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Blow up of the Cohen–Kuznetzov operator and an automorphic problem of K. Saito

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The main aim of my talk is to present a solution of one automorphic problem proposed by Kyoji Saito in 1991. This problems can be briefly formulated as follows: *to continue automorphic forms to an extension of the classical homogeneous domain of type IV.*

1. Set up. To give the exact formulation of the problem we have to introduce some notions. The type IV domains or the homogeneous domains of the orthogonal type are important in the theory of singularities, in the algebraic geometry and in the theory of Kac–Moody Lie algebras of Borcherds type. The general set-up is the following. Let L be an integral lattice with a quadratic form of signature $(2, n)$ $(n \geq 3)$,

$$\mathcal{D}_L = \{[\mathbf{w}] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\mathbf{w}, \mathbf{w}) = 0,\ (\mathbf{w}, \overline{\mathbf{w}}) > 0\}^+,$$

where "plus" denotes a connected component, is the associated n-dimensional Hermitian domain of type IV in the Cartan's classification. We denote by $\mathrm{O}^+(L)$ the index 2 subgroup of the integral orthogonal group $\mathrm{O}(L)$ preserving $\mathcal{D}_L$. A modular variety of the orthogonal type is the quotient space $\mathcal{F}_L(\Gamma) = \Gamma\backslash\mathcal{D}_L$ where Γ is a subgroup of $\mathrm{O}^+(L)$ of finite index. This is a n-dimensional quasi-projective variety. The most important geometric examples of such varieties are the moduli spaces of polarised K3 surfaces (dim = 19), the moduli spaces of polarised Abelian and Kummer surfaces (dim = 3), the moduli space of of Enriques surfaces (dim = 10), the moduli spaces of polarised irreducible symplectic varieties (dim = 20). The same modular varieties appear in the theory of singularities, in the theory of Frobenius structures, in some variants of mirror symmetry, etc. Using modular forms one can define birational invariants of the modular varieties, in particular its geometric genus or its Kodaira dimension (see [Fr], [G2], [GHS1], [GHS2], [Vo]). The automorphic forms on type IV domains are also related to partition functions of the different models in the string theory. The Fourier-Jacobi coefficients of the modular forms of the orthogonal type, the Jacobi modular forms, are the characters of the affine Lee algebras. It would be interesting to consider one parameter deformations of all these staff.

In 1983 K. Saito and E. Looijenga introduced extended period domains in the theory of deformations of special surface singularities (see [Sa1], [Lo]).

This is a one parameter extension of the homogeneous domain of type IV. By definition we have

$$\mathcal{D}_L^t = \{[\mathbf{w}] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\mathbf{w}, \overline{\mathbf{w}}) > |(\mathbf{w}, \mathbf{w})|\}^+. \qquad (1)$$

It is clear that $O^+(L \otimes \mathbb{R})$ acts on this domain. $\mathcal{D}_L^t$ is the period domain of e-hyperbolic weight systems in the K. Saito theory.

One can give a definition of modular forms on this non-classical domain (we call them *t-modular forms*) similar to the definition of the modular forms on $\mathcal{D}_L$.

Definition. A t-modular form of weight k and character χ for a subgroup $\Gamma < O^+(L)$ is a holomorphic function $F\colon (\mathcal{D}_L^t)^\bullet \to \mathbb{C}$ on the affine cone $(\mathcal{D}_L^t)^\bullet$ over $\mathcal{D}_L^t$ such that

$$F(\alpha v) = \alpha^{-k} F(v) \quad \forall \alpha \in \mathbb{C}^* \quad \text{and} \quad F(gv) = \chi(g)F(v) \quad \forall g \in \Gamma. \qquad (2)$$

If we take the domain $\mathcal{D}_L$ instead of $\mathcal{D}_L^t$ we get the Borcherds definition of the modular forms on type IV domain (see [Bo]).

We denote the linear space of the t-modular forms on $(\mathcal{D}_L^t)^\bullet$ of weight k and character χ for Γ by $M_k^t(\Gamma, \chi)$. By $M_k(\Gamma, \chi)$ we denote the finite dimensional space of the usual modular forms on $\mathcal{D}_L^\bullet$. We note that the dimension of the space $M_k^t(\Gamma, \chi)$ is not finite (see below).

Let L be of signature $(2, n)$ ($n \geq 3$) and u be a unimodular isotropic vector (i.e., there exists $v \in L$ such that $(u, v) = 1$). The tube realisation $\mathcal{H}_u$ of the homogeneous domain $\mathcal{D}_L$ at the standard 0-dimensional cusp determined by u is the following "upper half-space" defined by the hyperbolic sublattice $L_1 = u^\perp/\mathbb{Z}u$ of L:

$$\mathcal{H} = \mathcal{H}(L_1) = \{Z \in L_1 \otimes \mathbb{C} \mid (\operatorname{Im} Z, \operatorname{Im} Z) > 0\}^+$$

where $^+$ denotes a connected component of the domain (see [G1] for details). In a similar way we obtain a tube realisation of $\mathcal{D}_L^t$:

$$\mathcal{H}^t = \mathcal{H}^t(L_1) = \{(Z;t) \in (L_1 \otimes \mathbb{C}) \times \mathbb{C} \mid (\operatorname{Im} Z, \operatorname{Im} Z) > \frac{|t| - \operatorname{Re} t}{2}\}^+ \qquad (3)$$

(see [Ao]). The relation with the projective model $\mathcal{D}_L^t$ is given by the following correspondence

$$(Z;t) \mapsto v = \begin{pmatrix} \frac{t-(Z,Z)}{2} \\ Z \\ 1 \end{pmatrix} \in \mathcal{D}_L^t, \qquad t = (v,v) \text{ if } v \in \mathcal{D}_L^t.$$

The fractional linear action of $O^+(L \otimes \mathbb{R})$ on the tube domain $\mathcal{H}^t$ and the automorphic factor $j(g; Z, t)$ of this action are defined as follows

$$g \cdot v = j(g; Z, t) \begin{pmatrix} \frac{t'-(Z',Z')}{2} \\ Z' \\ 1 \end{pmatrix} = j(g; Z, t) g\langle (Z, t) \rangle.$$

Example. *The time form.* The parameter $t = (v, v)$ (*"the time"*) for $v \in (\mathcal{D}_L^t)^\bullet$ is the first example of the t-modular forms. According to our definition this is a modular form of weight -2 with respect to $\mathrm{O}^+(L)$ because t is a holomorphic function on $(\mathcal{D}_L^t)^\bullet$ of homogeneous degree 2 which is invariant with respect to $\mathrm{O}^+(L\otimes\mathbb{R})$. In principle we can make our definition of modular forms more restrictive adding the condition that F should be invariant only with respect to a discrete subgroup of $\mathrm{O}^+(L\otimes\mathbb{R})$. In any case the "time" modular form t is a rather natural object in the Saito's theory.

The most natural modular group in the theory of the automorphic forms on type IV domain is the so-called *stable orthogonal group.* For every non-degenerate even integral lattice we denote by $L^* = \mathrm{Hom}(L, \mathbb{Z})$ its dual lattice. The finite group $A_L = L^*/L$ carries a discriminant quadratic form q_L and a discriminant bilinear form b_L, with values in $\mathbb{Q}/2\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ respectively. We define

$$\widetilde{\mathrm{O}}(L) = \{g \in \mathrm{O}(L) \mid g|_{A_L} = \mathrm{id}\}, \qquad \widetilde{\mathrm{O}}^+(L) = \widetilde{\mathrm{O}}(L) \cap \mathrm{O}^+(L).$$

In the case of indefinite quadratic forms we usually have that $\mathrm{O}^+(L)/\widetilde{\mathrm{O}}^+(L) \cong \mathrm{O}(L^*/L)$ (see [Nik]).

2. The problem on the modular forms with a parameter and the main result. Now we can give the exact formulation of the automorphic problem of K. Saito.

Problem. *Let $F(Z) \in M_k(\widetilde{\mathrm{O}}^+(L), \chi)$. To construct a non trivial extension $F(Z; t) \in M_k^t(\widetilde{\mathrm{O}}^+(L), \chi)$ such that*

$$F(Z; t)|_{t=0} = F(Z).$$

Let assume for simplicity that L contains two hyperbolic planes

$$L = 2U \oplus L_0 \quad \text{where} \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L_1 = U \oplus L_0. \tag{4}$$

L_0 is an even integral negative definite lattice of rank n_0, L_1 is a hyperbolic lattice and $\operatorname{sign}(L) = (2, n_0 + 2)$. The modular group $\widetilde{\mathrm{O}}^+(L)$ acting on $\mathcal{H} = \mathcal{H}(L_1)$ contains all translations $Z \to Z + l$ ($l \in L_1$). Therefore the Fourier expansion at the standard 0-dimensional cusp defined by the first copy of U in L of any $\widetilde{\mathrm{O}}^+(L)$-modular form F has the following form

$$F(Z) = \sum_{l \in L_1^*,\ (l,l) \geq 0} a(l) \exp(2\pi i (l, Z)). \tag{5}$$

We note that the stable orthogonal group of a lattice with two hyperbolic planes is an analogue of the full modular group $\mathrm{SL}_2(\mathbb{Z})$ or $\mathrm{Sp}_2(\mathbb{Z})$. The

Fake Monster Lie algebra discovered by R. Borcherds is determined by the Borcherds modular form Φ_{12} (see [Bo]) which is a modular form with respect to the orthogonal group of the even unimodular lattice $II_{2,26} = 2U \oplus 3E_8(-1)$. For an unimodular lattice $\widetilde{\mathrm{O}}^+(L) = \mathrm{O}^+(L)$. The moduli space of the K3 surfaces of degree $2d$ is the modular varietie of the stable orthogonal group of the lattice $L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d\rangle$ of signature $(2,19)$. The modular forms with respect to $\widetilde{\mathrm{O}}^+(L_{2d})$ play the crucial role in the solution of the classical problem about the general type of the moduli spaces of K3 surfaces (see [GHS1] and [Vo]).

The main result of the talk is the following theorem which gives the answer on the K. Saito problem formulated above.

Main Theorem. *Let $L = 2U \oplus L_0$ be a lattice of signature $(2, n_0+2)$ where $n_0 = \operatorname{rank} L_0 > 0$. Let*

$$F(Z) = \sum_{l\in L_1^*,\ (l,l)\geq 0} a(l)\exp(2\pi i(l,Z)) \in M_k(\widetilde{\mathrm{O}}^+(L),\chi)$$

where $k > \frac{n_0}{2}$. Then

$$F(Z;t) = F(Z) + \sum_{l\in L_1^*}\sum_{\nu\geq 1} \frac{a(l)\,(l,l)^\nu\,(-\pi^2 t)^\nu}{(k-\frac{n_0}{2})\dots(k-\frac{n_0}{2}+\nu-1)\nu!}\exp(2\pi i(l,Z))$$

is a t-modular form of type $M_k^t(\widetilde{\mathrm{O}}^+(L),\chi)$.

3. The differential operator of Cohen–Kuznetzov. The function $F(Z;t)$ can be obtained by action on $F(Z)$ of a formal power series of quasi-modular differential operators. We make an illustration of this method in the case of $\mathrm{SL}_2(\mathbb{Z})$. It is known that $\mathrm{SL}_2(\mathbb{Z})/\{\pm E_2\}$ is isomorphic to $SO^+(L)$ where $L = U \oplus \langle 2\rangle$ is of signature $(2,1)$. This example corresponds to $n_0 = -1$ in our notations. So we are in a degenerate situation: a modular form for $\mathrm{O}(2,1)$-group has no Fourier–Jacobi expansion which is one of the main tools of our proof. Nevertheless we can explain the main idea using SL_2. In particular in this case our method gives a new construction of the Cohen–Kuznetzov differential operator (see [Co], [Ku], [EZ], [CMZ]).

We consider the quasi-modular Eisenstein series of weight 2

$$G_2(\tau) = -D(\log(\eta(\tau))) = -\frac{1}{24} + \sum_{n\geq 1}\sigma_1(n)q^n, \qquad q = e^{2\pi i\tau}$$

where

$$D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}.$$

The graded ring $M_*[G_2]$ of the quasi-modular forms is generated by G_2 over the graded ring $M_* = \bigoplus_{k\geq 0} M_k(\mathrm{SL}_2(\mathbb{Z}))$ of the modular forms.

A Jacobi type form of weight k and index m is a holomorphic function $\phi : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$ which satisfies

$$\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = e^{2\pi i m \frac{cz^2}{c\tau+d}} (c\tau+d)^k \phi(\tau,z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(see [EZ], [KZ]). We denote the space of all such functions by $JT_{k,m}$. For $m=0$ the Jacobi type form of index 0 is a formal power series over the rings of modular forms: $JT_{k,0} = M_{k+*}[[z]]$. We can define the following operator of *the automorphic correction* (see [G3]) for $\phi \in JT_{k,m}$:

$$\mathrm{AC}_m : \phi(\tau,z) \mapsto e^{-8\pi^2 m G_2(\tau) z^2} \phi(\tau,z) = \sum_{n\ge 0} f_{k+n}(\tau) z^n \in JT_{k,0} \qquad (6)$$

where $f_{k+n}(\tau) \in M_{k+n}(\mathrm{SL}_2(\mathbb{Z}))$. The operator AC_m gives us one line proof of the well known fact (see [EZ]) that the Taylor coefficients of Jacobi type forms are quasi-modular forms. Let us put the following question:
to find a differential operator from M_k to $JT_{k,m}$ "dual" to the operator of the automorphic correction AC_m.

In the ring $M_*[G_2]$ we fix two natural operators: multiplication by G_2 and the differential operator D

$$G_2\bullet, \ D : M_*[G_2] \to M_*[G_2].$$

We have $D(G_2) = -2G_2^2 + \frac{5}{6}G_4$. Therefore

$$D(G_2\bullet) \equiv -2G_2^2 \bullet + G_2 \cdot D \mod M_*. \qquad (7)$$

This means that the difference is an operator which transforms M_* into M_*. The standard quasi-modular operators are

$$D_k = D + 2kG_2\bullet \ : M_k \to M_{k+2},$$

$$D_{k,n} = D_{k+2(n-1)} \circ \cdots \circ D_{k+2} \circ D_k : M_k \to M_{k+2n}.$$

Proposition 2. *The major quasi-modular part $E_{k,n}$ of $D_{k,n}$ is given by the following sum*

$$E_{k,n} = \sum_{\nu=0}^{n} \frac{n!\,\Gamma(k+n)}{\nu!(n-\nu)!\,\Gamma(k+\nu)} (2G_2)^{n-\nu} D^{\nu} : M_k \to M_{k+2n}.$$

(We use Γ-functions in the formulation in order to apply the same calculus in the case of negative or half integral weights.)

Proof. Using only (!) the relation (7) we obtain we obtain

$$D_{k+2l}(E_{k,l}) = E_{k,l+1} + \frac{5}{3}G_4 \cdot E_{k,l-1} \equiv E_{k,l+1} \mod M_*$$

where the degree of $E_{k,l-1}$ in G_2 and D is equal to $l-1$.

Now we can construct the operator dual to the operator of the automorphic correction AC_m.

Corollary 3. *We set*

$$\nabla(X) = 1 + \sum_{n\geq 1} \frac{E_{k,n}}{n!\,\Gamma(k+n)} X^n = e^{2G_2X}\nabla_D(X)$$

where

$$\nabla_D(X) = \sum_{\nu\geq 0} \frac{D^\nu}{\nu!\,\Gamma(k+\nu)} X^\nu.$$

If $X = -4\pi^2 m z^2$ then the last series defines the operator from $M_k(\mathrm{SL}_2(Z))$ to $JT_{k,m}$

$$\nabla_D(X)(f) = \sum_{\nu\geq 0} \frac{D^\nu(f)}{\nu!\,\Gamma(k+\nu)} X^\nu \in JT_{k,m}.$$

Proof. The result follows form the diagram

$$\begin{array}{ccc} M_k & \xrightarrow{\nabla(X)} & JT_{k,0} \\ & {\scriptstyle \nabla_D(X)}\searrow & \downarrow {\scriptstyle e^{-2G_2X}} \\ & & JT_{k,1}. \end{array}$$

Remarks. $\nabla_D(X)$ coincides with the Cohen–Kuznetzov differential operator. Corollary 3 gives a new simple construction of this operator. In [G3], [G4] we introduced two types of the automorphic corrections of Jacobi forms using the logarithmic derivatives of the Dedekind eta-function $\eta(\tau)$ (the Jacobi type correction) and of the Weierstrass function (the full Jacobi correction). The second correction gives us another type of differential operators on the Jacobi forms of one or several variables. We are planning to consider them in a separate paper.

We note that we can apply the same purely algebraic arguments to automorphic forms of negative weights and to quasi-modular forms.

Corollary 4. *Let $k \in \mathbb{Z}_{<0}$ and $f(\tau)$ be an automorphic form of weight k. Then*

$$\sum_{\nu\geq|k|+1} \frac{D^\nu(f)}{\nu!\,\Gamma(k+\nu)} X^{\nu-|k|-1} \in JT_{|k|+2,m}$$

is a Jacobi type form.

Proof. We take into account that $\Gamma(k+\nu)$ has a pole for $\nu = 0, 1, \ldots, |k|$.

The first non-zero Taylor coefficient of a Jacobi type form of weight k (positive, negative or zero) is a SL_2-automorphic form of the same weight.

Therefore Corollary 4 gives us a simple algebraic proof of the classical Bol's identity:

$$\left(D^{|k|+1}f\right)|_{|k|+2}M = \left(D^{|k|+1}f\right)$$

for any meromorphic modular form of negative weight k. We note that in the case of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ or for half-integral weights there are no principle changes in the results considered in this section. The case of the quasi-modular form G_2 is more interesting.

Corollary 5. *For any $l \geq 1$ we have that $Q_l(G_2) \in M_{2l}(\mathrm{SL}_2(\mathbb{Z}))$ where*

$$Q_l(G_2) = \sum_{\nu=1}^{l} \frac{l!(l-1)!}{\nu!(\nu-1)!(l-\nu)!}(2G_2)^{l-\nu}D^{\nu-1}(G_2) - \frac{(l-1)!}{2}(2G_2)^l.$$

In particular $Q_1(G_2) = 0$, $Q_2(G_2) = D(G_2) + 2G_2^2$, etc.

Proof. Q_l is the major quasi-modular part of the differential operator $D_{2(l-1)} \circ \cdots \circ D_4 \circ (D + 2G_2^2)$ acting on $G_2(\tau)$. In the proof of Proposition 2 we have to change the constant in the first operator D_2. It gives us a translation of the weights from 2 two 0 in the formula for $E_{k,n}$, i.e.,

$$D_{2l} \circ Q_l = Q_{l+1} + l(l-1)\frac{5}{3}G_4 \cdot Q_{l-1} \equiv Q_{l+1} \mod M_*.$$

The same translation we have to make in the operator $\nabla_D(X)$ which gives us a Jacobi type form of weight 0.

Corollary 6. *We have*

$$1 - 2\sum_{l\geq 1}\frac{Q_l(G_2)}{l!(l-1)!}X^l = e^{2G_2X}\nabla'_D(X)(G_2)$$

where

$$\nabla'_D(X)(G_2) = 1 - 2\sum_{\nu\geq 1}\frac{D^{\nu-1}(G_2)}{\nu!(\nu-1)!}X^\nu \in JT_{0,m}$$

and $X = (2i\pi mz)^2$.

Remark. The Jacobi type form $\nabla'_D(X)(G_2)$ was constructed in [Ao, §5] using the recurrent calculation like in [EZ]. Our approach is different.

4. Blow up of the operator $\nabla_D(X)$. Let us assume that L contains two hyperbolic planes and $F \in M_k(\widetilde{\mathrm{O}}^+(L), \chi)$. The modular variety $\widetilde{\mathrm{O}}^+(L)\backslash \mathcal{D}(L)$ has the cusps of dimension 0 and 1. The Fourier expansion of F at the standard zero dimensional cusp is given in (5). The Fourier-Jacobi expansion is determined by the splitting (4) (see [G1] for details). The same type of Fourier-Jacobi expansion can be determined for an extended t-modular form $F(Z;t) \in M_k^t(\widetilde{\mathrm{O}}^+(L), \chi)$

$$F(Z;t) = \phi_0(\tau;t) + \sum_{m\geq 1}\phi_{k,m}(\tau,\mathfrak{z};t)e^{2\pi i m\omega}, \quad Z = \begin{pmatrix}\omega\\ \mathfrak{z}\\ \tau\end{pmatrix} \in \mathcal{H},\ \mathfrak{z} \in L_0 \otimes \mathbb{C}.$$

The Fourier-Jacobi coefficient $\phi_{k,m}(\tau,\mathfrak{z};t)$ is a Jacobi form of weight k and index m with many abelian variables $\mathfrak{z}\in L_0\otimes\mathbb{C}$ with a parameter t, i.e., it is a Jacobi form in τ and $\mathfrak{z}$ and a Jacobi type form with respect to t. The only difference with our definition of Jacobi type forms is that the variable t is a modular parameter of degree 2 with respect to the $\mathrm{SL}_2(\mathbb{Z})$-component of the Jacobi group

$$t\mapsto \frac{t}{(c\tau+d)^2},\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})\subset\Gamma^J(L_0).$$

Definition. A Jacobi form of weight k and index m with parameter t with respect to an even integral negative definite lattice L_0 is a holomorphic function $\phi(\tau,\mathfrak{z};t)$ on $\mathbb{H}_1\times(L_0\otimes\mathbb{C})\times\mathbb{C}$ which satisfies two functional equations

$$\phi(\frac{a\tau+b}{c\tau+d},\frac{\mathfrak{z}}{c\tau+d};\frac{t}{(c\tau+d)^2})=(c\tau+d)^k\exp\big(\pi i m\frac{c(t-(\mathfrak{z},\mathfrak{z}))}{c\tau+d}\big)\phi(\tau,\mathfrak{z};t)$$

for any $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$,

$$\phi(\tau,\mathfrak{z}+\lambda\tau+\mu;t)=\exp\big(\pi i m((\lambda,\lambda)\tau+2(\lambda,\mathfrak{z}))\big)\phi(\tau,\mathfrak{z};t),\qquad \forall\,\lambda,\mu\in L_0.$$

Moreover the form $\phi(\tau,z;t)$ is holomorphic at infinity

$$\phi(\tau,\mathfrak{z};t)=\sum_{\substack{n\in\mathbb{Z},\,l\in L_0^*\\ 2nm+(l,l)\ge 0}} a(n,l;t)\exp\big(2\pi i(n\tau+(l,\mathfrak{z}))\big).$$

We denote the space of all such Jacobi forms by $J^t_{k,m}(L_0)$. If we put $t=0$ we get the definition of the usual Jacobi forms $J_{k,m}(L_0)$. For details see [G1] where one more interpretation of Jacobi forms is given: the complete function $\widetilde{\phi}_{k,m}(Z)=\phi_{k,m}(\tau,\mathfrak{z})e^{2\pi i m\omega}$ is a modular form on $\mathcal{H}$ with respect to the parabolic subgroup $\Gamma^J(L_0)$ (the Jacobi group of L_0). The same interpretation we have for $J^t_{k,m}(L_0)$. Similar to (6) we define the automorphic correction of Jacobi t-forms

$$\phi(\tau,\mathfrak{z};t)\mapsto e^{-4\pi^2 mG_2(\tau)t}\phi(\tau,\mathfrak{z};t)=\sum_{n\ge 0}\psi_{k+2n}(\tau,\mathfrak{z})\,t^n\in J_{k+2*,m}(L_0)[[t]].$$

In [G1] we constructed some examples of modular forms of singular weight $k=\frac{n_0}{2}$. This is the minimal possible weight of modular forms with respect to congruence subgroups of $\mathrm{O}^+(L)$. If $F\in M_{\frac{n_0}{2}}(\widetilde{\mathrm{O}}^+(L))$ then it has the Fourier expansion of a rather special type

$$F(Z)=\sum_{l\in L_1^*,\ (l,l)=0} a(l)\exp(2\pi i(l,Z)). \tag{8}$$

The modular forms of singular weight belong to the kernel of the $O^+(L_1 \otimes \mathbb{R})$-invariant heat operator

$$H = 2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\omega} + S_0[\frac{\partial}{\partial\mathfrak{z}}]$$

where S_0 is the matrix of the negative definite quadratic form of L_0 (see [G1]). We add the variable ω in the classical heat operator because we consider Jacobi forms as functions on the tube domain $\mathcal{H}$. Using this operator we can define a quasi-modular operator

$$H_k = H - 8\pi^2 m(2k - n_0)G_2\bullet \,:\, J_{k,m}(L_0) \to J_{k+2,m}(L_0).$$

The proof of SL_2-invariance of H_k is similar to D_k. The Heisenberg invariance follows from the fact that H is $O^+(L_1 \otimes \mathbb{R})$-invariant. We set $G_2' = -8\pi^2 m G_2$. Then we have

$$H(G_2'\,\bullet) \equiv -2(G_2')^2 \bullet + G_2' H \mod J_{*,m}(L_0).$$

Without any problems and *without any additional calculation* we can generalise the operator $\nabla_D(X)$ to the case of Jacobi forms in many variables. Our construction of $\nabla_D(X)$ is based only on the structure constants of the non-commutative ring of the quasi-modular differential operators generated by D and G_2. The permutation of the generators is defined by (7). Now we can consider a similar algebra with other structure constants. We make the following changes

$$D \mapsto H, \quad k \mapsto k - \frac{n_0}{2}, \quad G_2 \mapsto G_2' = -8\pi^2 m G_2.$$

Therefore we obtain the following reformulations of Proposition 2 and Corollary 3 (no additional proof!):

$$E_{k,n}^{(H)} = \sum_{\nu=0}^{n} \frac{n!\,\Gamma(k - \frac{n_0}{2} + n)}{\nu!(n-\nu)!\,\Gamma(k - \frac{n_0}{2} + \nu)} (2G_2')^{n-\nu} H^\nu$$

defines the operator $E_{k,n}^{(H)} : J_{k,m}(L_0) \to J_{k+2n,m}(L_0)$. if $k - \frac{n_0}{2} > 0$. Moreover we have the following analogue of $\nabla_D(X)$:

$$\nabla_H(t) = \sum_{\nu \geq 0} \frac{H^\nu}{\Gamma(k - \frac{n_0}{2} + \nu)\,\nu!} \left(\frac{t}{4}\right)^\nu \tag{9}$$

transforms $\phi(\tau, \mathfrak{z}) \in J_{k,m}(L_0)$ $(k > \frac{n_0}{2})$ in a Jacobi form of the same type with parameter t

$$\nabla_H(t)(\widetilde{\phi}) = \sum_{\nu \geq 0} \frac{H^\nu(\widetilde{\phi})}{\Gamma(k - \frac{n_0}{2} + \nu)\,\nu!} \left(\frac{t}{4}\right)^\nu \in J_{k,m}^t(L_0) \tag{10}$$

where $\widetilde{\phi}(Z) = \phi(\tau, \mathfrak{z})e^{2\pi i m\omega}$. In the case of SL_2-modular forms Corollary 4 gives us a variant of $\nabla_D(X)$ operator for negative weight k. In the orthogonal case we have to change the weight 0 with the singular weight $\frac{n_0}{2}$. Let assume that $k - \frac{n_0}{2}$ is a negative integer and $\phi \in J_{k,m}$ is a (nearly holomorphic) Jacobi form of weight k. Then similar to Corollary 4

$$\nabla_{H,k}(t)(\widetilde{\phi}) = \sum_{\nu \geq 1+\frac{n_0}{2}-k} \frac{H^\nu(\widetilde{\phi})}{\Gamma(k - \frac{n_0}{2} + \nu)\,\nu!} \left(\frac{t}{4}\right)^{\nu-(1+\frac{n_0}{2}-k)} \in J^t_{n_0-k+2,m}(L_0).$$

Therefore we have an analogue of the Bol's identity for Jacobi forms of weight k such that $k - \frac{n_0}{2}$ is negative integral:

$$(H^{\frac{n_0}{2}-k+1}(\widetilde{\phi}))|_{n_0-k+2}M = H^{\frac{n_0}{2}-k+1}(\widetilde{\phi}), \qquad \forall M \in \mathrm{SL}_2(\mathbb{Z}). \tag{11}$$

We note that this identity reflects the structure of the formal non-commutative ring generated by two elements with a relation of type (7) and no additional calculation are needed.

Now we fix a Jacobi form $\phi(\tau, \mathfrak{z}) \in J_{k,m}(L_0)$ of weight $k > \frac{n_0}{2}$. Then

$$\widetilde{\phi}(Z) = \phi(\tau, \mathfrak{z})\, e^{2\pi i m\omega} = \sum_{\substack{l=(n,l_0,m)\in L_1^* \\ (l,l)\geq 0}} a(l)\exp\bigl(2\pi i(l, Z)\bigr).$$

Let us calculate the action of the operator (10). First we note that

$$H^\nu\bigl(a(l)e^{2\pi i(l,Z)}\bigr) = (2\pi i)^{2\nu}(l,l)^\nu a(l), \qquad \forall\, l \in L_1^*.$$

Then we use the following Bessel function of order n

$$J_n(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!\,\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{n+2\nu}$$

which is a regular function in $z \in \mathbb{C}$. We put $t = X^2$. Then we have

$$\Gamma(k - \frac{n_0}{2})\nabla_H(X^2)(\widetilde{\phi}) = \sum_{\substack{l=(n,l_0,m)\in L_1^* \\ (l,l)=0}} a(l)\, e^{2\pi i(l,Z)}$$

$$+\Gamma(k - \frac{n_0}{2}) \sum_{\substack{l=(n,l_0,m)\in L_1^* \\ (l,l)>0}} a(l)\, \frac{J_{k-\frac{n_0}{2}-1}(2\pi\sqrt{(l,l)}\,X)}{(\pi\sqrt{(l,l)}\,X)^{k-\frac{n_0}{2}-1}}\, e^{2\pi i(l,Z)}.$$

The function $e^{2\pi i x z} J_\mu(4\pi v\sqrt{x})$ decreases faster than any fixed power of x. Therefore the last series converges for any Z.

5. Proof of the main theorem. The main idea of the proof of the theorem is to apply $\nabla_H(X^2)$ to a modular form of non singular weight

$$F(Z) = \phi_0(\tau) + \sum_{m\geq 1} \phi_{k,m}(\tau,\mathfrak{z})e^{2\pi i m\omega} \in M_k(\widetilde{\mathrm{O}}^+(L)), \quad (k > \frac{n_0}{2}).$$

More exactly we consider

$$F(Z;X^2) = \phi_0(\tau) + \sum_{m\geq 1} \Gamma(k-\frac{n_0}{2})\nabla_H(X^2)\big(\phi_{k,m}(\tau,\mathfrak{z})e^{2\pi i m\omega}\big). \tag{12}$$

Then

$$F(Z;X^2) =$$

$$\sum_{\substack{l\in L_1^*\\(l,l)=0}} a(l)\, e^{2\pi i(l,Z)} + \Gamma(k-\frac{n_0}{2}) \sum_{\substack{l\in L_1^*\\(l,l)>0}} c(l)\, \frac{J_{k-\frac{n_0}{2}-1}(2\pi\sqrt{(l,l)}\,X)}{(\pi\sqrt{(l,l)}\,X)^{k-\frac{n_0}{2}-1}}\, e^{2\pi i(l,Z)}.$$

This series converges for any Z in the homogeneous domain $\mathcal{H}$ because the Bessel functions have a good asymptotic (see the previous section). According to (10) and (12) $F(Z;X^2)$ is invariant with weight k with respect to the action of the Jacobi group $\Gamma^J(L_0)$. We can also calculate its Fourier expansion

$$F(Z;X^2) = F(Z) + \sum_{l\in L_1^*}\sum_{\nu\geq 1} \frac{a(l)\,(l,l)^\nu\,(-\pi^2X^2)^\nu}{(k-\frac{n_0}{2})\dots(k-\frac{n_0}{2}+\nu-1)\nu!}\, e^{2\pi i(l,Z)}$$

where $l=(n,l_0,m)\in L_1^*$ and $Z=(\tau,\mathfrak{z},\omega)$. Therefore $F(Z;X^2)$ is invariant with respect to the transformation $V:(\tau,\mathfrak{z},\omega)\to(\omega,\mathfrak{z},\tau)$. But the stable orthogonal group $\widetilde{O}^+(L)$ is generated by $\Gamma^J(L_0)$ and V (see [G1]).

The same arguments work if we consider a modular form $F(Z)$ with a character χ. In this case the Fourier-Jacobi coefficients are invariant with the character $\chi|_{\Gamma^J(L_0)}$ and the permutation on n and m in the Fourier coefficient $a(n,l_0,m)$ gives us the factor $(-1)^k\chi(V)$.

6. Comments. At the end of this talk we would like to make some remarks and comments.

1. Characters. If L contains two hyperbolic planes (the case of $\mathrm{SL}_2(\mathbb{Z})$-Jacobi forms) and its rank over $\mathbb{F}_3$ and $\mathbb{F}_2$ is at least 5 or 6 respectively, then $\widetilde{\mathrm{O}}^+(L)$ has the only non trivial character det (see [GHS3]). Therefore non-trivial characters appear mainly for Siegel modular forms (see [G5]).

2. The congruence subgroups. The case of the Jacobi forms with respect to the Hecke congruence subgroup $\Gamma_0(N)$ corresponds to the lattice of type $U\oplus U(N)\oplus L_0$. The main theorem is also valid in this case. The proof is nearly the same because the construction of the differential operators works for any subgroups. It is interesting to consider the t-extension of the reflective modular forms, e.g., the Siegel modular forms with the simplest divisor (see [GN2], [GH] and [GC]). These modular forms are related to special modular varieties and to partition functions of the CHL models in the string theory.

3. The singular weight. The weight $k = \frac{n_0}{2}$ is called singular. This is the minimal possible weight of modular forms with respect to an orthogonal group of signature $(2, n_0 + 2)$ (see [G1]). In this case the Fourier expansion of $F(Z)$ is very special (see (8)). (For SL_2 a modular form of singular weight is a constant.) We cannot obtain a t-deformation of $F(Z)$ of singular weight using the method based on the operator $\nabla_H(X^2)$ because the modular forms of singular weight belong to the kernel of the extended heat operator H. In particular we cannot deform the Siegel theta-constant $\Delta_{1/2}$ (see [GN2]) or the Borcherds function Φ_{12} with respect to $\mathrm{O}^+(II_{2,26})$ (see [Bo]). For such modular forms we are planning to give another constructions.

4. The example of H. Aoki. The first example of t-modular forms was constructed in [Ao]. He applied the lifting construction of [G1] to some special Jacobi forms from $J^t_{k,1}(L_0)$. More exactly, let $L = 2U \oplus L_0$ and $\phi \in J_{k,m}(L_0)$. Then the multiplication by the Jacobi type form $\nabla'_D(t)(G_2)$ of weight 0 defines a t-extension of Jacobi forms

$$\phi \mapsto \phi^{(t)} = \phi \cdot \nabla'_D(t)(G_2) \in J^t_{k,m}(L_0).$$

Then one can apply the lifting construction of [G1] to this function

$$\mathrm{Lift}(\phi^{(t)}) \in M^t_k(\widetilde{\mathrm{O}}^+(L)).$$

In [Ao] it was proved for an unimodular L_0 but the same result is true for any even integral $2U \oplus L_0$. The lifting works for the Jacobi theta-series of singular weight. In particular it gives us a t-extension of the modular form of singular weight (the simplest modular forms) introduced in [G1] but the Borcherds form of singular weight Φ_{12} for $II_{2,26}$ and the Siegel theta-constant $\Delta_{1/2}$ are not of this type. For a fixed k the liftings $\mathrm{Lift}(\phi)$ form only a small subspace (the Maass subspace) of the space $M_k(\widetilde{\mathrm{O}}^+(L))$. The main theorem of this talk gives a nontrivial t-deformation for any modular form of non-singular weight. In particular for the liftings we have two different t-extensions because the t-modular form from the main theorem does not coincide in general with the lifting of $\phi^{(t)}$.

5. T-generalisation. The t-extension proposed in this paper have a more general variant. We can say that the present t-extension is defined by the root system A_1 because $t = X^2$. We can propose a formal series of differential operators which will give a T-extension of modular forms where the parameter space T is defined by a root system of a semi-simple Lie algebra.

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