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CAP automorphic representations of inner forms of $Sp(2)$

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1 Introduction

For a reductive group G defined over a number field k, a cusp form ϕ on $G(\mathbb{A}_k)$ is said to be a CAP form if there exists an element ϕ' of an irreducible component of the residual spectrum such that ϕ and ϕ' share the same absolute values of Hecke eigenvalues at almost all places of k. In case of $GSp(2)$, Piatetski-Shapiro constructed the Saito-Kurokawa representations as examples of CAP forms [6], and Soudry determined the other CAP forms [7]. The residual spectrum for G is decomposed into the spaces of residues of Eisenstein series for the cuspidal representations of Levi subgroups of parabolic subgroups from the Langlands' spectral theory of Eisenstein series. Similarly the space of CAP forms for G should be decomposed into subspaces along parabolic subgroups of G. The Saito-Kurokawa representations and the examples constructed by Soudry are related to the Siegel and Klingen parabolic subgroup, respectively.

In this note, we treat the case that G is an inner form of $Sp(2)$. This G has only one unique proper parabolic subgroup P up to $G(k)$-conjugate, which corresponds to the Siegel parabolic subgroup of $Sp(2)$ via an inner twist. Therefore any irreducible component of the residual spectrum of G is associated to P. However, there exists an irreducible component of the space of the CAP forms which is associated to the Klingen parabolic subgroup or the Borel subgroup of $Sp(2)$. An aim in this note is to construct such an example of CAP forms associated to the Klingen parabolic subgroup.

Generally, for an elliptic Arthur parameter there should exist a non-zero set of irreducible automorphic representations corresponding to it, which is called its Arthur packet, and such packets should exhaust the discrete spectrum of space of automorphic forms in case of a quasisplit group [2]. However in case of non-quasisplit group like as G it is possible that an Arthur packet is empty. Since G and $Sp(2)$ share the Arthur parameters, there may exist an Arthur packet which shares an Arthur parameter with an irreducible component of the residual spectrum of $Sp(2)$ associated to the Klingen parabolic subgroup. Since it is not included in the residual spectrum of G, it is included in the space of the CAP forms associated to the Klingen parabolic subgroup. It is what we want. I tried to construct this CAP form by the analogy of the lift considered by Howe and Piatetski-Shapiro [3]. They used the theta lift from $O(2)$ to $Sp(2)$, which is included in the subspace associated to the Klingen parabolic subgroup of the residual spectrum. Therefore its analogy is the theta lift from the unitary group of one-dimensional skew-hermitian space over a quaternion algebra to G. The failure of the Hasse principle for skew-hermitian spaces causes the failure of the multiplicity one property for the theta lift. This phenomenon does not occur in case of $Sp(2)$. These multiplicities is consistent with the Arthur's multiplicity conjecture [2].

2 Inner forms of $Sp(2)$

Let k be a number field and $\mathbb{A}$ its adele ring. $|\ |_{\mathbb{A}}$ denotes the idele norm of $\mathbb{A}^\times$. For any place v of k, we write k_v for the completion of k at v and $|\ |_v$ for the v-adic norm. Let ψ be a non-trivial character of $\mathbb{A}$ which is trivial on k, and for any place v of k ψ_v denotes the v-component of ψ.

Let D be a quaternion division algebra over k. We write ν, τ and ι for the reduced norm, the reduced trace and the main involution of D, respectively. We write $D_- = \{x \in D \mid \tau(x) = 0\}$. Also we write S_D for the set of places v of k at which D is ramified, and s_D for the number of its elements, which is non-zero, finite and even. Let $W = D^{\oplus 2}$ be the free left module over D with rank two, and we equip it with a hermitian form $\langle\ ,\ \rangle$ given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1\, {}^\iota y_2 + y_1\, {}^\iota x_2 \qquad (x_1, x_2, y_1, y_2 \in D).$$

Let G be the unitary group of this form, so that

$$G = \left\{ g \in GL(2, D) \;\middle|\; g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^*g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Here we write ${}^*(a_{i,j}) = ({}^\iota a_{j,i})$ for $(a_{i,j}) \in \mathbb{M}(2, D)$. It can be regarded as a reductive group defined over k. It is non-quasisplit and an inner form of $Sp(2)$ with respect to a quadratic extension k' of k such that all $v \in S_D$ do not split fully in k'/k. Fix a k-parabolic subgroup P and its Levi factor M as

$$P = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in G \right\}, \quad M = \left\{ \begin{pmatrix} x & 0 \\ 0 & ({}^\iota x)^{-1} \end{pmatrix} \;\middle|\; x \in D^\times \right\},$$

P is the unique proper parabolic subgroup of G up to $G(k)$-conjugate and corresponds to the Siegel parabolic subgroup via an inner twist. We write again ν for the character of M corresponding to the reduced norm. U denotes the unipotent radical of P, so that

$$U = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \;\middle|\; y \in D_- \right\}.$$

H denotes either the quaternionic unitary group G or $Sp(2)$. $H(k)\backslash H(\mathbb{A})$ becomes locally compact Hausdorff space and has a non-zero $H(\mathbb{A})$-invariant measure up to scalars. Fix such a measure dh. Then the space $L^2(H(k)\backslash H(\mathbb{A}))$ of square-integrable functions on $H(k)\backslash H(\mathbb{A})$ is defined and the representation ρ of $H(\mathbb{A})$ on $L^2(H(k)\backslash H(\mathbb{A}))$ is defined by

$$\rho(h)f(g) = f(gh) \qquad (h, g \in H(\mathbb{A})).$$

This representation has an orthogonal decomposition;

$$L^2(H(k)\backslash H(\mathbb{A})) = L^2_{\text{disc}}(H) \oplus L^2_{\text{cont}}(H),$$

where $L^2_{\text{disc}}(H)$ is the maximal completely reducible closed subspace of $L^2(H(k)\backslash H(\mathbb{A}))$ and $L^2_{\text{cont}}(H)$ is its orthogonal complement. For $\phi \in L^2(H(k)\backslash H(\mathbb{A}))$ its constant term ϕ_Q along a k-parabolic subgroup $Q = M_Q U_Q$ of H is defined by

$$\phi_Q(g) = \int_{U_Q(k)\backslash U_Q(\mathbb{A})} \phi(ug)du \quad (g \in H(\mathbb{A}))$$

where du is a Haar measure of $U_Q(k)\backslash U_Q(\mathbb{A})$. $L_0^2(H)$ denotes the space of cuspidal elements of $L^2(H(k)\backslash H(\mathbb{A}))$, that is, elements whose constant terms along all the proper k-parabolic subgroups vanish. It is known that $L_0^2(H)$ is a $H(\mathbb{A})$-invariant closed subspace contained in $L^2_{\mathrm{disc}}(H)$. We write $L^2_{\mathrm{res}}(H)$ for its orthogonal complement in $L^2_{\mathrm{disc}}(H)$, which is called the residual spectrum. In this note, we call an irreducible component of $L^2_{\mathrm{disc}}(H)$ an irreducible automorphic representation of $H(\mathbb{A})$. Any irreducible automorphic representation π of $H(\mathbb{A})$ has a decomposition into a restricted tensor product $\pi \simeq \bigotimes_v' \pi_v$.

From the Langlands' spectral theory of Eisenstein series, the residual spectrum of H has an orthogonal decomposition of the form

$$L^2_{\mathrm{res}}(H) = \bigoplus_Q L^2_{\mathrm{res}}(H)_Q.$$

Here Q runs over the set of k-parabolic subgroup of H up to $H(k)$-conjugate and $L^2_{\mathrm{res}}(H)_Q$ is the space of residues of Eisenstein series associated to the cuspidal representations of a Levi factor of Q.

3 Residual spectrum of G and $Sp(2)$

The irreducible decomposition of residual spectrum of G and $Sp(2)$ has been determined ([4], [8]). We review it in this section.

First, we see the case of $Sp(2)$. $Sp(2)$ has three standard parabolic subgroups; $P_S = M_SU_S$, $P_K = M_KU_K$ and B which are the Siegel, Klingen and Borel parabolic subgroup respectively. Therefore the residual spectrum of $Sp(2)$ has a decomposition of three spaces;

$$L^2_{\mathrm{res}}(Sp(2)) = L^2_{\mathrm{res}}(Sp(2))_{P_S} \oplus L^2_{\mathrm{res}}(Sp(2))_{P_K} \oplus L^2_{\mathrm{res}}(Sp(2))_B.$$

Theorem 3.1 ([4]). *Let k be a totally real number field. For a standard parabolic subgroup Q of $Sp(2)$, $L^2_{\mathrm{res}}(Sp(2))_Q$ is isomorphic to the direct sum of the following irreducible representations. Each occurs with multiplicity one.*

(P_S case) *The unique irreducible quotient $J_{P_S}^{Sp(2)}(\pi)$ of $\mathrm{Ind}_{P_S(\mathbb{A})}^{Sp(2,\mathbb{A})}(\pi|\det|_{\mathbb{A}}^{1/2})$. Here π runs over irreducible self-dual cuspidal representations of $M_S(\mathbb{A}) \simeq GL(2,\mathbb{A})$ whose standard L-functions $L(s,\pi)$ do not vanish at $s = 1/2$.*

(B case) *The trivial representation $\mathbf{1}_{Sp(2)}$, and the theta lift $R(T)$ from the trivial representation of the orthogonal group $O(T,\mathbb{A})$ under the Weil representation $\omega_{T,\psi}$. Here T runs over isometry classes of 2-dimensional non-degenerate quadratic spaces over k.*

(P_K case) *The unique irreducible quotient $J_{P_K}^{Sp(2)}(k',\theta,\psi)$ of $\mathrm{Ind}_{P_K(\mathbb{A})}^{Sp(2,\mathbb{A})}(\omega_{k'/k}|\cdot|_{\mathbb{A}}\otimes\pi_\psi(\theta))$. Here k'/k is a non-trivial quadratic extension of k. $\pi_\psi(\theta)$ denotes the endoscopic lift from an automorphic character θ of $U_{k'/k}(1,\mathbb{A})$ which is ψ-generic [5].*

Next, we see the case of G. Since P is the unique proper k-parabolic subgroup of G up to $G(k)$-conjugate, $L^2_{\mathrm{res}}(G) = L^2_{\mathrm{res}}(G)_P$.

Theorem 3.2 ([8]). *Let k be a totally real number field. The irreducible components of the residual spectrum of G consist of the following representations.*

(1) The trivial representation $\mathbf{1}_G$,

(2) The unique irreducible quotient $J_P^G(\pi)$ *of* $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi|\nu|_{\mathbb{A}}^{1/2})$. *Here* π *runs over the set of infinite dimensional irreducible self-dual cuspidal representations of* $M(\mathbb{A})$ *whose standard* L*-functions* $L(s,\pi)$ *do not vanish at* $s = 1/2$, *and*

(3) The theta lift $R(V)$ *from the trivial representation of* $G(V_{\mathbb{A}})$ *under the Weil representation* $\omega_{V,\psi}$. *Here* V *runs over the set of local isometry classes of (-1)-hermitian right* D*-spaces of dimension one, and* $G(V)$ *is the unitary group of* V.

In the case (1) and (2), the multiplicity of each representation is one. In the case (3), the multiplicity of each representation is $2^{\#S_R-2}$.

The representations appearing in the two theorems above should be associated to Arthur parameters from the point of view of Arthur's conjecture. We will describe the associated Arthur parameters. Suppose the existence of the hypothetical global Langlands group $\mathcal{L}_k$ of k. By an Arthur parameter is meant a continuous homomorphism $\phi : \mathcal{L}_k \times SL(2,\mathbb{C}) \to {}^LH$ where ${}^LH$ is the L-group of H such that

(i) writing $p_k : \mathcal{L}_k \to W_k$ for the conjectural homomorphism and $p_2 : {}^LH \to W_k$ the canonical projection where W_k is the Weil group of k, $p_2 \circ \phi = p_k$,

(ii) its restriction to $\mathcal{L}_k$ is a Langlands parameter with bounded image, and

(iii) its restriction to $SL(2,\mathbb{C})$ is analytic.

The Arthur's conjecture says that for any elliptic Arthur parameter, a set of irreducible automorphic representations of $H(\mathbb{A})$ which is called its Arthur packet is assigned (this packet is possible to be empty), and such packets exhausts the irreducible components of $L^2_{\mathrm{disc}}(H)$.

When $H = G$ or $Sp(2)$, ${}^LH$ is isomorphic to $SO(5,\mathbb{C}) \times W_k$. We realize $SO(5,\mathbb{C})$ as the special orthogonal group defined by the quadratic form over $\mathbb{C}$ given by

$$I_5 = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

ρ denotes the standard representation of $SL(2,\mathbb{C})$. The image of n-th symmetric power $\mathrm{Sym}^n\rho$ can be embedded in $SO(n+1,\mathbb{C})$ for even integer n, $Sp((n+1)/2,\mathbb{C})$ for odd integer n. The Arthur parameters associated to (the packets containing) automorphic representations appearing in Theorem 3.1 and 3.2 is described as follows.

- $\mathbf{1}_{Sp(2)}$ and $\mathbf{1}_G$ correspond to

$$\phi = \mathbf{1}_5 \otimes \mathrm{Sym}^4\rho \times p_k.$$

- $J_{P_S}^{Sp(2)}(\pi)$ and $J_P^G(\pi)$ correspond to

$$\phi = \left((\phi_\pi \otimes \rho) \oplus (\mathbf{1}_{\mathcal{L}_k} \otimes \mathbf{1}_{SL(2,\mathbb{C})})\right) \times p_k.$$

  Here ϕ_π is the Langlands parameter associated to π, whose image is contained by $SL(2,\mathbb{C})$.

- $R(T)$ and $R(V)$ correspond to

$$\phi = \left((\mathrm{Ind}_{W_{k'}}^{W_k} \mathbf{1}_{W_{k'}} \otimes \mathbf{1}_{SL(2,\mathbb{C})}) \oplus (\omega_{k'/k} \otimes \mathrm{Sym}^2\rho)\right) \circ p_k \times p_k.$$

  Here $\omega_{k'/k}$ is the quadratic character of W_k associated to k'/k. Remark the image of $\mathrm{Ind}_{W_{k'}}^{W_k} \mathbf{1}_{W_{k'}}$ is contained by $O(2,\mathbb{C})$.

- $J_{P_K}^{Sp(2)}(k',\theta,\psi)$ corresponds to

$$\phi = \left((\phi_{\pi_\psi(\theta)} \otimes \mathbf{1}_{SL(2,\mathbb{C})}) \oplus (\omega_{k'/k} \otimes \mathrm{Sym}^2\rho)\right) \circ p_k \times p_k. \tag{3.1}$$

  Here $\phi_{\pi_\psi(\theta)}$ is the Langlands parameter associated to $\pi_\psi(\theta)$, whose image is contained by $O(2,\mathbb{C})$.

When we observe the above description of Arthur parameters we have a natural question. Is the Arthur packet corresponding to the last Arthur parameter empty in case of $H = G$? The answer is no. In fact, the packet is realized in $L_0^2(G)$. An aim in this note is to construct it using theta correspondence.

4 Theta correspondence from unitary group of skew-hermitian spaces

Let $V = V_\xi$ be the one-dimensional skew-hermitian space over (D,ι) defined by $\xi \in D_-$. Let $\delta = \det V_\xi = \nu_D(\xi) = -\xi^2 \mod (k^\times)^2$ and $k' = k(\xi) \simeq k(\sqrt{-\delta})$. $G(V)$ and $G_0(V)$ denote the unitary group and special unitary group of V, respectively. Then $G_0(V)$ is isomorphic to the norm torus for the quadratic extension k'/k. Since $(G(V), G)$ is a dual reductive pair we can consider the Weil representation $\omega_{V,\psi}$ of $G(V_\mathbb{A}) \times G(\mathbb{A})$.

Let $\chi = \prod_v \chi_v$ be a non-trivial character of $G_0(V_k)\backslash G_0(V_\mathbb{A})$ and put $S_\chi = \{v \mid \chi_v^2 = 1\}$. Since

$$\mathrm{Ind}_{G_0(V_\mathbb{A})}^{G(V_\mathbb{A})}\chi \subset L^2_{\mathrm{disc}}(G(V)) = L^2(G(V))$$

we want to construct an irreducible automorphic representation of $G(\mathbb{A})$ by the theta lift from $\mathrm{Ind}_{G_0(V_\mathbb{A})}^{G(V_\mathbb{A})}\chi$. However $\mathrm{Ind}_{G_0(V_\mathbb{A})}^{G(V_\mathbb{A})}\chi$ is not irreducible. Therefore the description of its irreducible decomposition is needed. As for its local component we have

$$\mathrm{Ind}_{G_0(V_v)}^{G(V_v)}\chi_v \simeq \begin{cases} \tilde{\chi}_v^+ \oplus \tilde{\chi}_v^- & v \in S_\chi \cap S_D{}^c, \\ \tilde{\chi}_v & \text{otherwise}. \end{cases}$$

Here $\tilde{\chi}_v^+$, $\tilde{\chi}_v^-$ are characters not isomorphic to each other, and $\tilde{\chi}_v$ is χ_v or a two-dimensional irreducible representation. Fix a $\gamma_0 \in O(k', N_{k'/k})\backslash SO(k', N_{k'/k})$ and embed γ_0 in $G(V_v) \simeq$

$O(k'_v, N_{k'_v/k_v})$ for all $v \notin S_D$. For $v \in S_\chi \cap S_D{}^c$ we may assume $\tilde{\chi}_v^+(\gamma_0) = 1$, which characterizes $\tilde{\chi}_v^+$ and $\tilde{\chi}_v^-$. Then an irreducible component of the global induction is of form,

$$\tau = (\otimes_{v\in S}\tilde{\chi}_v^-)\otimes(\otimes'_{v\in S_\chi\setminus S}\tilde{\chi}_v^+)\otimes(\otimes'_{v\notin S_\chi}\tilde{\chi}_v)$$

for some finite set $S \subset S_\chi \cap S_D{}^c$. For any $v \in S_\chi \cap S_D{}^c$ define

$$\mathcal{S}^{\pm}(V_v) = \{f \in \mathcal{S}(V_v) \mid f(\gamma_0\,\cdot) = \pm f\}$$

where $\mathcal{S}(V_v)$ is the space of Schwartz-Bruhat functions on V_v.

The theta lift from τ is defined as follows.

(1) Case of $\chi^2 \neq 1$

We adopt the usual definition as that of theta kernel and theta integral;

$$\theta(f,h,g) = \sum_{x\in V_k}\omega_{V,\psi}(h,g)f(x) \quad (g \in G(\mathbb{A}), h \in G(V_{\mathbb{A}}), f \in \mathcal{S}(V_{\mathbb{A}})),$$

$$\theta(f,g) = \int_{G_0(V_k)\backslash G_0(V_{\mathbb{A}})}\theta(f,h,g)\chi(h)dh.$$

The theta lift $\Theta(V,\chi,S)$ from τ is defined by

$$\Theta(V,\chi,S) = \{\theta(f,g) \mid f \in \mathcal{S}^S(V_{\mathbb{A}})\}$$

where $\mathcal{S}^S(V_{\mathbb{A}}) = (\otimes_{v\in S}\mathcal{S}^-(V_v))\otimes(\otimes'_{v\in S_\chi\setminus S}\mathcal{S}^+(V_v))\otimes(\otimes'_{v\notin S_\chi}\mathcal{S}(V_v))$.

(2) Case of $\chi^2 = 1$

In this case τ is one-dimensional. The theta integral is defined by

$$\theta(f,g) = \int_{G(V_k)\backslash G(V_{\mathbb{A}})}\theta(f,h,g)\tau(h)dh,$$

where $\theta(f,h,g)$ is the same one as above. The theta lift $\Theta(V,\chi,S)$ from τ is defined by

$$\Theta(V,\chi,S) = \{\theta(f,g) \mid f \in \mathcal{S}(V_{\mathbb{A}})\}.$$

To state the main theorem in this note we prepare the definition of CAP representation with respect to the Klingen parabolic subgroup.

Definition 4.1. *Let $\pi \simeq \otimes'_v \pi_v$ be an irreducible cuspidal representation of $G(\mathbb{A})$. We say that π is a CAP representation with respect to P_K if there exists an irreducible cuspidal representation $\sigma \simeq \otimes'_v \sigma_v$ and a quadratic character $\omega = \prod_v \omega_v$ of $\mathbb{A}^\times$ such that for almost all v, π_v is isomorphic to a composition factor of $\mathrm{Ind}_{P_K(k_v)}^{Sp(2,k_v)}(\sigma_v \otimes \omega_v|\cdot|_v)$.*

Theorem 4.2. *1. $\Theta(V,\chi,S)$ is non-zero, irreducible and cuspidal.*

2. $\Theta(V,\chi,S)$ is a CAP representation with respect to P_K.

3. For the restricted tensor product $\Theta(V,\chi,S) \simeq \otimes'_v \Theta(V,\chi,S)_v$, $\Theta(V,\chi,S)_v$ can be determined as a representation for any v. (As for the description of local factors, see the next section.)

We shall mention the proof of (2) only. In quasisplit case, as for the theta lift from $O(k', N_{k'/k})$ we have the following diagram.

$$\begin{array}{llll} & & \text{quotient of } \mathrm{Ind}_{P_K(\mathbb{A})}^{Sp(2,\mathbb{A})}(\theta(k',\chi) \otimes \omega_{k'/k}|\cdot|_{\mathbb{A}}) & : Sp(2) \\ & \nearrow^{\Theta^{Sp(2)}} & & \\ O(k', N_{k'/k}) : & \chi \xmapsto[\text{Shalika-Tanaka}]{} & \theta(k',\chi) : \text{cuspidal} & : SL(2). \end{array}$$

$\Theta^{Sp(2)}(\chi)$ is included in $L^2_{\mathrm{res}}(Sp(2))_{P_K}$ if $\chi^2 \neq 1$. Since $\Theta(V,\chi,S)$ is an inner form analogue of $\Theta^{Sp(2)}(\chi)$, its local components is isomorphic to the those of $\Theta^{Sp(2)}(\chi)$ for almost all places. This implies (2) in the theorem.

$\Theta(V,\chi,S)$ is the automorphic representation associated to the Arthur parameter given by (3.1). Adams conjectured the correspondent of Arthur parameters under the theta correspondence [1]. According to this conjecture the correspondence of Arthur parameters for our theta lift is given by Φ in the following diagram.

$$\begin{array}{llll} & & (\mathrm{Ind}_{W_{k'}}^{W_k}\chi \otimes \mathbf{1}) \oplus (\omega_{k'/k} \otimes \mathrm{Sym}^2\rho) & : Sp(2) \\ & \nearrow^{\Phi} & & \\ O(k', N_{k'/k}) : & \chi \otimes \mathbf{1} \xmapsto[\text{Shalika-Tanaka}]{} & \mathrm{Ind}_{W_{k'}}^{W_k}\chi \otimes \mathbf{1} & : SL(2). \end{array}$$

Here the Langlands parameter associated to χ is also written by χ.

5 Multiplicity conjecture

We write $m(\Theta(V,\chi,S))$ for the multiplicity of $\Theta(V,\chi,S)$ in $L^2_{\mathrm{disc}}(G)$.

Proposition 5.1.

$$m(\Theta(V,\chi,S)) \geq \begin{cases} 2^{\#(S_\chi \cap S_D)-1} & S_D \not\subset S_\chi, \\ 2^{\#S_D - 2} & S_D \subset S_\chi. \end{cases}$$

This result is caused by the failure of Hasse's principle for skew-hermitian spaces. (The last statementt in Theorem 3.2 is also caused by the same reason.) We can find one-dimensional skew-hermitian spaces V_1 and V_2 which are locally isometric, but not globally isometric. Then $\Theta(V_1,\chi,S)$ and $\Theta(V_1,\chi,S)$ have the same local components for all places. On the other hand, by the calculation of Fourier coefficient we have that they are different as spaces of automorphic forms. Therefore the multiplicity one property does not hold.

Arthur also conjectured the multiplicity for $L^2_{\mathrm{disc}}(H)$ [2]. We shall compare the proposition above with the conjecture. To describe the conjecture the description of the local Arthur packets for the Arthur parameter given by (3.1) is needed. For any v, Π_{ϕ_v} denotes the local Arthur packet for the Arthur parameter

$$\phi = \left((\mathrm{Ind}_{W_{k'}}^{W_k}\chi \otimes \mathbf{1}) \oplus (\omega_{k'/k} \otimes \mathrm{Sym}^2\rho)\right) \circ p_k \times p_k.$$

(1) Case of $v \notin S_D$

$$\Pi_{\phi_v} = \begin{cases} \{\theta(V_v^{\pm}, \tilde{\chi}_v)\} & \chi_v^2 \neq 1 \text{ and } \delta_v \neq -1, \\ \{\theta(\mathbb{H}_v, \tilde{\chi}_v)\} & \chi_v^2 \neq 1 \text{ and } \delta_v = -1, \\ \{\theta(V_v^{\pm}, \tilde{\chi}_v^{\pm})\} & \chi_v^2 = 1 \text{ and } \delta_v \neq -1, \\ \{\theta(\mathbb{H}_v, \tilde{\chi}_v^{\pm})\} & \chi_v^2 = 1 \text{ and } \delta_v = -1. \end{cases}$$

Here $V_v^{\pm}$ is the two-dimensional quadratic space over k_v with determinant δ_v and Hasse invariant ± 1, $\mathbb{H}_v$ is the two-dimensional hyperbolic space over k_v, and $\theta(V_v, \lambda_v)$ denotes the Howe correspondent of the representation λ_v of $G(V_v)$. The correspondent from $\tilde{\chi}_v^-$ is supercuspidal and the others are of the form of a quotient of $\mathrm{Ind}_{P_K(k_v)}^{Sp(2,k_v)}(\omega_{k'_v/k_v}|\cdot|_v \otimes \tau_v)$ for some irreducible representation τ_v of $SL(2, \mathbb{A})$.

(2) Case of $v \in S_D$

$$\Pi_{\phi_v} = \begin{cases} \{\theta(V_v, \chi_v), \theta(V_v, \chi_v^{-1})\} & \chi_v^2 \neq 1, \\ \{\theta(V_v, \chi_v)\} & \chi_v^2 = 1. \end{cases}$$

Elements of Π_{ϕ_v} are supercuspidal except for $\chi_v = 1$.

Writing $\mathcal{S}_\phi$ for the S-group for ϕ,

$$\mathcal{S}_\phi = \begin{cases} <s_1> \simeq \mathbb{Z}/2\mathbb{Z} & \chi^2 \neq 1, \\ <s_1> \oplus <s_2> \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \chi^2 = 1. \end{cases}$$

A pairing $\langle\, , \rangle_v$ on $\mathcal{S}_\phi \times \Pi_{\phi_v}$ is defined as follows.

(1) Case of $v \notin S_D$

$$\langle s_1, \theta(V_v^{\eta}, \tilde{\chi}_v^{\varepsilon})\rangle_v = \eta \cdot 1,$$
$$\langle s_2, \theta(V_v^{\eta}, \tilde{\chi}_v^{\varepsilon})\rangle_v = \varepsilon \cdot 1,$$

where we regard $\mathbb{H}_v = V_v^+$ and $\tilde{\chi}_v = \tilde{\chi}_v^+$.

(2) Case of $v \in S_D$

$$\langle s_1, \theta(V_v, \chi_v^{\kappa})\rangle_v = \kappa \qquad \chi_v^2 \neq 1,$$

$$\langle s, \theta(V_v, \chi_v)\rangle_v = \begin{cases} 2 & s = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \chi_v^2 = 1.$$

The global Arthur packet Π_ϕ is included in $\bigotimes_v' \Pi_{\phi_v}$. Arthur conjectured that for $\pi \in \Pi_\phi$ its multiplicity in $L^2_{\mathrm{disc}}(G)$ is given by

$$\#\mathcal{S}_\phi^{-1} \textstyle\sum_{s \in \mathcal{S}_\phi} \varepsilon(s) \langle s, \pi \rangle$$

where ε is a character of $\mathcal{S}_\phi$ and $\langle s, \pi\rangle = \prod_v \langle s, \pi_v\rangle_v$. In our case, $\varepsilon = \mathbf{1}$ and

$$\#\mathcal{S}_\phi^{-1} \textstyle\sum_{s \in \mathcal{S}_\phi} \varepsilon(s) \langle s, \theta(V, \chi, S)\rangle = \begin{cases} 2^{\#(S_\chi \cap S_D) - 1} & S_D \not\subset S_\chi, \\ 2^{\# S_D - 2} & S_D \subset S_\chi. \end{cases}$$

Therefore this conjecture is consistent with Proposition 5.1.

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