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PRINCIPAL SERIES WHITTAKER FUNCTIONS ON $GL(3,\mathbb{C})$

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1. INTRODUCTION

The precise analytic properties of Whittaker functions are utilized in the study of the Fourier expansions of automorphic forms and their related topics such as L-functions. In this note, we give explicit formulas for (general) principal series Whittaker functions on $GL(3,\mathbb{C})$. Also, we give a propagation formula which is an expression of the Whittaker functions on $GL(3,\mathbb{C})$ via those on $GL(2,\mathbb{C})$: This is an analogue of the recent result of Ishii-Stade [5] for the class one cases.

This note is based on our recent paper [3]. See it for details.

2. DEFINITION OF WHITTAKER FUNCTIONS

Let $G = NAK$ be an Iwasawa decomposition of a real reductive group G. For an irreducible admissible representation (π, H_π) of G, we choose a K-type (τ^*, V_{τ^*}) in π which occurs with multiplicity one and fix an injective K-homomorphism $i \in \mathrm{Hom}_K(\tau^*, \pi|_K)$. Here (τ^*, V_{τ^*}) means the contragradient representation of (τ, V_τ). Moreover, take a non-degenerate character η of N. Let us consider the intertwining space

$$\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi, C^\infty\mathrm{Ind}_N^G(\eta))$$

between $(\mathfrak{g}_{\mathbb{C}}, K)$-modules consisting of all K-finite vectors, where $C^\infty\mathrm{Ind}_N^G(\eta)$ is the induced representation of G from η as C^∞-induction. For each $T \in \mathcal{I}_{\eta,\pi}$, we define a V_τ-valued function T_i on G by

$$T(i(v^*))(g) = \langle v^*, T_i(g)\rangle, \quad v^* \in V_{\tau^*}, \quad g \in G.$$

Here $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_{\tau^*} \times V_\tau$. The function T_i means a restriction of $T \in \mathcal{I}_{\eta,\pi}$ to K and satisfies

$$T_i(ngk) = \eta(n)\tau(k)^{-1}T_i(g), \quad (n, g, k) \in N \times G \times K.$$

Then we put

$$\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}} = \bigcup_{i\in\mathrm{Hom}_K(\tau^*,\pi|_K)} \{T_i \mid T \in \mathcal{I}_{\eta,\pi},\ T_i \text{ is moderate growth}\}.$$

(Here the term "moderate growth" is by means of [9] §8.1.) According to the multiplicity one theorem of Shalika [8], the dimension of the space $\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ is at most one. A unique (up to constant) element in $\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ is called *a primary Whittaker function* with respect to (π,η,τ).

3. Whittaker functions on $GL(3, \mathbf{C})$

In this section, we determine the primary Whittaker function on $GL(3, \mathbf{C})$ for principal series representations and their minimal K-types.

3.1. Groups and representations. Let $G = GL(3, \mathbf{C})$ be the complex general linear group of degree 3, which is viewed as a real reductive group, with the center

$$Z_G = \{ru1_3 \mid r \in \mathbf{R}_{>0}, u \in U(1)\} \simeq \mathbf{C}^\times.$$

Here 1_n is the unit matrix of degree n. Let $K = U(3)$ be a maximal compact subgroup of G, and define subgroups A and N of G by

$$\begin{aligned} A &= \{\mathrm{diag}\,(a_1, a_2, a_3) \in G \mid a_i \in \mathbf{R}_{>0}, i = 1, 2, 3\}, \\ N &= \left\{ n(x) = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in G \,\middle|\, x = (x_i) \in \mathbf{C}^3 \right\}. \end{aligned}$$

Then we have an Iwasawa decomposition $G = NAK$. If we put

$$M = \{\mathrm{diag}\,(u_1, u_2, u_3) \mid u_i \in U(1),\ i = 1, 2, 3\} \simeq U(1)^3,$$

then M is the centralizer of A in K and $P = NAM$ gives the upper triangular subgroup of G, which is a minimal parabolic subgroup of G.

The equivalence classes of irreducible continuous representations of K are parameterized by the set of the highest weights

$$\Lambda = \{\mu = (\mu_1, \mu_2, \mu_3) \mid \mu \in \mathbf{Z}^3, \mu_1 \geq \mu_2 \geq \mu_3\}.$$

We denote by (τ_μ, V_μ) the representation of K associated with $\mu \in \Lambda$. The representation space V_μ has the (normalized) GZ-basis $\{f(M)\}_{M \in G(\mu)}$ parameterized by the set $G(\mu)$ of all G-patterns of type μ (*cf.* [1], [2]). Here a G-pattern $M \in G(\mu)$ is a triangle

$$M = \begin{pmatrix} \mu_1 \; \mu_2 \; \mu_3 \\ \alpha_1 \; \alpha_2 \\ \beta \end{pmatrix}$$

consisting of 6 integers satisfying the inequalities

$$\mu_1 \geq \alpha_1 \geq \mu_2 \geq \alpha_2 \geq \mu_3, \quad \alpha_1 \geq \beta \geq \alpha_2.$$

Let us take a character $\sigma_{\mathbf{n}}$ of M defined by

$$\sigma_{\mathbf{n}}(\mathrm{diag}(u_1, u_2, u_3)) = u_1^{n_1} u_2^{n_2} u_3^{n_3}, \quad \mathbf{n} = (n_1, n_2, n_3) \in \mathbf{Z}^3,$$

and an element ν in the dual $\mathfrak{a}_{\mathbf{C}}^*$ of $\mathfrak{a}_{\mathbf{C}}$ identified with $(\nu_1, \nu_2, \nu_3) \in \mathbf{C}^3$ via $\nu_i = \nu(E_{ii})$ for $1 \leq i \leq 3$. Here $\mathfrak{a}_{\mathbf{C}}$ is the complexification of the Lie algebra of A and E_{ii} is the diagonal matrix unit with (i, i)-entry 1 and the remaining entries 0. Then the induced representation

$$\pi = \pi(\nu, \sigma_{\mathbf{n}}) = \mathrm{Ind}_P^G(1_N \otimes e^{\nu+\rho} \otimes \sigma_{\mathbf{n}})$$

of G from the parabolic subgroup $P = NAM$ is called *the principal series representation* of G. Here ρ is the half-sum of the positive restricted roots, i.e.,

$$e^\rho(\mathrm{diag}\,(a_1, a_2, a_3)) = \left(\frac{a_1}{a_3}\right)^2, \quad \mathrm{diag}\,(a_1, a_2, a_3) \in A.$$

The central character of π is given by

$$Z_G \ni ru1_3 \mapsto r^{\tilde{\nu}} u^{\tilde{n}}, \quad r \in \mathbf{R}_{>0},\ u \in U(1),$$

with $\tilde{\nu} = \nu_1 + \nu_2 + \nu_3$ and $\tilde{n} = n_1 + n_2 + n_3$, and the minimal K-type of π is the representation $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ of K associated with the dominant permutation $\mathbf{m} \in \Lambda$ of $\mathbf{n}$.

Finally, we take a non-degenerate character η of N defined by

$$\eta(n(\boldsymbol{x})) = \exp\left(2\pi\sqrt{-1}\mathrm{Im}\,(x_1 + x_3)\right).$$

3.2. **Differential equations.** Let us take an irreducible principal series representation $\pi = \pi(\nu, \sigma_{\mathbf{n}})$ of G with the minimal K-type $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ and a non-degenerate unitary character η of N defined in the previous subsection. In this subsection, we consider a system of differential equations for the functions ϕ in $\mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$. This is described as that for the M-components $\phi(M)$, since the Whittaker functions are V_τ-valued. Here the M-component $\phi(M)$ of ϕ corresponding to $M \in G(\mathbf{m})$ is defined by

$$\phi(M; g) = \langle \phi(g), f(M) \rangle, \quad g \in G,$$

for the GZ-basis $\{f(M)\}_{M \in G(\mathbf{m})}$ of $V_{\mathbf{m}}$.

It is well known that each element C in the center $Z(\mathfrak{g}_{\mathbf{C}})$ of the universal enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$ of $\mathfrak{g}_{\mathbf{C}}$ acts as a scalar on the K-finite vectors in π. If we take an injection $\jmath \in \mathrm{Hom}_K(\tau_{\mathbf{m}}, \pi|_K)$, then we have the equation

$$(1) \qquad C \cdot \jmath(f(M)) = \chi_C \jmath(f(M)), \quad M \in G(\mathbf{m})$$

for a scalar χ_C. Therefore, each M-component $\phi(M)$ of $\phi \in \mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$ satisfies the equations

$$(2) \qquad C\phi(M) = \chi_C \phi(M), \quad C \in Z(\mathfrak{g}_{\mathbf{C}}).$$

Here we remark that the generators of $Z(\mathfrak{g}_{\mathbf{C}})$ can be constructed from the Capelli elements in $U(\mathfrak{g})$ (*cf.* [4]) via the identification of $U(\mathfrak{g}_{\mathbf{C}})$ and $U(\mathfrak{g}) \otimes_{\mathbf{C}} U(\mathfrak{g})$.

Let $\mathfrak{k}$ (resp. $\mathfrak{p}$) be the $+1$ (resp. the -1) eigenspace of the Cartan involution θ of $\mathfrak{g}$ defined by $\theta(X) = -{}^tX$. Then the complexification $\mathfrak{p}_{\mathbf{C}}$ of $\mathfrak{p}$ becomes a K-module via the adjoint action and its irreducible decomposition is $\mathfrak{p}_{\mathbf{C}} = Z_{\mathfrak{p},\mathbf{C}} \oplus \mathfrak{p}_{0,\mathbf{C}}$ with $Z_{\mathfrak{p},\mathbf{C}} \simeq V_{(0,0,0)}$ and $\mathfrak{p}_{0,\mathbf{C}} \simeq V_{(1,0,-1)}$. In the tensor product $\mathfrak{p}_{0,\mathbf{C}} \otimes V_\mu$ with a general irreducible representation V_μ of K, V_μ occurs with multiplicity two as the irreducible component. Take an injector ι from $V_{\mathbf{m}}$ into $\mathfrak{p}_{0,\mathbf{C}} \otimes V_{\mathbf{m}} \simeq V_{(1,0,-1)} \otimes V_{\mathbf{m}}$ and fix an injection $\jmath \in \mathrm{Hom}_K(\tau_{\mathbf{m}}, \pi|_K)$. Since the minimal K-type $\tau_{\mathbf{m}}$ occurs with multiplicity one in $\pi|_K$, the composition

$$V_{\mathbf{m}} \xrightarrow{\iota} \mathfrak{p}_{0,\mathbf{C}} \otimes V_{\mathbf{m}} \xrightarrow{\alpha} \pi(\mathfrak{p}_{0,\mathbf{C}})\jmath(V_{\mathbf{m}}) \subset L^2_{(M,\sigma_{\mathbf{n}})}(K)$$

is a scalar multiple of $\jmath$, where α is the evaluation map. Thus, if we write

$$\iota(f(M)) = \sum_{M' \in G(\mathbf{m})} X^{(\iota)}_{M,M'} \otimes f(M'), \quad X^{(\iota)}_{M,M'} \in \mathfrak{p}_{0,\mathbf{C}},$$

for the GZ-basis $\{f(M)\}_{M \in G(\mathbf{m})}$ of $V_{\mathbf{m}}$ then we have the equation

$$(3) \qquad \sum_{M' \in G(\mathbf{m})} X^{(\iota)}_{M,M'} \cdot \jmath(f(M')) = \lambda_\iota \jmath(f(M)), \quad M \in G(\mathbf{m})$$

with a scalar λ_ι. We call this equation (3) *the Dirac-Schmid eigen-equation.* Then, for each injector ι, we have the difference-differential equation

$$\sum_{M' \in G(\mathbf{m})} X^{(\iota)}_{M,M'} \phi(M') = \lambda_\iota \phi(M), \tag{4}$$

among the M-components $\{\phi(M)\}_{M \in G(\mathbf{m})}$ of $\phi \in \mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$.

3.3. **Explicit integral formulas.** Let $\pi = \pi(\nu, \sigma_{\mathbf{n}})$, $(\tau^*, V_{\tau^*}) = (\tau_{\mathbf{m}}, V_{\mathbf{m}})$, and η be as in the previous subsection. We give two explicit integral formulas for the primary Whittaker function $\phi \in \mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$, in this subsection.

The Whittaker functions are determined by its A-radial parts (i.e. its restriction to A) because of the Iwasawa decomposition of G. Moreover, the values on the center Z_G of G are given by the central character of π, i.e.,

$$\phi(rug) = r^{\tilde{\nu}} u^{\tilde{n}} \phi(g), \quad \phi \in \mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}, \ r \in \mathbf{R}_{>0}, \ u \in U(1), \ g \in G.$$

Therefore, we can describe them as functions of two variables with the coordinates

$$y_1 = \frac{a_1}{a_2}, \quad y_2 = \frac{a_2}{a_3},$$

for $\mathrm{diag}\,(a_1, a_2, a_3) = a_3 \cdot \mathrm{diag}\,(y_1 y_2, y_2, 1) \in A$.

To state our results, we need some notations. If we write $\mathbf{m} = (n_a, n_b, n_c) \in \Lambda$, then we put $(\lambda_1, \lambda_2, \lambda_3) = \left(\nu_c - \frac{\tilde{\nu}}{3}, \nu_a - \frac{\tilde{\nu}}{3}, \nu_b - \frac{\tilde{\nu}}{3}\right)$. For each G-pattern $M = \begin{pmatrix} m_1\, m_2\, m_3 \\ \alpha_1\, \alpha_2 \\ \beta \end{pmatrix} \in G(\mathbf{m})$, we put $\delta(M) = \alpha_1 + \alpha_2 - m_2 - \beta$ and

$$\begin{aligned} \zeta_1^{(1)}(M) &= \lambda_1 - m_3 + \beta, & \zeta_1^{(2)}(M) &= -\lambda_1 + m_1 - \beta - \delta(M), \\ \zeta_2^{(1)}(M) &= \lambda_2 + m_1 - \beta, & \zeta_2^{(2)}(M) &= -\lambda_2 - m_3 + \beta + \delta(M), \\ \zeta_3^{(1)}(M) &= \lambda_3 + \alpha_1 - \alpha_2 - |\delta(M)|, & \zeta_3^{(2)}(M) &= -\lambda_3 + m_1 - m_3 - \alpha_1 + \alpha_2. \end{aligned}$$

Now we can state our main result, that is, two explicit integral formulas for the primary Whittaker function with respect to the triple (π, η, τ).

Theorem 3.1. *Let $W_3(y)$ be the A-radial part of the primary Whittaker function in $\mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$ and $W_3(M; y) = y_1^2 y_2^2 \tilde{W}_3(M; y)$ be its M-component. Then $\tilde{W}_3(M; y)$ has the following integral expressions:*

$$\begin{aligned} \tilde{W}_3(M; y) &= \frac{1}{(2\pi\sqrt{-1})^2} \int_{s_1} \int_{s_2} V_3(M; s_1, s_2) (\pi y_1)^{-s_1} (\pi y_2)^{-s_2} ds_1 ds_2 \\ &= 2^4 (\pi y_1)^{\frac{-\lambda_3 + m_1 - m_3}{2}} (\pi y_2)^{\frac{\lambda_3 + m_1 - m_3}{2}} \\ &\quad \times \int_0^\infty K_A\left(2\pi y_1 \sqrt{1 + \frac{1}{v}}\right) K_{A+\delta(M)}\left(2\pi y_2 \sqrt{1+v}\right) v^B (1+v)^C \frac{dv}{v}. \end{aligned}$$

Here, in the first integral expression of Mellin-Barnes type, the paths s_i of integrations are the vertical lines from $\mathrm{Re}\, s_i - \sqrt{-1}\infty$ to $\mathrm{Re}\, s_i + \sqrt{-1}\infty$ with enough large *real part and the integrand $V_3(M;s_1,s_2)$ is defined by*

$$V_3(M;s_1,s_2)=\prod_{i=1}^{2}\prod_{j=1}^{3}\Gamma\left(\frac{s_i+\zeta_j^{(i)}(M)}{2}\right)\Big/\Gamma\left(\frac{s_1+s_2+\zeta_3^{(1)}(M)+\zeta_3^{(2)}(M)}{2}\right).$$

Also, in the second integral expression of Euler type, K_ν is the K-Bessel function and the parameters A, B and C are given as follows.

$$A=\frac{\zeta_1^{(1)}(M)-\zeta_2^{(1)}(M)}{2},\quad B=\frac{2\zeta_3^{(1)}(M)-\zeta_1^{(1)}(M)-\zeta_2^{(1)}(M)}{4},\quad C=\frac{|\delta(M)|}{2}.$$

This theorem is obtained by solving the system of difference-differential equations (2) and (4) for the M-components of the Whittaker function in the previous subsection, explicitly.

Remark 3.2. *The holonomic system of differential equations given in §3.2 has regular singularities along 2 divisors $y_1=0$ and $y_2=0$ which are of simple normal crossing at $(y_1,y_2)=(0,0)$. The power series solutions of the system at $(y_1,y_2)=(0,0)$ are called the secondary Whittaker functions. The secondary Whittaker functions play an important role in constructing the Poincaré series (cf.* [6], [7]*). Our proof of the main theorem requires the factorization theorem of the primary Whittaker functions by the secondaries.*

Remark 3.3. *In the explicit description of the Dirac-Scmid eigen-equations, we used the Clebsch-Gordan coefficients for the injectors $\iota: V_\mu\to V_\mu\otimes V_{(1,0,-1)}$ with respect to the GZ-basis. Our paper* [2] *discussed its dual, that is, the Clebsch-Gordan coefficients for the projectors from $V_\mu\otimes V_{(1,0,-1)}$ to V_μ.*

4. Propagation formula for Whittaker functions

In this section, we give an expression of the primary Whittaker function on $GL(3,\mathbf{C})$ in terms of that on $GL(2,\mathbf{C})$. This is an analogue of the formula obtained by Ishii-Stade [5].

4.1. Whittaker functions on $GL(2,\mathbf{C})$.

First we recall two explicit integral formulas of the principal series Whittaker functions on $GL(2,\mathbf{C})$.

Let $G'=GL(2,\mathbf{C})$ and take subgroups $K'=U(2)$, and

$$A'=\left\{\begin{pmatrix}a_1&0\\0&a_2\end{pmatrix}\,\middle|\,a_i\in\mathbf{R}_{>0},i=1,2\right\},\quad N'=\left\{n(x)=\begin{pmatrix}1&x\\0&1\end{pmatrix}\,\middle|\,x\in\mathbf{C}\right\},$$

of G'. Then we have an Iwasawa decomposition $G'=N'A'K'$ of G'. The upper triangular subgroup $P'=N'A'M'$ of G' with the centralizer M' of A' in K' given by

$$M'=\left\{\begin{pmatrix}u_1&0\\0&u_2\end{pmatrix}\,\middle|\,u_i\in U(1),\,i=1,2\right\}\simeq U(1)^2,$$

is the minimal parabolic subgroup.

We can parameterize the equivalence classes of irreducible continuous representations of $K'=U(2)$ by the set

$$\Lambda'=\{\mu'=(\mu_1',\mu_2')\mid\mu'\in\mathbf{Z}^2,\mu_1'\geq\mu_2'\},$$

from the highest weight theory. The representation space $V_{\mu'}$ of the representation $\tau_{\mu'}$ associated with $\mu' = (\mu_1', \mu_2') \in \Lambda'$ has the (normalized) GZ-basis $\{f'(M')\}_{M' \in G(\mu')}$ as in the case of $U(3)$. Here

$$G(\mu') = \left\{ M' = \left(\begin{smallmatrix} \mu_1' & \mu_2' \\ & \alpha' \end{smallmatrix} \right) \,\middle|\, \alpha' \in \mathbf{Z},\ \mu_1' \geq \alpha' \geq \mu_2' \right\}.$$

A principal series representation

$$\pi' = \pi'(\nu', \sigma_{\mathbf{n}'}) = \mathrm{Ind}_{P'}^{G'}(1_{N'} \otimes e^{\nu'+\rho'} \otimes \sigma_{\mathbf{n}'}),$$

of G' with data $\nu' = (\nu_1', \nu_2') \in \mathbf{C}^2$ and $\mathbf{n}' = (n_1', n_2') \in \mathbf{Z}^2$ is defined similarly to that of $GL(3, \mathbf{C})$. Here the half-sum ρ' of the positive restricted roots is given by

$$e^{\rho'}(\mathrm{diag}\,(a_1, a_2)) = \frac{a_1}{a_2}, \quad \mathrm{diag}\,(a_1, a_2) \in A'.$$

As in the case of $GL(3, \mathbf{C})$, the central character of π' is

$$Z_{G'} = \{ru1_2 \,|\, r \in \mathbf{R}_{>0}, u \in U(1)\} \ni ru1_2 \mapsto r^{\tilde{\nu}'} u^{\tilde{n}'}, \quad r \in \mathbf{R}_{>0},\ u \in U(1),$$

with $\tilde{\nu}' = \nu_1' + \nu_2'$ and $\tilde{n}' = n_1' + n_2'$, and the minimal K'-type of π' is the representation $(\tau_{\mathbf{m}'}, V_{\mathbf{m}'})$ associated with the dominant permutation $\mathbf{m}' \in \Lambda'$ of $\mathbf{n}'$.

Also, we take a non-degenerate character η' of N' defined by

$$\eta'(n(x)) = \exp\left(2\pi\sqrt{-1}\mathrm{Im}\,(x)\right).$$

By virtue of the Iwasawa decomposition of G' and the central character of π', the Whittaker functions can be described as functions of a variable

$$y = \frac{a_1}{a_2}, \quad \text{for } \mathrm{diag}\,(a_1, a_2) = a_2 \cdot \mathrm{diag}\,(y, 1) \in A'.$$

Theorem 4.1. *Let $\pi' = \pi'(\nu', \sigma_{\mathbf{n}'})$ be an irreducible principal series representation of G' with the minimal K'-type $(\tau_{\mathbf{m}'}, V_{\mathbf{m}'})$ associated with the dominant permutation $\mathbf{m}' = (m_1', m_2') = (n_a', n_b') \in \Lambda'$ of $\mathbf{n}'$, and let η' be a non-degenerate unitary character of N'. Moreover let $W_2(y) \in \mathrm{Wh}(\pi', \eta', \tau')^{\mathrm{mod}}$ be the (A'-radial part of) primary Whittaker function with M'-components $W_2(M'; y) = y\tilde{W}_2(M'; y)$ for each G-pattern $M' = \left(\begin{smallmatrix} m_1' & m_2' \\ & \alpha' \end{smallmatrix} \right) \in G(\mathbf{m}')$ of weight $(w_1', w_2') = (\alpha', m_1' + m_2' - \alpha')$. Then the function $\tilde{W}_2(M'; y)$ has the following expressions:*

$$\tilde{W}_2(M'; y) = \frac{1}{2\pi\sqrt{-1}} \int_s V_2(M'; s)(\pi y)^{-s} ds = 4(\pi y)^A K_B(2\pi y).$$

Here, the path of integration is the vertical line from $\mathrm{Re}\, s - \sqrt{-1}\infty$ *to* $\mathrm{Re}\, s + \sqrt{-1}\infty$ *with enough large real part and the integrand $V_2(M'; s)$ is defined by*

$$V_2(M'; s) = \Gamma\left(\frac{s + \lambda_2' + m_1' - \alpha'}{2}\right) \Gamma\left(\frac{s + \lambda_1' + \alpha' - m_2'}{2}\right),$$

with

$$\lambda_1' = \nu_b' - \frac{\tilde{\nu}'}{2}, \quad \lambda_2' = \nu_a' - \frac{\tilde{\nu}'}{2},$$

and the parameters A and B are given by

$$A = \frac{m_1' - m_2'}{2}, \quad B = \frac{\lambda_1' - \lambda_2' + w_1' - w_2'}{2}.$$

This theorem is obtained by solving a system of differential equations for the Whittaker functions, as in the case of $GL(3,\mathbf{C})$. But the required calculation is much simpler than that of $GL(3,\mathbf{C})$.

Remark 4.2. *As in the case of $GL(3,\mathbf{C})$, we can show the factorization theorem of the primary Whittaker functions by the secondaries, which is essentially the expression of the K-Bessel function by the I-Bessel functions.*

4.2. **Some integral formulas.** The modified Bessel function $K_\nu(z)$ of the second kind has several integral expressions. Among them, we recall two expressions: One is the integral expression of Mellin-Barnes type

$$K_\nu(z) = \frac{1}{4}\cdot\frac{1}{2\pi\sqrt{-1}}\int_s \Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)\left(\frac{z}{2}\right)^{-s} ds.$$

Here, the path of integration is the vertical line from $\mathrm{Re}\, s - \sqrt{-1}\infty$ to $\mathrm{Re}\, s + \sqrt{-1}\infty$ with enough large real part. Another is that of Euler type

$$K_\nu(z) = \frac{1}{2}\int_0^\infty \exp\left(\frac{-z(t+t^{-1})}{2}\right) t^\nu \frac{dt}{t},$$

which is valid only for $\mathrm{Re}\, z > 0$.

Also we recall the following integral formula so-called Barnes' lemma

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}&\int_z \Gamma(z+a)\Gamma(z+b)\Gamma(-z+c)\Gamma(-z+d)dz\\ &= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.\end{aligned}$$

Here the path of integration is the vertical line from $\mathrm{Re}\, z - \sqrt{-1}\infty$ to $\mathrm{Re}\, z + \sqrt{-1}\infty$ with enough large real part.

4.3. **Propagation formula.** Let $\pi = \pi(\nu, \sigma_{\mathbf{n}})$ be an irreducible principal series representation of $G = GL(3,\mathbf{C})$ with data $\nu = (\nu_1,\nu_2,\nu_3) \in \mathbf{C}^3$ and $\mathbf{n} = (n_1,n_2,n_3) \in \mathbf{Z}^3$ and let η be a non-degenerate unitary character of N defined in §3. For simplicity, we assume that the parameter $\mathbf{n}$ satisfies the regularity condition

$$n_1 \geq n_2 \geq n_3.$$

Then $\mathbf{n} \in \Lambda$ and the minimal K-type of π is $(\tau_{\mathbf{m}}, V_{\mathbf{m}}) = (\tau_{\mathbf{n}}, V_{\mathbf{n}})$.

Let $W_3(y) \in \mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ be the (A-radial part of) primary Whittaker function with M-components $W_3(M;y) = y_1^2 y_2^2 \tilde{W}_3(M;y)$. Under the regularity condition on $\mathbf{n}$, we have the parameters $(\lambda_1,\lambda_2,\lambda_3) = \left(\nu_3 - \frac{\tilde{\nu}}{3}, \nu_1 - \frac{\tilde{\nu}}{3}, \nu_2 - \frac{\tilde{\nu}}{3}\right)$ which appear in the integrand $V_3(M;s_1,s_2)$ of the integral expression of Mellin-Barnes type for $\tilde{W}_3(M;y)$ in Theorem 3.1.

Theorem 4.3. *Let* $M = \begin{pmatrix} m_1\, m_2\, m_3 \\ \alpha_1\, \alpha_2 \\ \beta \end{pmatrix} \in G(\mathbf{m})$. *Then the integrand* $V_3(M; s_1, s_2)$ *has the following expression.*

$$\begin{aligned} V_3(M; s_1, s_2) &= \Gamma\left(\frac{s_1 + \zeta_j^{(1)}(M)}{2}\right)\Gamma\left(\frac{s_2 + \zeta_j^{(2)}(M)}{2}\right) \\ &\times \frac{1}{2\pi\sqrt{-1}}\int_z \Gamma\left(\frac{z + s_1 + \mu_1}{2}\right)\Gamma\left(\frac{z + s_2 + \mu_2}{2}\right) V_2(M'; -z)dz, \end{aligned}$$

where $V_2(M'; s)$ *is the integrand of the integral expression of* $\tilde{W}_2(M'; y)$ *in Theorem 4.1 for a triple* $(\pi'(\nu', \sigma_{\mathbf{n}'}), \eta', \tau_{\mathbf{m}'})$ *and a G-pattern* $M' \in G(\mathbf{m}')$ *and the path of integration is the vertical line from* $\mathrm{Re}\, z - \sqrt{-1}\infty$ *to* $\mathrm{Re}\, z + \sqrt{-1}\infty$ *with large enough real part. The parameters and the representations are given in the following table.*

	j	μ_1	μ_2	ν'	$\mathbf{n}'$	M'
$\delta(M) \geq 0$	2	$-\frac{\lambda_2}{2} - \alpha_2 + \beta$	$\frac{\lambda_2}{2} + m_1 - \alpha_1$	(ν_2, ν_3)	(m_2, m_3)	$\begin{pmatrix} m_2\, m_3 \\ \alpha_2 \end{pmatrix}$
$\delta(M) \leq 0$	1	$-\frac{\lambda_1}{2} + \alpha_1 - \beta$	$\frac{\lambda_1}{2} + \alpha_2 - m_3$	(ν_1, ν_2)	(m_1, m_2)	$\begin{pmatrix} m_1\, m_2 \\ \alpha_1 \end{pmatrix}$
$\delta(M) = 0$	3	$-\frac{\lambda_3}{2}$	$\frac{\lambda_3}{2}$	(ν_1, ν_3)	(m_1, m_3)	$\begin{pmatrix} m_1\, m_3 \\ \beta \end{pmatrix}$

Proof. Assume $\delta(M) \geq 0$. Then, since $\zeta_1^{(1)}(M) + \zeta_1^{(2)}(M) = \zeta_3^{(1)}(M) + \zeta_3^{(2)}(M)$, Barnes' lemma leads the equation

$$\begin{aligned} V_3(M; s_1, s_2) &= \Gamma\left(\frac{s_1 + \zeta_2^{(1)}(M)}{2}\right)\Gamma\left(\frac{s_2 + \zeta_2^{(2)}(M)}{2}\right) \\ &\times \frac{1}{2\pi\sqrt{-1}}\int_z \Gamma\left(\frac{z + s_1 + \mu_1}{2}\right)\Gamma\left(\frac{z + s_2 + \mu_2}{2}\right) \\ &\times \Gamma\left(\frac{-z + \mu_3}{2}\right)\Gamma\left(\frac{-z + \mu_4}{2}\right) dz, \end{aligned}$$

where the parameters μ_1 and μ_2 are given in the assertion of theorem and μ_3 and μ_4 are

$$\mu_3 = \frac{-\nu_2 + \nu_3}{2} + \alpha_2 - m_3, \quad \mu_4 = \frac{\nu_2 - \nu_3}{2} - \alpha_2 + m_2.$$

Here we use the relations $\lambda_1 + \frac{\lambda_2}{2} = \frac{-\nu_2 + \nu_3}{2}$ and $\lambda_3 + \frac{\lambda_2}{2} = \frac{\nu_2 - \nu_3}{2}$.

The assertion for the other cases of $\delta(M)$ can be obtained similarly. □

Corollary 4.4. *We have the following expression of* $\tilde{W}_3(M; y)$.

$$\begin{aligned} \tilde{W}_3(M; y) &= 4\pi^{a_1 + a_2} y_1^{a_1 + A_1} y_2^{a_2 - A_2} \int_0^\infty \int_0^\infty \exp\left(-\pi\left(y_1^2 u_1 + \frac{1}{u_1} + y_2^2 u_2 + \frac{1}{u_2}\right)\right) \\ &\times u_1^{A_1} u_2^{-A_2} \tilde{W}_2\left(M'; \pi y_2 \sqrt{\frac{u_2}{u_1}}\right) \frac{du_1}{u_1}\frac{du_2}{u_2}. \end{aligned}$$

Here

$$a_k = \frac{1}{2}\left\{\zeta_j^{(k)}(M) + \mu_k\right\}, \quad A_k = \zeta_j^{(k)}(M) - a_k, \quad k = 1, 2,$$

and the parameters and the representations are given in Theorem 4.3.

Proof. This corollary is obtained from the integral expression of Mellin-Barnes type for $\tilde{W}_3(M; y)$ in Theorem 3.1 and Theorem 4.3 by using the integral expressions of $K_\nu(z)$ in §4.2. □

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