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PERIODS OF AUTOMORPHIC FORMS AND L-VALUES

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1. Introduction

This article is a proceeding of an expository talk, in which I discussed a possibility to relate a period integral to some L-values.

Let G be a connected reductive algebraic group defined over an algebraic number field k. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Let $H \subset G$ be a connected algebraic subgroup. Let $\theta : H(\mathbb{A}) \to \mathbb{C}^\times$ a character which is trivial on $H(k)$.

Definition 1.1. An integral of the form

$$\mathcal{P}_{H,\theta}(\varphi) = \int_{H(k)\backslash H(\mathbb{A})} \varphi(h)\overline{\theta(h)}\, dh$$

is called an (H, θ)-period.

Remark 1.2. Some people say that the terminology "period" is inadequate in this context.

The automorphic representation π is said to be (H, θ)-distinguished if $\mathcal{P}_{H,\theta}(\phi) \neq 0$ for some $\varphi \in \pi$. If there is no fear of confusion, we simply say that π is distinguished.

If $\dim_{\mathbb{C}}\mathrm{Hom}_{H_v}(\pi_v, \theta_v) < \infty$ for all v, then it is believed that the period integral $\mathcal{P}_{H,\theta}(\varphi)$ is related to some L-values. More precisely, we are looking for a formula, which is of the form

$$\frac{|\mathcal{P}_{H,\theta}(\varphi)|^2}{\langle\varphi,\varphi\rangle} = \frac{1}{\sharp\mathcal{S}_\pi}\cdot C_H \frac{\Delta_G}{\Delta_H}\frac{L(1/2,\pi,\rho)}{L(1,\pi,\mathrm{Ad})}\prod_v l_v(\varphi_{1,v},\bar{\varphi}_{1,v}).$$

Here, $\mathcal{S}_\pi$ is a certain finite group depending only on the L-packet of π. The constant Δ_H (reps. Δ_G) is a product of certain L-value determined by the motive (see Gross [8]) of reductive part of H (resp. G). The constant C_H is a constant depending only on the choice of the local and global Haar measure on $H(\mathbb{A})$. The representation ρ is a finite dimensional symplectic representation of ${}^LG$. The local homomorphism $l_v \in \mathrm{Hom}_{H_v\times H_v}(\pi_v \times \tilde{\pi}_v, \theta \times \bar{\theta})$ should depends only on local data. We call this kind of equation a "period formula" in this manuscript. A typical (conjectural) example of a period formula is the Gross-Prasad type conjecture for orthogonal groups (joint work with Ichino [15]), which we recall in the next section.

2. Gross-Prasad type conjectures

Let k be a global field with $\mathrm{char}(k) \neq 2$. Let (V_1, Q_1) and (V_0, Q_0) be quadratic forms over k with rank $n+1$ and n, respectively. We assume $n \geq 2$. When $n = 2$, we also assume (V_0, Q_0) is not isomorphic to the hyperbolic plane over k. We denote the special orthogonal group of (V_i, Q_i) by G_i $(i = 0, 1)$. In this section, the subscript i will indicate either 0 or 1, except for some obvious situation. We assume there is an embedding $\iota : V_0 \hookrightarrow V_1$ of quadratic spaces. Then we have an embedding of the corresponding special orthogonal group $\iota : G_0 \hookrightarrow G_1$. We regard G_0 as a subgroup of G_1 by this embedding. The group $G_i(k_v)$ of k_v-valued points of G_i is denoted by $G_{i,v}$.

For even-dimensional quadratic form (V, Q), the discriminant field K_Q is defined by $K_Q = k(\sqrt{(-1)^{\dim V/2} \det Q})$. We put $K = K_{Q_0}$ (resp. $K = K_{Q_1}$), if $\dim V_0$ is even (resp. if $\dim V_1$ is even). We call K the discriminant field for the pair (V_1, V_0). Let $\chi = \chi_{K/k}$ be the Hecke character associated to K/k by the class field theory.

Put

$$\Delta_{G_{i,v}} = \begin{cases} \zeta_v(2)\zeta_v(4)\cdots\zeta_v(2l) & \text{if } \dim V_i = 2l+1, \\ \zeta_v(2)\zeta_v(4)\cdots\zeta_v(2l-2)\cdot L_v(l,\chi) & \text{if } \dim V_i = 2l, \end{cases}$$

$$\Delta_{G_i} = \begin{cases} \zeta(2)\zeta(4)\cdots\zeta(2l) & \text{if } \dim V_i = 2l+1, \\ \zeta(2)\zeta(4)\cdots\zeta(2l-2)\cdot L(l,\chi) & \text{if } \dim V_i = 2l. \end{cases}$$

Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible square-integrable automorphic representation of $G_i(\mathbb{A})$. There is a canonical inner product $\langle *, * \rangle$ on forms on $G_i(k)\backslash G_i(\mathbb{A})$ defined by

$$\langle \varphi_i, \varphi_i' \rangle = \int_{G_i(k)\backslash G_i(\mathbb{A})} \varphi_i(g_i)\overline{\varphi_i'(g_i)}dg_i,$$

where dg_i is the Tamagawa measure on $G_i(\mathbb{A})$. We choose a Haar measure $dg_{i,v}$ on $G_{i,v}$ for each v. There exist a positive numbers C_i such that $dg_i = C_i \prod_v dg_{i,v}$, when the right hand side is well-defined. Since $\pi_{i,v}$ is an unitary representation, there is an inner product $\langle *, * \rangle_v$ on $\pi_{i,v}$ for any place v of k. We put $\|\varphi_{i,v}\| = \langle \varphi_{i,v}, \varphi_{i,v} \rangle_v^{1/2}$, as usual. There exists a positive constant C_{π_i} such that $\langle \varphi_i, \varphi_i' \rangle = C_{\pi_i} \prod_v \langle \varphi_{i,v}, \varphi_{i,v}' \rangle_v$ for any decomposable vectors $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$ and $\varphi_i' = \otimes_v \varphi_{i,v}' \in \otimes_v \pi_{i,v}$

We fix maximal compact subgroups $\mathcal{K}_1 = \prod_v \mathcal{K}_{1,v} \subset G_1(\mathbb{A})$ and $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v} \subset G_0(\mathbb{A})$ such that $[\mathcal{K}_0 : \mathcal{K}_1 \cap \mathcal{K}_0] < \infty$. We choose a $\mathcal{K}_i$-finite decomposable vector $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$. In this section, we consider the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ where $\varphi_1|_{G_0}$ is the restriction of φ_1 to $G_0(\mathbb{A})$.

Let S be a finite set of bad places containing all archimedean places. We may and do assume the following conditions hold for $v \notin S$:

(U1) G_i is unramified over k_v.
(U2) $\mathcal{K}_{i,v}$ is a hyperspecial maximal compact subgroup of $G_{i,v}$.
(U3) $\mathcal{K}_{0,v} \subset \mathcal{K}_{1,v}$.
(U4) $\pi_{i,v}$ is an unramified representation of $G_{i,v}$.
(U5) The vector $\varphi_{i,v}$ is fixed by $\mathcal{K}_{i,v}$ and $\|\varphi_{i,v}\| = 1$.
(U6) $\int_{\mathcal{K}_{i,v}} dg_{i,v} = 1$.

When G_i is unramified over k_v, we shall say that a Haar measure on $G_{i,v}$ is the standard Haar measure if the volume of a hyperspecial maximal compact subgroup is 1. Thus the condition (U6) means that the measure $dg_{i,v}$ is the standard Haar measure.

The L-group ${}^LG_i$ of G_i is a semi-direct product $\hat{G}_i \rtimes W_k$. Here, W_k is the Weil group of k and

$$\hat{G}_i = \begin{cases} \mathrm{Sp}_l(\mathbb{C}) & \text{if } \dim V_i = 2l+1, \\ \mathrm{SO}(2l, \mathbb{C}) & \text{if } \dim V_i = 2l. \end{cases}$$

We denote by st the standard representation of ${}^LG_i$. The completed standard L-function for π_i is denoted by $L(s, \pi_i, \mathrm{st})$ for an irreducible automorphic representation π_i of $G_i(\mathbb{A})$. For simplicity, we sometimes denote $L(s, \pi_i, \mathrm{st})$ by $L(s, \pi_i)$. For $v \notin S$, the Euler factor for $L(s, \pi_i)$ is given by $\det(1 - \mathrm{st}(A_{\pi_{i,v}}) \cdot q_v^{-s})^{-1}$, where, $A_{\pi_{i,v}}$ is the Satake parameter of $\pi_{i,v}$. We consider the tensor product L-function $L(s, \pi_1 \boxtimes \pi_0)$. The Euler factor of $L(s, \pi_1 \boxtimes \pi_0)$ for $v \notin S$ is given by $\det(1 - \mathrm{st}(A_{\pi_{1,v}}) \otimes \mathrm{st}(A_{\pi_{0,v}}) \cdot q_v^{-s})^{-1}$.

Consider the adjoint representation $\mathrm{Ad} : {}^LG_i \to \mathrm{GL}(\mathrm{Lie}(\hat{G}_i))$. The associated L-function $L(s, \pi_i, \mathrm{Ad})$ is called the adjoint L-function. We assume that $L(s, \pi_1 \boxtimes \pi_0)$ and $L(s, \pi_i, \mathrm{Ad})$ can be analytically continued to the whole s-plane.

We put

$$\mathcal{P}_{\pi_1, \pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s + (1/2), \pi_1, \mathrm{Ad}) L(s + (1/2), \pi_0, \mathrm{Ad})}.$$

Let $\pi_{i,v}$ be an irreducible admissible representation of $G_{i,v}$. We denote the complex conjugate of $\pi_{i,v}$ by $\bar{\pi}_{i,v}$. It is believed that

$$\text{(MF)} \qquad \dim_{\mathbb{C}} \operatorname{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \leq 1$$

for non-archimedean place v of k. Recently, Aizenbud, Gourevitch, Rallis, and Schiffmann wrote a preprint, in which they obtained closely related results. For archimedean place, (MF) is verified in many cases, but not in general.

We consider the matrix coefficient

$$\Phi_{\varphi_{i,v},\varphi'_{i,v}}(g_i) = \langle \pi_{i,v}(g_i)\varphi_{i,v}, \varphi'_{i,v}\rangle_v, \quad g_i \in G_{i,v}$$

for a $\mathcal{K}_{1,v}$-finite vector $\varphi_{1,v}, \varphi'_{1,v} \in \pi_{1,v}$ and a $\mathcal{K}_{0,v}$-finite vector $\varphi_{0,v}, \varphi'_{0,v} \in \pi_{0,v}$. Put

$$I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \int_{G_{0,v}} \Phi_{\varphi_{1,v},\varphi'_{1,v}}(g_{0,v})\overline{\Phi_{\varphi_{0,v},\varphi'_{0,v}}(g_{0,v})}\, dg_{0,v},$$

$$\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \Delta_{G_{1,v}}^{-1} \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)^{-1} I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}).$$

When $\varphi_{1,v} = \varphi'_{1,v}$ and $\varphi_{0,v} = \varphi'_{0,v}$, we simply denote these objects by $I(\varphi_{1,v}, \varphi_{0,v})$ and $\alpha_v(\varphi_{1,v}, \varphi_{0,v})$, respectively. If both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered, then the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$. Moreover, if v is a non-archimedean place, and the conditions (U1), (U2), (U3), (U4), (U5), and (U6) hold, then we can show that $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$.

Conjecture 2.1. Assume that both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered. Then $\dim_{\mathbb{C}} \operatorname{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now let $\pi_i \simeq \otimes_v \pi_{i,v}$ be irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$. We shall say that π_i is almost locally generic if π_i satisfies the following condition (ALG).

(ALG) For almost all v, the constituent $\pi_{i,v}$ is generic.

It is believed that π_i is almost locally generic if and only if π_i is tempered (generalized Ramanujan conjecture).

Conjecture 2.2. Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$. We assume both π_1 and π_0 are almost locally generic. Then

(1) The integral $I(\varphi_{1,v}, \varphi_{0,v})$ should be absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.
(2) $\dim_{\mathbb{C}} \operatorname{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $\mathcal{K}_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now we state our global conjecture.

Conjecture 2.3. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ are irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume π_1 and π_0 are almost locally generic. Then there should be an integer β such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle\langle \varphi_0, \varphi_0\rangle} = 2^\beta C_0 \Delta_{G_1} \mathcal{P}_{\pi_1,\pi_0}(1/2) \prod_{v\in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

It seems that the integer β is related to the order of the groups, which appear in the theory of endoscopy.

It is possible to formulate a similar conjecture for non-tempered automorphic representations (cf. [15]).

3. The relative trace formula

For low rank groups, some periods formula are proved by using theta correspondence and Rankin-Selberg formulas (see, e.g, [3], [12], [13], [14], [19], [22]). For higher rank groups, it seems some sophisticated tool such as relative trace formula is necessary. In this section, we will discuss how a relative trace formula can be applied to period formulas.

Let G be a connected reductive algebraic group defined over k. We assume, for simplicity, $G(k)\backslash G(\mathbb{A})$ is compact.

We recall the Selberg trace formula. Let $f \in C_0^\infty(G(\mathbb{A}))$ be a test function. The kernel function $K_f(g_1, g_2)$ is defined by

$$K_f(g_1, g_2) = \sum_{\gamma\in G(k)} f(g_1^{-1}\gamma g_2).$$

For an automorphic form φ on $G(\mathbb{A})$,

$$\begin{aligned}\rho(f)\varphi(g_2) = (\varphi * f)(g_2) &= \int_{G(\mathbf{A})} \varphi(g_1) f(g_1^{-1} g_2)\, dg_1 \\ &= \int_{G(k)\backslash G(\mathbf{A})} \varphi(g_1) \sum_{\gamma\in G(k)} f(g_1^{-1}\gamma g_2)\, dg_1 \\ &= \int_{G(k)\backslash G(\mathbf{A})} \varphi(g_1) K_f(g_1, g_2)\, dg_1.\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{tr}\rho(f) &= \int_{G(k)\backslash G(\mathbb{A})} K_f(g,g)\,dg \\ &= \int_{G(k)\backslash G(\mathbb{A})} \sum_{\gamma\in G(k)} f(g^{-1}\gamma g)\,dg \\ &= \sum_{\{\gamma\}} \int_{G(k)\backslash G(\mathbb{A})} \sum_{\gamma'\in G_\gamma(k)\backslash G(k)} f(g^{-1}\gamma'^{-1}\gamma\gamma' g)\,dg \\ &= \sum_{\{\gamma\}} \mathrm{Vol}(G_\gamma(k)\backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\,dg.\end{aligned}$$

Here, $\{\gamma\}$ is a conjugacy class of $\gamma \in G(k)$ and G_γ is the centralizer of γ. Set $a(\gamma) = \mathrm{Vol}(G_\gamma(k)\backslash G_\gamma(\mathbb{A}))$.

Note that the orbital integral $O(\gamma, f) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\,dg$ is decomposed as a local product

$$\int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\,dg = \prod_v \int_{G_\gamma(k_v)\backslash G(k_v)} f(g_v^{-1}\gamma g_v)\,dg_v.$$

The right regular representation ρ is a sum of automorphic representations $\rho = \oplus_\pi m_\rho(\pi)\cdot\pi$. Here, $m_\rho(\pi)$ is the multiplicity of π. The distribution character $\chi_\pi(f)$ is defined by $\chi_\pi(f) = \mathrm{tr}\pi(f)$ for a test function $f \in C_0^\infty(G(\mathbb{A}))$. Then we have

$$\mathrm{tr}\rho(f) = \sum_\pi m_\rho(\pi)\chi_\pi(f).$$

Thus we have the Selberg trace formula

$$\sum_{\{\gamma\}} a(\gamma)O(\gamma.f) = \sum_\pi m_\rho(\pi)\chi_\pi(f).$$

Note that in the right hand side, π extends over the isomorphism classes of irreducible automorphic representations.

Now, we consider the relative trace formula. Let $H_1, H_2 \subset G$ be connected algebralic subgroups of G. Let $\theta_i : H_i(\mathbb{A}) \to \mathbb{C}^\times$ be a character which is trivial on $H_i(k)$ for $i = 1, 2$. As before, the kernel function $K_f(g_1, g_2)$ is defined by

$$K_f(g_1,g_2) = \sum_{\gamma\in G(k)} f(g_1^{-1}\gamma g_2)$$

for a test function $f \in C_0^\infty(G(\mathbb{A}))$.

Consider the integral

$$\int_{H_1(k)\backslash H_1(\mathbf{A})}\int_{H_2(k)\backslash H_2(\mathbf{A})} K_f(h_1,h_2)\theta_1(h_1)\overline{\theta_2(h_2)}\,dh_1\,dh_2$$
$$= \sum_{\gamma\in H_1(k)\backslash G(k)/H_2(k)} \int_{H_1(\mathbf{A})}\int_{H_{2,\gamma}(k)\backslash H_2(\mathbf{A})} f(h_1^{-1}\gamma h_2)\theta_1(h_1)\overline{\theta_2(h_2)}\,dh_1\,dh_2.$$

Here, $H_{2,\gamma} = \gamma^{-1}H_1\gamma \cap H_2$. In this sum, γ contributes only when $\theta_1(\gamma h_2\gamma^{-1}) = \theta_2(h_2)$ for any $h_2 \in H_{2,\gamma}(\mathbb{A})$, in which case γ is said to be (θ_1,θ_2)-relevant (or simply "relevant"). Set

$$a(\gamma) = \mathrm{Vol}(H_{2,\gamma}(k)\backslash H_{2,\gamma}(\mathbb{A})),$$
$$I_\gamma(\theta_1,\theta_2;f) = \int_{H_1(\mathbf{A})}\int_{H_{2,\gamma}(\mathbf{A})\backslash H_2(\mathbf{A})} f(h_1^{-1}\gamma h_2)\theta_1(h_1)\overline{\theta_2(h_2)}\,dh_1\,dh_2.$$

Then we have

$$\int_{H_1(k)\backslash H_1(\mathbf{A})}\int_{H_2(k)\backslash H_2(\mathbf{A})} K_f(h_1,h_2)\theta_1(h_1)\overline{\theta_2(h_2)}\,dh_1\,dh_2$$
$$= \sum_{\substack{\gamma\in H_1\backslash G/H_2\\ \text{relevant}}} a(\gamma)I_\gamma(\theta_1,\theta_2;f).$$

On the other hand, note that

$$\rho(f)\varphi_1(g_2) = \int_{G(k)\backslash G(\mathbf{A})} K_f(g_1,g_2)\varphi_1(g_1)\,dg_1$$
$$= \sum_\pi \sum_{\substack{\varphi_2\in\pi\\ \text{CONS}}} \int_{G(k)\backslash G(\mathbf{A})} K_f(g_1,g_2)\varphi_1(g_1)\varphi_2(g_2')\,dg_1\,dg_2'\cdot\overline{\varphi_2(g_2)}$$
$$= \sum_\pi \sum_{\substack{\varphi_2\in\pi\\ \text{CONS}}} \langle K_f,\bar\varphi_1\times\bar\varphi_2\rangle\cdot\overline{\varphi_2(g_2)}.$$

Here, φ_2 extends over a complete orthonormal system (CONS) for π. It follows that

$$K_f(g_1,g_2) = \sum_\pi \sum_{\substack{\varphi_1,\varphi_2\in\pi\\ \text{CONS}}} \langle K_f,\bar\varphi_1\times\bar\varphi_2\rangle\cdot\overline{\varphi_1(g_1)\varphi_2(g_2)}$$
$$= \sum_\pi \sum_{\substack{\varphi_1\in\pi\\ \text{CONS}}} \overline{\varphi(g_1)}\cdot\rho(f)\varphi(g_2).$$

Therefore we have

$$\int_{H_1(k)\backslash H_1(\mathbb{A})}\int_{H_2(k)\backslash H_2(\mathbb{A})} K_f(h_1,h_2)\theta_1(h_1)\overline{\theta_2(h_2)}\,dh_1\,dh_2$$

$$=\int_{H_1(k)\backslash H_1(\mathbb{A})}\int_{H_2(k)\backslash H_2(\mathbb{A})}\left[\sum_{\pi}\sum_{\substack{\varphi\in\pi\\ \text{CONS}}}\overline{\varphi(g_1)}\cdot\rho(f)\varphi(g_2)\right]\theta_1(h_1)\overline{\theta_2(h_2)}\,dh_1\,dh_2$$

$$=\sum_{\pi}\sum_{\substack{\varphi\in\pi\\ \text{CONS}}}\overline{\mathcal{P}_{H_1,\theta_1}(\varphi)}\mathcal{P}_{H_2,\theta_2}(\rho(f)\varphi).$$

Set

$$I_\pi(\theta_1,\theta_2;f)=\sum_{\substack{\varphi\in\pi\\ \text{CONS}}}\overline{\mathcal{P}_{H_1,\theta_1}(\varphi)}\mathcal{P}_{H_2,\theta_2}(\rho(f)\varphi).$$

The automorphic representation π is said to be (θ_1,θ_2)-distinguished (or simply "distinguished") if it is (H_1,θ_1)-distinguished and (H_2,θ_2)-distinguished. Then we have the relative trace formula

$$\sum_{\substack{\gamma\in H_1\backslash G/H_2\\ \text{relevant}}} a(\gamma)I_\gamma(\theta_1,\theta_2;f)=\sum_{\pi:\text{distinguished}} I_\pi(\theta_1,\theta_2;f).$$

Note that in the right hand side, π extends over some orthogonal decomposition $\rho=\sum_\pi \pi$. (Therfore different π's can be isomorphic.)

Remark 3.1. Assume that G is the product $G=G'\times G'$. Let H_1 be the diagonal subgroup $H_1=\Delta(G')=\{(g',g')\,|\,g'\in G'\}$ and H_2 be the second factor $H_2=\{(1,g')\,|\,g'\in G'\}$. Set $\theta_1=\theta_2=1$. Then the double coset $H_1\backslash G/H_2$ can be identified with the conjugacy calsses of G'. If $\gamma\in H_1\backslash G/H_2$ correspond to the conjugacy class γ' of G', then we have

$$I_\gamma(\theta_1,\theta_2;f)=O(\gamma',f'),$$

where

$$f'(g')=\int_{G'(\mathbb{A})} f(g_1',g_1'g')\,dg_1.$$

Moreover, an irreducible automorphic representation $\pi=\pi_1'\boxtimes\pi_2'$ is (θ_1,θ_2)-distinguished if and only if $\pi_1'\simeq\tilde{\pi}_2'$. In this case, we have $I_\pi(\theta_1,\theta_2;f)=\mathrm{tr}\pi_2'(f')$. Thus the Selberg trace formula can be considered as a special case of the relative trace formula.

Let G', H_1', θ_1', H_2', and θ_2' be another set of data. We assume there exists a bijection

$$\{\gamma\in H_1\backslash G/H_2\,|\,\gamma:\ \text{relevant}\}\simeq\{\gamma'\in H_1'\backslash G'/H_2'\,|\,\gamma':\ \text{relevant}\}$$

with the following properties:

(1) (matching) For each test function $f \in C_0(G(\mathbb{A}))$, there exists a test function $f' \in C_0(G'(\mathbb{A}))$ such that $I_\gamma(\theta_1, \theta_2; f) = I_{\gamma'}(\theta_1, \theta_2'; f')$.

(2) (fundamental lemma) For almost all unramified v, there exists a Hecke algebra homomorphism

$$\mathcal{H}(K_{G,v}\backslash G_v/K_{G,v}) \to \mathcal{H}(K_{G',v}\backslash G'_v/K_{G',v})$$

which is compatible with the matching.

Then it is expected that there exists a correspondence for the L-packets of $G(\mathbb{A})$ and $G'(\mathbb{A})$ such that

$$I_\Pi^\kappa(\theta_1, \theta_2; f) = I_{\Pi'}^{\kappa'}(\theta_1', \theta_2'; f').$$

Here, Π is an L-packet for $G(\mathbb{A})$, and κ is certain function on the L-packet and

$$I_\Pi^\kappa(\theta_1, \theta_2; f) = \sum_{\pi \in \Pi} \kappa(\pi) I_\pi(\theta_1, \theta_2; f).$$

In the right hand side, Π' is the L-packet of $G'(\mathbb{A})$ corresponding to Π, and $I_{\Pi'}^{\kappa'}(\theta_1', \theta_2'; f')$ is defined in a similar way.

This equation would imply that there exists a certain relation between period integrals for $G(\mathbb{A})$ and $G'(\mathbb{A})$. In this way, it would be possible to reduce a period formula for $G(\mathbb{A})$ to an analogous formulas for $G'(\mathbb{A})$.

Recently, H. Jacquet [16] proposed a program to attack an analogue of the Gross-Prasad type conjecture for the unirary groups.

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