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Construction of approximate solutions for singular integral equations by using the theory of reproducing kernels

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1 Introduction

In this paper, we report a new concept and method of construction of approximate solutions for linear singular integral equations by using the two theories of the Tikhonov regularization and reproducing kernels.

The theory of reproducing kernels has been developed and investigated by many authors. Recently, Saitoh [10] and co-researchers [6] consider applications of the theory to construction of approximate solutions for bounded linear operator equations on Hilbert spaces by combining the theory with the linear operators. This method can be applied to singular integral equations since it is well known that singular integral operators on L^2 space are bounded linear operators, see [7].

Singular integral equations are presently encountered in a wide range of mathematical models, for instance in acoustics, fluid dynamics, elasticity and fracture mechanics. Particularly, crack problems in elasticity lead to singular integral equations by the potential theory and [2, 4] showed the uniquely existence of solution of the problem by proving the compactness of the singular integral operator. Moreover, about a problem for prediction of the direction of crack propagation in the elastic plate there only exist some criteria in engineering sense. Unfortunately, we can not know which criterion is true and the best because it is very difficult to measure the angle of crack propagation by experiment. Although the difference among the criteria has been discussed by many authors, it remains an open problem. One of difficulty comparing the criteria is to calculate the energy release rate at the crack tip, which is a rate of the energy, per unit length along the crack edge, that is supplied by the elastic energy in the body and by the loading system in creating the new crack surface. Therefore, in order to evaluate the energy release rate defined by the released potential energy as the crack increases a unit length we need to construct the solution in the elastic plate with virtual kinked crack extension. Itou [3] considered the formulation of the kinked crack problem in linearized elastic plate and introduced the procedure for reducing the problem to a singular integral equation by employing a conformal mapping technique (cf. [8]). Then, an explicit representation of the solution is required for discussing the difference among the criteria.

As a typical singular integral equation, we shall consider the Carleman's equation over a real interval, for any $L^2(-1,1)$ ($\equiv L^2$) function g and for real or complex valued continuous (or bounded integrable) functions a , b

$$(Ly)(t) \equiv a(t)y(t) + \frac{b(t)}{\pi i}\text{p.v.}\int_{-1}^{+1} \frac{y(\zeta)}{\zeta - t}\,\mathrm{d}\zeta = g(t) \quad \text{on} \quad -1 < t < 1 \quad (1.1)$$

in the class of the functions of the Paley-Wiener space. We denote by p.v. the Cauchy's principal value of the integral. According to [8], the operator L satisfying a condition $a^2(t) - b^2(t) \neq 0$ for $-1 < t < 1$ is called a regular type operator. It is well known that the equation (1.1) always has an explicit solution for a regular type operator, see also [7]. However, when $a^2(t) - b^2(t) = 0$ at some points $t \in (-1,1)$, special treatment is required, see [1]. Namely, in general it is impossible to represent the solution explicitly. Actually, the analysis of this case is important for the kinked crack problem. In [5] we introduced a new approach for (1.1), including the case where the condition of a regular type operator is violated, by transforming the integral equations to integral equations of Fredholm of the second type with sufficiently smooth coefficients and by using the two theories of the Tikhonov regularization and reproducing kernels. And we can deal with a general linear singular integral equation, however, for simplicity, we state the results for this most typical case (1.1).

A brief outline of this paper is as follows. We shall use the Paley-Wiener space as the approximate function space and so, in Section 2 we introduce their fundamental properties. In Section 3 using the properties, we construct an approximate solution for the equation (1.1) based on the Tikhonov regularization and the theory of reproducing kernels. In Section 4, in order to guarantee the validity of our method we introduce a concrete analytic representation of the approximate solutions for the Carleman's integral equation for the case of the whole space and with constant coefficients. In Section 5 we summarize our result.

2 Paley-Wiener space and reproducing kernels

At first we shall fix notations following [10, 11] and at the same time we shall show the basic relation of the sampling theory (sinc method) and the theory of reproducing kernels.

We consider the integral transform, for $g \in L^2\left(-\frac{\pi}{h}, \frac{\pi}{h}\right)$, $(h > 0)$

$$f(z) = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} g(t) e^{-izt}\, dt. \tag{2.1}$$

In order to identify the image space following the theory of reproducing kernels [9], we form the reproducing kernel

$$\begin{aligned} K_h(z, \overline{u}) &= \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-izt} \overline{e^{-iut}}\, dt \\ &= \frac{1}{\pi(z - \overline{u})} \sin \frac{\pi}{h}(z - \overline{u}). \end{aligned} \tag{2.2}$$

The image space of (2.1) is called the Paley-Wiener space $W\left(\frac{\pi}{h}\right)$ $(\equiv W_h)$ comprised of all entire functions of exponential type satisfying the following conditions:

$$|f(z)| \leq C \exp\left(\frac{\pi |z|}{h}\right)$$

and

$$\int_{\mathbb{R}} |f(x)|^2\, dx < \infty$$

for some constant C and as $|z| \to \infty$. From the identity we have

$$K_h(jh, j'h) = \frac{1}{h}\delta(j, j')$$

with the Kronecker's δ. Since $\delta(j, j')$ is the reproducing kernel for the Hilbert space ℓ^2, from the general theory of integral transforms and the Parseval's identity we have the isometric identities in (2.1)

$$\frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |g(t)|^2 \,\mathrm{d}t = h\sum_j |f(jh)|^2 = \int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}x.$$

This means that the reproducing kernel Hilbert space H_{K_h} with $K_h(z, \overline{u})$ is characterized as a space comprising the Paley-Wiener space W_h with the norm squares defined above. Here we used the well-known result that $\{jh\}_j$ is a unique set for the Paley-Wiener space W_h, that is, $f(jh) = 0$ for all j implies $f \equiv 0$. Then, the reproducing property of $K_h(z, \overline{u})$ states that

$$\begin{aligned} f(x) &= (f(\cdot), K_h(\cdot, x))_{H_{K_h}} \\ &= h\sum_j f(jh)K_h(jh, x) \\ &= \int_{\mathbb{R}} f(\xi)K_h(\xi, x)\,\mathrm{d}\xi, \end{aligned}$$

in particular, for $x \in \mathbb{R}$. This representation is the sampling theorem which represents the whole data $f(x)$ in terms of the discrete data $\{f(jh)\}_j$. Furthermore, refer to [9] for a general result of the sampling theory and error estimates for some finite points $\{hj\}_j$.

3 Construction of approximate solutions

Note that L in (1.1) is a bounded linear operator from W_h into L^2, as we see from the Cauchy-Schwarz inequality and a boundedness of the finite Hilbert transform, see [7]. Then, from [10] and [5] the application of reproducing kernels to the Tikhonov regularization is given by the following propositions:

Proposition 1. *For $\lambda > 0$ we introduce the inner product in W_h and denote it by $W_h(L;\lambda)$ as*

$$(f_1, f_2)_{W_h(L;\lambda)} \equiv \lambda(f_1, f_2)_{W_h} + (Lf_1, Lf_2)_{L^2},$$

then $W_h(L;\lambda)$ becomes the Hilbert space with the reproducing kernel $K_\lambda(p,q)$ and satisfying the equation

$$K_h(\cdot, q) = (\lambda I + L^*L)K_\lambda(\cdot, q),$$

where L^* is the adjoint of $L : W_h \to L^2$.

Proposition 2. *For any $\lambda > 0$ and $g \in L^2$, the extremal function $f^*_{\lambda,g}$ of the following problem*

$$\inf_{f \in W_h} \left(\lambda \|f\|^2_{W_h} + \|Lf - g\|^2_{L^2}\right) \tag{3.1}$$

exists uniquely and $f^*_{\lambda,g}$ is represented by

$$f^*_{\lambda,g}(p) = (g, LK_\lambda(., p))_{L^2} \tag{3.2}$$

which is the member of W_h attaining the infimum in (3.1).

In (3.2), when g contains errors or noises, we need its error estimation. For this we can obtain the general result, see [5].

Proposition 3. *In* (3.2), *we obtain the estimate*

$$|f^*_{\lambda,g}(p)| \le \frac{1}{\sqrt{\lambda}} \sqrt{K(p,p)} \|g\|_{L^2}.$$

For the properties and error estimates we can take the limit

$$\lim_{\lambda \to 0} f^*_{\lambda,g}(p).$$

In particular, if there exists the Moore-Penrose generalized solution for the operator equation

$$Lf = g$$

in the sense of (3.2), then the limit converges uniformly to the Moore-Penrose generalized solution.

From Proposition 1 the reproducing kernel $K_\lambda(t,t')$ is calculated by solving the following integral equation of Fredholm of the second kind:

$$\frac{1}{\lambda}K_h(t,t') = K_\lambda(t,t') + \frac{1}{\lambda}(LK_\lambda(\cdot,t')(p),(LK_h(\cdot,t))(p))_{L^2}. \tag{3.3}$$

Then, it follows from Proposition 2 the extremal function $f^*_{\lambda,g}$ in (3.1) is given by

$$f^*_{\lambda,g}(t) = (g, LK_\lambda(\cdot,t))_{L^2}. \tag{3.4}$$

By applying the operator L to (3.3) with respect to functions of t (3.3) is reduced to

$$\frac{1}{\lambda}L_tK_h(t,t') = L_tK_\lambda(t,t') + \frac{1}{\lambda}L_t(LK_\lambda(\cdot,t')(p),(LK_h(\cdot,t))(p))_{L^2}. \tag{3.5}$$

Therefore, $LK_\lambda(\cdot,t)$ is given as the solution of the integral equation of Fredholm of the second kind (3.5) for each fixed t. Note that functions in (3.5) $L_tK_h(t,t')$ and $L_t\overline{LK_h(\cdot,t)}$ are calculated by using Fourier's integral and the formula. Now we denote the Fourier transform by $\mathcal{F}$

$$\mathcal{F}[f](\xi) = \int_{-\infty}^{\infty} f(t)e^{-i\xi t}\,dt$$

and the Hilbert transform by $\mathcal{H}$

$$[\mathcal{H}y](t) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\frac{y(\zeta)}{\zeta - t}\,d\zeta.$$

Then it is well known that

$$\mathcal{H}f = \mathcal{F}^{-1}\left(-i\operatorname{sgn}\xi(\mathcal{F}f)(\xi)\right)$$

and the Hilbert transform of the function

$$\frac{\sin(ax)}{x} \quad (a > 0)$$

is

$$\frac{\cos(ax)-1}{x}.$$

4 Example (Carleman's equation for the case of the whole line and with complex constant coefficients)

In this section as a typical example of applying our method we consider approximate solutions for Carleman's equation for the case of the whole line and with complex constant coefficients $a,\ b \in \mathbb{C}$

$$(\tilde{L}y)(t) \equiv ay(t) + \frac{b}{\pi i}\ \text{p.v.} \int_{-\infty}^{\infty} \frac{y(\zeta)}{\zeta - t}\, \mathrm{d}\zeta = g(t) \quad \text{on} \quad -\infty < t < \infty. \tag{4.1}$$

In the same way as (3.5), we obtain

$$\frac{1}{\lambda}\tilde{L}_t K_h(t,t') = \tilde{L}_t K_\lambda(t,t') + \frac{1}{\lambda}\tilde{L}_t(\tilde{L}K_\lambda(\cdot,t'))(p), (\tilde{L}K_h(\cdot,t))(p))_{L^2}. \tag{4.2}$$

If we can find the solution $\tilde{L}K_\lambda(t,t')$ in (4.2), then we can obtain the approximate solution of (4.1) expressed by (3.4) as the extremal function of (3.1) by virtue of Propositions in Section 3.

Let find the solution of (4.2).
Firstly, (4.2) can be rewritten as

$$\frac{1}{\lambda}\tilde{L}_t K_h(t,t') = \tilde{L}_t K_\lambda(t,t') + \frac{1}{\lambda}\int_{\mathbb{R}} \tilde{L}_p K_\lambda(\cdot,t')(p)\tilde{L}_t\overline{\tilde{L}_p K_h(\cdot,t)(p)}\, \mathrm{d}p. \tag{4.3}$$

From (2.2) the left-hand side of (4.3) becomes

$$\frac{1}{\lambda}\tilde{L}_t K_h(t,t') = \frac{1}{\lambda}\left(a\frac{\sin\frac{\pi}{h}(t-t')}{\pi(t-t')} + \frac{b}{i}\frac{\cos\frac{\pi}{h}(t-t')-1}{\pi(t-t')}\right).$$

And the kernel of the integral on the right-hand side of (4.3) becomes

$$\begin{aligned}
&\tilde{L}_{t'}\overline{\tilde{L}_t K_h(t,t')} \\
&= \tilde{L}_{t'}\left(\bar{a}\frac{\sin\frac{\pi}{h}(t-t')}{\pi(t-t')} - \frac{\bar{b}}{i}\frac{\cos\frac{\pi}{h}(t-t')-1}{\pi(t-t')}\right) \\
&= (|a|^2+|b|^2)\frac{\sin\frac{\pi}{h}(t-t')}{\pi(t-t')} - (a\bar{b}+\bar{a}b)i\frac{\cos\frac{\pi}{h}(t-t')-1}{\pi(t-t')}.
\end{aligned}$$

Note that

$$\mathcal{F}\left[\frac{\sin\frac{\pi t}{h}}{\pi t}\right](\xi) \equiv p(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| < \frac{\pi}{h}, \\ \frac{1}{2} & \text{if} \quad |\xi| = \frac{\pi}{h}, \\ 0 & \text{if} \quad |\xi| > \frac{\pi}{h}, \end{cases} \tag{4.4}$$

$$\mathcal{F}\left[\frac{1}{\pi t}\right](\xi) \equiv q_1(\xi) = \begin{cases} i & \text{if} \quad \xi < 0, \\ -i & \text{if} \quad \xi \geq 0, \end{cases}$$

$$\mathcal{F}\left[\frac{\cos\frac{\pi t}{h}}{\pi t}\right](\xi) \equiv q_2(\xi) = \begin{cases} i & \text{if} \quad \xi < -\frac{\pi}{h}, \\ 0 & \text{if} \quad -\frac{\pi}{h} < \xi < \frac{\pi}{h}, \\ -i & \text{if} \quad \xi > \frac{\pi}{h}, \end{cases}$$

$$\mathcal{F}\left[\frac{\cos\frac{\pi t}{h} - 1}{\pi t}\right](\xi) \equiv q(\xi) = \begin{cases} 0 & \text{if} \quad |\xi| > \frac{\pi}{h}, \\ -i & \text{if} \quad -\frac{\pi}{h} < \xi < 0, \\ i & \text{if} \quad 0 < \xi < \frac{\pi}{h}. \end{cases} \tag{4.5}$$

Secondly, we suppose $\tilde{L}_t K_\lambda(t, t') \equiv \varphi(t-t')$. Then, by applying the Fourier transform to both sides of (4.3) with respect to t and using the convolution theorem we have

$$\begin{aligned} &\frac{e^{-i\xi t'}}{\lambda}(ap(\xi) - biq(\xi)) \\ = \quad & e^{-i\xi t'}\mathcal{F}[\varphi](\xi) + \frac{e^{-i\xi t'}}{\lambda}\left((|a|^2 + |b|^2)p(\xi) - (a\bar{b} + \bar{a}b)iq(\xi)\right)\mathcal{F}[\varphi](\xi) \end{aligned}$$

and so,

$$\mathcal{F}[\varphi](\xi) = \frac{ap(\xi) - biq(\xi)}{\lambda + (|a|^2 + |b|^2)p(\xi) - (a\bar{b} + \bar{a}b)iq(\xi)}. \tag{4.6}$$

From (4.4) and (4.5) one can see that (4.6) is rewritten as

$$\mathcal{F}[\varphi](\xi) = \begin{cases} 0 & \text{if} \quad |\xi| > \frac{\pi}{h}, \\ \frac{a-b}{\lambda + |a-b|^2} & \text{if} \quad -\frac{\pi}{h} < \xi < 0, \\ \frac{a+b}{\lambda + |a+b|^2} & \text{if} \quad 0 < \xi < \frac{\pi}{h}. \end{cases} \tag{4.7}$$

Taking the inverse Fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[f](\xi) e^{i\xi t}\, \mathrm{d}\xi$$

to (4.7) yields the solution of (4.3)

$$\begin{aligned} \tilde{L}_t K_\lambda(t,t') &= \frac{a-b}{\lambda + |a-b|^2} \frac{1}{2\pi i(t-t')} \cdot \left(1 - e^{-\frac{i\pi}{h}(t-t')}\right) \\ &\quad + \frac{a+b}{\lambda + |a+b|^2} \frac{1}{2\pi i(t-t')} \cdot \left(e^{\frac{i\pi}{h}(t-t')} - 1\right). \end{aligned}$$

Therefore, the reproducing kernel $K_\lambda(t,t')$ is represented as follows:

$$\begin{aligned} K_\lambda(t,t') &= \frac{1}{\lambda + |a+b|^2} \frac{1}{2\pi} \int_{-\pi/h}^{0} e^{-i\xi(t-t')}\, \mathrm{d}\xi \\ &\quad + \frac{1}{\lambda + |a-b|^2} \frac{1}{2\pi} \int_{0}^{\pi/h} e^{-i\xi(t-t')}\, \mathrm{d}\xi. \end{aligned}$$

Theorem 1. *In (4.1) for any function $g \in L^2$ the best approximate solution $f^*_{\lambda,h,g}$ is represented by*

$$\begin{aligned} f^*_{\lambda,h,g}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi) \left[\frac{\overline{(a+b)}}{\lambda + |a+b|^2} \int_{-\pi/h}^{0} e^{i\eta(\xi - t)}\, \mathrm{d}\eta \right. \\ &\qquad \left. + \frac{\overline{(a-b)}}{\lambda + |a-b|^2} \int_{0}^{\pi/h} e^{i\eta(\xi - t)}\, \mathrm{d}\eta \right] \mathrm{d}\xi. \end{aligned}$$

Moreover, if we take g as $\tilde{L} f = g$ for a function $f \in W_h$, then we obtain the result

$$\lim_{\lambda \to 0} f^*_{\lambda,h,g}(t) = f(t)$$

and this convergence is uniformly.

Note that in the regular type case, that is, $a^2 - b^2 \neq 0$, we can take $\lambda = 0$. Hence, we do not need the Tikhonov regularization in our problem. It is a trivial case. Furthermore, since it follows from Theorem 1 that

$$\begin{aligned}\tilde{L}_t f^*_{\lambda,h,g}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\left[\frac{|a+b|^2}{\lambda+|a+b|^2}\int_{-\pi/h}^{0} e^{i\eta(\xi-t)}\,d\eta \right.\\ &\qquad\left. +\frac{|a-b|^2}{\lambda+|a-b|^2}\int_{0}^{\pi/h} e^{i\eta(\xi-t)}\,d\eta\right] d\xi,\end{aligned}$$

one can see that for the cases $\lambda = 0$ and $a^2 - b^2 \neq 0$

$$\lim_{h\to 0} \tilde{L}_t f^*_{0,h,g}(t) = g(t)$$

at points t where g is continuous.

Now we will confirm the validity of Theorem 1. For this we consider the following examples.

Example 1 (the Hilbert transform ($a = 0$ and $b = i$)).
For the extremal problem

$$\inf_{f\in W_h}\left\{\|\mathcal{H}f - g\|^2_{L^2}\right\},$$

the extremal function $f^*_{\mathcal{H},h,g}$ attaining the infimum exists uniquely and it is given by

$$f^*_{\mathcal{H},h,g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(\xi)\frac{1}{\xi - t}\left[\cos\frac{\pi}{h}(\xi - t) - 1\right] d\xi.$$

Moreover, we have

$$\left[\mathcal{H}f^*_{H,h,g}\right](t) = (g, K_h(\cdot,t))_{L^2},$$

that is, $\left[\mathcal{H}f^*_{\mathcal{H},h,g}\right]$ is the orthogonal projection of g onto the Paley-Wiener space W_h.

Example 1 also means to give an approximate Hilbert transform for any L^2 function g by an ordinary integral

$$\lim_{h\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} g(\xi)\frac{1}{\xi - t}\left[1 - \cos\frac{\pi}{h}(\xi - t)\right] d\xi = [\mathcal{H}g](t)$$

at the points t where g is continuous.

In the case of regular operators ($a^2 - b^2 \neq 0$) it is obvious to derive the exact solution. Indeed, in this case (4.1) becomes

$$(\tilde{L}y)(t) = ay(t) + \frac{b}{i}\mathcal{H}y(t) = g(t) \quad \text{on} \quad -\infty < t < \infty.$$

Applying $\mathcal{H}$ to both sides of this yields

$$a\mathcal{H}y(t) + \frac{b}{i}\mathcal{H}(\mathcal{H}y)(t) = \mathcal{H}g(t).$$

Since

$$\mathcal{H}\mathcal{H}y = -y,$$

one knows that

$$a\mathcal{H}y(t) - \frac{b}{i}y(t) = \mathcal{H}g(t).$$

Hence, we obtain an explicit solution

$$y(t) = \frac{1}{a^2 - b^2}\left\{ag(t) - \frac{b}{i}\mathcal{H}g(t)\right\}. \tag{4.8}$$

On the other hand, it follows from Theorem 1 and Example 1

$$\begin{aligned} f^*_{0,h,g}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\left[\frac{1}{a+b}\int_{-\frac{\pi}{h}}^{0} e^{i\eta(\xi-t)}\, d\eta + \frac{1}{a-b}\int_{0}^{\frac{\pi}{h}} e^{i\eta(\xi-t)}\, d\eta\right] d\xi \\ &= \frac{1}{\pi(a^2-b^2)}\int_{-\infty}^{\infty} \frac{g(\xi)}{\xi - t}\left[a \sin\frac{\pi}{h}(\xi - t) + bi\left(1 - \cos\frac{\pi}{h}(\xi - t)\right)\right] d\xi \\ &= \frac{1}{a^2-b^2}\left[a(g, K_h(\cdot, t))_{L^2} - bif^*_{\mathcal{H},h,g}(t)\right]. \end{aligned}$$

Consequently, as $h \to 0$,

$$f^*_{0,h,g}(t) \to y(t) = \frac{1}{a^2-b^2}\left\{ag(t) - \frac{b}{i}\mathcal{H}g(t)\right\}$$

which is equivalent to (4.8).

Next, for the irregular cases, $a^2 - b^2 = 0$, the integral equations have the solutions only for very special functions g.

For example, we consider a case of $a = b = \frac{1}{2}$, that is,

$$(\tilde{L}y)(t) = \frac{1}{2}y(t) + \frac{1}{2i}\mathcal{H}y(t) = g(t) \quad \text{on} \quad -\infty < t < \infty. \tag{4.9}$$

This equation implies that $g(t)$ is the boundary value of an analytic function in the upper half-plane of $\mathbb{C}$.

Since $\mathcal{H}(\mathcal{H}y) = -y$, we see that

$$ig(t) = \frac{1}{2}\mathcal{H}y - \frac{1}{2i}y = \mathcal{H}g.$$

Therefore, we know that the solvability condition for (4.9) is

$$-\frac{1}{2}g(t) - \frac{i}{2}\mathcal{H}g(t) = 0,$$

see, pp. 270 [1] for the details. And then it can be easily seen that solutions of (4.9) can be described by

$$y(t) = g(t) + H_-(t), \tag{4.10}$$

where H_- is the boundary value of any analytic function in the lower half-plane of $\mathbb{C}$.

Example 2 (an irregular case $(a = b = \frac{1}{2})$).
From Theorem 1 *and* Example 1 *we obtain the extremal function related with* (4.9) *as follows* ;

$$\begin{aligned} f^*_{+0,h,g}(t) &\equiv \lim_{\lambda\to 0} f^*_{\lambda,h,g}(t) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\frac{1}{i(\xi - t)}\left[1 - e^{-i\frac{\pi}{h}(\xi - t)}\right] d\xi \\ &= \frac{1}{2}\left[if^*_{\mathcal{H},h,g}(t) + (g, K_h(\cdot, t))_{L^2}\right]. \end{aligned}$$

Furthermore, if the condition $\mathcal{H}g = ig$ for existence of the solutions is fulfilled, then

$$\begin{aligned} \lim_{h\to 0} f^*_{+0,h,g}(t) &= \frac{1}{2}\left[i(-\mathcal{H}g(t)) + g\right] = g(t), \\ \lim_{h\to 0} \tilde{L}_t f^*_{+0,h,g}(t) &= \frac{1}{2}g(t) + \frac{1}{2i}\mathcal{H}g(t) = g(t) \end{aligned}$$

at the points t where $g(t)$ is continuous. This result coincides with (4.10).

Similarly, in a case of $a = -b = \frac{1}{2}$, that is,

$$(\tilde{L}y)(t) = \frac{1}{2}y(t) - \frac{1}{2i}\mathcal{H}y(t) = g(t) \quad \text{on} \quad -\infty < t < \infty. \tag{4.11}$$

It implies that $g(t)$ is the boundary value of an analytic function in the lower half-plane of $\mathbb{C}$. Then, since $\mathcal{H}(\mathcal{H}y) = -y$, the solvability condition for (4.11) is

$$-\frac{1}{2}g(t) + \frac{i}{2}\mathcal{H}g(t) = 0.$$

Therefore, solutions of (4.11) can be given by

$$y(t) = g(t) + H_+(t), \tag{4.12}$$

where H_+ is the boundary value of any analytic function in the upper half-plane of $\mathbb{C}$.

Example 3 (an irregular case ($a = -b = \frac{1}{2}$)).
From Theorem 1 *and* Example 1 *we obtain the extremal function related with* (4.11) *as follows* ;

$$\begin{aligned} f^*_{+0,h,g}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\frac{1}{i(\xi - t)}\left[e^{i\frac{\pi}{h}(\xi-t)} - 1\right] d\xi \\ &= \frac{1}{2}\left[-if^*_{\mathcal{H},h,g}(t) + (g, K_h(\cdot,t))_{L^2}\right]. \end{aligned}$$

Furthermore, if the condition $\mathcal{H}g = -ig$ *for existence of the solutions is satisfied, then*

$$\begin{aligned} \lim_{h\to 0} f^*_{+0,h,g}(t) &= \frac{1}{2}\left[-i(-\mathcal{H}g(t)) + g\right] = g(t), \\ \lim_{h\to 0} \tilde{L}_t f^*_{+0,h,g}(t) &= \frac{1}{2}g(t) - \frac{1}{2i}\mathcal{H}g(t) = g(t) \end{aligned}$$

at the points t where $g(t)$ *is continuous. This result coincides with* (4.12).

Therefore, in Theorem 1 we obtain the explicit representations of the approximate solutions of (4.1) even in the irregular cases. Surprisingly enough, we can obtain the explicit representations of the "solutions" for any L^2 function g. In [5] we gave also a new algorithm with error estimates in order to escape the Fredholm's integral equation and derive an effective discretization.

5 Conclusion

In this paper we gave a new method for construction of approximate solutions of Carleman's Equation (1.1) which is deeply related to the problem for determining the direction of crack propagation in the elastic plate mentioned in Section 1. By employing the theory of the Reproducing Kernel we considered the approximate solutions as the extremal function of that problem (3.1). Then, finding the function is reduced to solve the Fredholm integral equation of the 2nd kind (3.5).

Lastly, in Section 4 we considered a concrete example of Carleman's Equation in the whole line with complex constant coefficients and compared our result with exact solutions. As a result we saw that they coincide in some examples including both cases of regular and irregular type operators.

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