mathbb{Q}(\xi)}^\times$. Our main result is an explicit formula for F_ξ^χ when $F = \mathcal{L}(f,f')$.

2.2

To state the main theorem, we let $(f, f') \in S_\kappa(D) \times \mathcal{A}_\kappa$ be Hecke eigenforms. We further assume that f and f' are eigenforms for the "Atkin-Lehler involution": For every $p|D$,

$$f\left(h\begin{pmatrix}0 & 1\\ -p & 0\end{pmatrix}\right) = \epsilon_p f(h), \qquad f'(h'\Pi_p) = \epsilon'_p f'(h')$$

with ϵ_p, $\epsilon'_p \in \{\pm 1\}$. Here Π_p is a prime element of B_p. Note that $\mathcal{L}(f, f') \equiv 0$ unless $\epsilon_p = \epsilon'_p$ for any $p|D$.

For $p < \infty$, let $\mathfrak{a}_p := \begin{cases}\mathcal{O}_p & (p \not| d_B \text{ or } p|D)\\ \mathfrak{P}_p & (p|D^{-1}d_B)\end{cases}$. We say that $\xi \in B^- \setminus \{0\}$ is *primitive* if $\xi \in \mathfrak{a}_p \setminus p\mathfrak{a}_p$ for each finite prime p. We note that a Fourier coefficient F_ξ of an automorphic form F on $G_{\mathbf{A}_\mathbb{Q}}$ satisfies

$$F_\xi\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix} g\right) = F_{t\xi}(g) \quad (t \in \mathbb{Q}^\times).$$

It follows that the calculation of the Fourier expansion of F is reduced to that of F_ξ for primitive ξ.

2.3

We next introduce several notations. Let $\xi \in B^- \setminus \{0\}$ and d_ξ denote the discriminant of an imaginary quadratic field $E := \mathbb{Q}(\xi)$. We put

$$a := \begin{cases}2\sqrt{-n(\xi)}\sqrt{d_\xi} & (d_\xi \text{ is odd})\\ \sqrt{-n(\xi)}\sqrt{d_\xi} & (d_\xi \text{ is even})\end{cases}, \quad b := \xi^2 - \frac{a^2}{4}.$$

With these a and b we define an embedding $\iota_\xi : E^\times \hookrightarrow GL_2(\mathbb{Q})$ by

$$\iota_\xi(x + y\xi) = x \cdot 1_2 + y \cdot \begin{pmatrix}a/2 & b\\ 1 & -a/2\end{pmatrix} \quad (x, y \in \mathbb{Q}).$$

The completion E_∞ of E at ∞ is identified with $\mathbb{C}$ by

$$\delta_\xi : E_\infty \ni x + y\xi \mapsto x + y\sqrt{-n(\xi)} \in \mathbb{C} \ (x, y \in \mathbb{R}).$$

For a Hecke character $\chi = \prod_{v \le \infty} \chi_v$ of E, we define $w_\infty(\chi) \in \mathbb{Z}$ to be

$$\chi_\infty(u) = (\delta_\xi(u)/|\delta_\xi(u)|)^{w_\infty(\chi)} \qquad (u \in E_\infty^\times).$$

Furthermore, for each finite prime $p < \infty$, $i_p(\chi)$ denotes the exponent of the conductor of χ at p and

$$\mu_p := \frac{\mathrm{ord}_p(2\xi)^2 - \mathrm{ord}_p(d_\xi)}{2}.$$

We then have the following (cf. [M-N-2, Theorem 5.1.1]):

Proposition 2.1. $\mathcal{L}(f,f')^{\chi}_{\xi} \equiv 0$ *unless*

$$i_p(\chi) = 0 \text{ for any } p|d_B \text{ and } w_\infty(\chi) = -\kappa. \tag{1}$$

2.4 Statement of the main theorem.

In what follows, we assume that (1) holds. We need further notations to state the main theorem.

Define $\gamma_0 = (\gamma_{0,p})_{p\le\infty} \in GL_2(\mathbb{A}_\mathbb{Q})$ and $\gamma'_0 = (\gamma'_{0,p})_{p<\infty} \in B^\times_{\mathbb{A}_{\mathbb{Q},f}}$ as follows:

$$\gamma_{0,p} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p+i_p(\chi)} \end{pmatrix} & (p \not| D), \\ 1_2 & (p|D \text{ and } p \text{ is inert in } E), \\ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (p|D \text{ and } p \text{ ramifies in } E), \\ \begin{pmatrix} 1 & a/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} N(\xi)^{1/4} & 0 \\ 0 & N(\xi)^{-1/4} \end{pmatrix} & (p=\infty), \end{cases}$$

$$\gamma'_{0,p} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p+i_p(\chi)} \end{pmatrix} & (p \not| d_B), \\ \Pi_{B,p}^{-1} & (p|d_B). \end{cases}$$

Here $\Pi_{B,p}$ is a prime element of B_p for $p|d_B$. We furthermore define the following local constants:

$$C_p(f,\xi,\chi) := \begin{cases} p^{2\mu_p - i_p(\chi)}(1-\delta(i_p(\chi)>0)e_p(E)p^{-1}) & (p \not| d_B), \\ 1 & (p|D^{-1}d_B), \\ 2\epsilon_p & (p|D \text{ and } p \text{ is inert in } E), \\ (p+1)^{-1} & (p|D \text{ and } p \text{ ramifies in } E), \end{cases}$$

where $\delta(P) = 1$ (resp. 0) if a condition P holds (resp. does not hold), and

$$e_p(E) = \begin{cases} -1 & (p \text{ is inert in } E), \\ 0 & (p \text{ ramifies in } E), \\ 1 & (p \text{ splits in } E). \end{cases}$$

For $(f,f') \in S_\kappa(D) \times \mathcal{A}_\kappa$ we introduce their toral integrals with respect to a Hecke character χ of E:

$$P_\chi(f;h) := \int_{\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times} f(\iota_\xi(s)h)\chi(s)^{-1}ds, \; P_\chi(f';h') := \int_{\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times} f(sh')\chi(s)^{-1}ds,$$

where $(h,h') \in GL_2(\mathbb{A}_{\mathbf{Q}}) \times B^\times_{\mathbb{A}_{\mathbf{Q}}}$. Here we normalize the measure ds of $\mathbb{A}_E^\times$ so that

$$\mathrm{vol}(\mathcal{O}_{E_p}^\times) = \mathrm{vol}(E_\infty^{(1)}) = 1$$

for each finite prime p, where $\mathcal{O}_{E_p}$ is the p-adic completion of the integer ring of E and $E_\infty^{(1)}$ denotes the group of elements in E_∞ with norm 1.

We denote by $\mathbf{h}(E)$ and $\mathbf{w}(E)$ the class number of E and the number of roots of unity in E respectively. We are now able to state our main result (cf. [M-N-2, Proposition 2.4.1, Theorem 5.2.1]).

Theorem 2.2. (1) *When* $\xi = 0$, $\mathcal{L}(f,f')_\xi \equiv 0$.
(2) *Let* ξ *be a primitive element in* $B^- \setminus \{0\}$. *Suppose that* χ *satisfies (1) and that* $\epsilon_p = \epsilon'_p$ *for any* $p|D$. *We then have the following formula:*

$$\mathcal{L}(f,f')_\xi^\chi \left(g_0 \begin{pmatrix} \sqrt{\eta_\infty} & 0 \\ 0 & \sqrt{\eta_\infty}^{-1} \end{pmatrix} \right)$$
$$= 2^{\kappa-1} N(\xi)^{\kappa/4} \frac{\mathbf{w}(E)}{\mathbf{h}(E)} \cdot \left(\prod_{p<\infty} C_p(f,\xi,\chi) \right) \eta_\infty^{\kappa/2+1} \exp(-4\pi\sqrt{N(\xi)}\eta_\infty)\, \overline{P_\chi(f;\gamma_0)} P_\chi(f';\gamma'_0).$$

Here $\eta_\infty \in \mathbb{R}_+^\times$ *and* $g_0 = (g_{0,p})_{p<\infty} \in G_{\mathbb{A}_{\mathbf{Q}},f}$ *is given by*

$$g_{0,p} := \begin{cases} \mathrm{diag}(p^{i_p(\chi)-\mu_p}, p^{2(i_p(\chi)-\mu_p)}, 1, p^{i_p(\chi)-\mu_p}) & (p \nmid d_B), \\ 1_2 & (p|d_B) \end{cases}$$

Remark 2.3. (1) $\mathcal{L}(f,f')_\xi^\chi$ is determined by the value at $g_0 \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$ due to Sugano's result ([Su, Proposition 2-5]).
(2) Murase and Sugano have obtained a similar formula for "Kudla lifting", i.e. a theta lift from $U(1,1)$ to $U(2,1)$ (cf. [M-S]).

Remark 2.4. Let Π (resp. Π') be the base change to $GL_2(\mathbb{A}_E)$ of the Jacquet-Langlands lift π_f (resp. $\pi_{f'}$) of the automorphic representation attached to f (resp. f'). Waldspurger [Wa-2, Proposition 7] proved the following formula:

$$\frac{||P_\chi(f;\gamma_0)||^2}{\langle f,f\rangle} = C_{f,\chi} \cdot L(\Pi \otimes \chi^{-1}, \frac{1}{2}),$$
$$\frac{||P_\chi(f';\gamma'_0)||^2}{\langle f',f'\rangle} = C_{f',\chi} \cdot L(\Pi' \otimes \chi^{-1}, \frac{1}{2}),$$

where

$$C_{\varphi,\chi} = \frac{\sqrt{|d_\xi|}}{4\pi} \cdot \frac{\zeta(2)}{2L(\pi_\varphi, \mathrm{Ad}, 1)} \prod_{v:\text{"bad"}} C_{\varphi,\chi,v}$$

for $\varphi = f$ or f', with the adjoint L-function $L(\pi_\varphi, \mathrm{Ad}, s)$ of $\varphi = f$ or f' and where $C_{\varphi,\chi,v}$ is a ratio of a local period and L-values. We now remark that there does not appear $\frac{\sqrt{|d_\xi|}}{4\pi}$ in Waldspurger's formula [Wa-2, Proposition 7]. This is due to the difference between normalizations of Waldspurger's measure and ours for $\mathbb{A}_E^\times$.

Our theorem and Waldspurger's formula then imply

$$\frac{||\mathcal{L}(f,f')_\xi^\chi(g_{0,f})||^2}{\langle f,f\rangle\langle f',f'\rangle} = C_{f,f',\chi} L(\Pi\otimes\chi^{-1},\frac{1}{2})L(\Pi'\otimes\chi^{-1},\frac{1}{2})$$

with

$$C_{f,f',\chi} := 2^{2(\kappa-1)}N(\xi)^{\frac{\kappa}{2}}\frac{w(E)}{\mathbf{h}(E)}|\prod_{p<\infty}C_p(f,\xi,\chi)|^2\exp(-8\pi\sqrt{N(\xi)})\cdot C_{f,\chi}\cdot C_{f',\chi}.$$

It would be interesting to find a more explicit form of the constant $C_{f,f',\chi}$

3 Application (Non-vanishing lifts)

A general approach to verify the non-vanishing of theta lifts is to study their Petersson inner products. This technique is due to S. Rallis [R] and J. S. Li [L] etc. Via the Siegel-Weil formula (cf. [We]), it reduces the problem to the non-vanishing of a special value of the standard L-function for the preimages of the theta lifts. This method is useful when the Siegel-Weil formula is available, but this is not the case for our theta lifts.

Our approach to show the existence of the non-vanishing Arakawa lifts is to find examples of (f,f') with non-vanishing toral integrals involved in our formula for Fourier coefficients of the lifts (Theorem 2.2).

3.1 Result

We now specialize the situation. Let $B = \mathbb{Q} + \mathbb{Q}\cdot i + \mathbb{Q}\cdot j + \mathbb{Q}\cdot ij$ with $i^2 = j^2 = -1$ and $ij = -ji$. It is known that $d_B = 2$ and the class number of B is one. We note that $D = 1$ or 2. Let $\mathcal{O} = \mathbb{Z}1 + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}(1+i+j+k)/2$, and put $\xi = \frac{1}{2}i$, which is primitive. Suppose that the Hecke character χ of E is unramified at all finite places. This assumption implies that $w_\infty(\chi)$ is divisible by 4.

Proposition 3.1. *Let B, ξ and χ be as above.*
Then there exist Hecke eigenforms (f,f') such that

$$\overline{P_\chi(f;\gamma_0)}P_\chi(f';\gamma_0') \neq 0$$

for every $\kappa \geq \begin{cases} 12 & (D=1) \\ 8 & (D=2) \end{cases}$ *with* $4|\kappa$.

Theorem 3.2. *Let (f,f') be as above. Then $\mathcal{L}(f,f') \not\equiv 0$.*

3.2 Outline of the proof

Theorem 3.2 is a direct consquence of Proposition 3.1 and Theorem 2.2. This subsection is thus devoted to the outline of our proof of Proposition 3.1. If one finds a pair (f, f') such that $\overline{P_\chi(f;\gamma_0)}P_\chi(f';\gamma_0') \neq 0$, there exists a pair of Hecke eigenforms with the same property. This follows from the fact that $\mathcal{S}_\kappa(\Gamma_0(D))$ and $\mathcal{A}_\kappa(B_{\mathbf{A_Q}}^\times)$ have basis consisting of Hecke eigenforms.

To begin with, we find $f' \in \mathcal{A}_\kappa(B_{\mathbf{A_Q}}^\times)$ such that $P_\chi(f';\gamma_0') \neq 0$. Eichler's trace formula of Brandt matrices (cf. [E, Theorem 5]) says that

$$\dim_{\mathbf{C}} \mathcal{A}_\kappa(B_{\mathbf{A_Q}}^\times) = \begin{cases} \frac{\kappa+12}{12} & (\kappa \equiv 0 \mod 12), \\ \frac{\kappa-4}{12} & (\kappa \equiv 4 \mod 12), \\ \frac{\kappa+4}{12} & (\kappa \equiv 8 \mod 12) \end{cases}$$

and hence $\dim_{\mathbf{C}} \mathcal{A}_\kappa(B_{\mathbf{A_Q}}^\times) \neq 0$ if $\kappa \geq 8$. By a direct calculation we see that $P_\chi(f';\gamma_0') = \pm 1 \times \langle f'(1), v_\kappa^* \rangle v_\kappa$, where v_κ is a hightest weight vector of V_κ. Since the class number of B is one, $f' \mapsto f'(1)$ induces an isomorphism $\mathcal{A}_\kappa(B_{\mathbf{A_Q}}^\times) \simeq V_\kappa^{\mathcal{O}^\times}$. Let f' be an element of $\mathcal{A}_\kappa(B_{\mathbf{A_Q}}^\times)$ corresponding to $\sum_{u \in \mathcal{O}^\times} \sigma_\kappa(u) v_\kappa$. We then have $P_\chi(f';\gamma_0') \neq 0$.

Next let us find $f \in \mathcal{S}_\kappa(\Gamma_0(D))$ such that $P_\chi(f;\gamma_0) \neq 0$. We view f as a modular form on the complex upper half plane. A direct calculation shows that the non-vanishing of $P_\chi(f;\gamma_0)$ is reduced to that of

$$\begin{cases} f(\sqrt{-1}) & (D = 1), \\ f(\frac{1+\sqrt{-1}}{2}) & (D = 2). \end{cases}$$

When $D = 1$, set

$$f = \begin{cases} \Delta^{\kappa/12} & (\kappa \equiv 0 \mod 12), \\ \Delta^{(\kappa-4)/12} E_4 & (\kappa \equiv 4 \mod 12), \\ \Delta^{(\kappa-8)/12} E_4^2 & (\kappa \equiv 8 \mod 12), \end{cases}$$

where Δ denotes the Ramanujan delta function and E_4 the Eisenstein series of weight 4. We then have $P_\chi(f;\gamma_0) \neq 0$. When $D = 2$, set

$$f = \left(\frac{\eta^{16}(2z)}{\eta^8(z)} \right)^{\kappa/4}$$

with the Dedekind eta function η. Since $\eta^{16}(2z)/\eta^8(z) \in S_4(\Gamma_0(2))$ (cf. [C, §2.1]) and $\eta(z)$ has no zero on the upper half plane, we have $f \in \mathcal{S}_\kappa(\Gamma_0(2))$ and $P_\chi(f;\gamma_0) \neq 0$.

Remark 3.3. The level raising of the modular forms of level $D = 1$ introduced above to forms of level 2 also yields modular forms with non-vanishing toral integrals.

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