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# ラプラス変換の実逆変換への再生核空間の応用

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In this note we describe Tikhonov regularization in Part I and its application to the real inverse of the Laplace transform in Part II.

# Part I

# Fundamentals on Tikhonov regularization

## 1 Best approximation problems

Let $K : E \times E \to \mathbf{C}$ be positive definite and $\mathcal{H}$ be a Hilbert space. Suppose that we are given a bounded linear operator $L : H_K \to \mathcal{H}$. Let us consider the following minimizing problem

$$\min_{f \in H_K} \|Lf - d\|_{\mathcal{H}} \tag{1}$$

for $d \in \mathcal{H}$.

If $L$ has an inverse, then it is easy to see that $f = L^{-1}d$ is a unique minimizer. Therefore, our main concern goes to the case when $L$ is not invertible. Let us set

$$k(p,q) = \langle L^*LK(\cdot,q), L^*LK(\cdot,p)\rangle_{H_K} = (L^*LL^*L[K_q])\,(p).$$

Let us set

$$P = \mathrm{Proj}(\,H \to \mathrm{N}(L)^{\perp}\,) = \mathrm{Proj}(\,H \to \overline{\mathrm{Ran}(L^*L)}\,).$$

**Proposition 1.1.** *Under the above notation, we have*

$$H_k = \{L^*Lf \ : \ f \in H_K\}$$

*and the inner product is given by*

$$\langle L^*Lf, L^*Lg\rangle_{H_k} = \langle Pf, g\rangle_{H_K}$$

*for* $f, g \in H_K$.

*Proof.* Let us set $\mathcal{H}_0$ as the Hilbert space in the right-hand side. Then it is easy to see the following.

1. $\mathcal{H}_0 \subset \mathcal{F}(E)$ in the sense of continuous embedding.
2. $k(\cdot, q) = L^*L\,[L^*LK_q] \in \mathcal{H}_0$ for all $q \in \mathcal{H}_0$ and
$$\{k(\cdot, q) \ : \ q \in E\}$$
is dense in $\mathcal{H}_0$. Indeed, let $L^*Lf \in \mathcal{H}_0$ be an element perpendicular to all $k(\cdot, q)$, $q \in E$. Then we have
$$\langle L^*Lf, k(\cdot, q)\rangle_{\mathcal{H}_0} = \langle Pf, L^*LK_q\rangle_{H_K} = \langle L^*LPf, K_q\rangle_{H_K},$$
which implies $L^*LPf = 0$ and hence $Pf \in \mathrm{R}(L^*L) \cap \mathrm{N}(L^*L) = \{0\}$. Thus, we conclude that $\{k(\cdot, q) \ : \ q \in E\}$ spans a dense subspace in $\mathcal{H}_0$.
3. If $p, q \in \mathcal{H}_0$, then we have
$$\begin{aligned}\langle k(\cdot, q), k(\cdot, p)\rangle_{\mathcal{H}_0} &= \langle\, L^*L\,[L^*LK_q]\,, L^*L\,[L^*LK_q]\,\rangle_{\mathcal{H}_0} \\ &= \langle L^*LK_q, L^*LK_p\rangle_{H_K} = k(p, q).\end{aligned}$$

In view of these three facts, we see that $\mathcal{H} = H_k$ with norm coincidence. □

**Theorem 1.2.** (1) *admits a solution if and only if* $L^*d \in H_k$. *If this is the case, then we have* $L^*d = L^*L\tilde{f}$ *for some* $\tilde{f} \in H_K$ *and* $\tilde{f}$ *is a solution to* (1).

*Proof.* Suppose that (1) admits a solution $\tilde{f}$. Then we have

$$\|Lf - d\|_{\mathcal{H}}^2 = \|L(f - \tilde{f})\|_{\mathcal{H}}^2 + 2\mathrm{Re}(\,\langle f - \tilde{f}, L^*(L\tilde{f} - d)\rangle_{H_K}) + \|L\tilde{f} - d\|_{\mathcal{H}}^2.$$

In view of this formula, we see that $\tilde{f}$ is a solution to (1) if and only if $L^*d = L^*L\tilde{f}$. From Proposition 1.1, we see that (1) has a solution if and only if $L^*d \in H_k$. □

**Theorem 1.3.** *Keep to the same assumption as above. Suppose further that* $0 \in \rho(L^*L)$. *Then we have*

$$\tilde{f} = (L^*L)^{-1}L^*d$$

*is a unique element that attains the minimum.*

**Definition 1.4.** Let $f_d \in H_K$ be the element such that $L^*d = L^*Lf_d$ with $f_d \in \mathrm{N}(L)^\perp$.

**Theorem 1.5.** *Keep to the same assumption as above. Then we have*

$$f_d(p) = \langle L^*d, L^*LK(\cdot, p)\rangle_{H_k}.$$

*Proof.* From the definition of $H_K$, we have

$$f_d(p) = \langle f_d, K(\cdot, p)\rangle_{H_K}.$$

Since $f_d \in \mathrm{N}(L)^\perp$, it follows that

$$\langle f_d, K(\cdot, p)\rangle_{H_K} = \langle P f_d, P K(\cdot, p)\rangle_{H_K}.$$

In view of the definition of $H_k$, we see that

$$\langle P f_d, P K(\cdot, p) \rangle_{H_K} = \langle L^*Lf_d, L^*LK(\cdot, p) \rangle_{H_k}.$$

Since $L^*Lf_d = L^*d$, we conclude, together with the observations above,

$$f_d(p) = \langle L^*d, L^*LK(\cdot, p) \rangle_{H_k}.$$

This is the desired result. □

**Definition 1.6.** One defines an unbounded operator $L^\dagger : \mathcal{H} \to H_K$ by

$$L^\dagger d = f_d,$$

where $f_d$ is given in Definition 1.4.

Let $\{E_\lambda\}_{\lambda \in \mathbf{R}}$ be a spectral family of $L^*L$.

**Definition 1.7.** Let $\alpha > 0$. Then define

$$f_{d,\alpha} := \left(\int_{\mathbf{R}} \frac{1}{\lambda + \alpha}\, dE_\lambda\right) L^*d.$$

**Proposition 1.8.** *Let $d \in \mathrm{D}(L^\dagger)$. Then we have*

$$\lim_{\alpha \downarrow 0} (L^*L + \alpha)^{-1} L^*d = f_d$$

*in the topology of $H_K$.*

*Proof.* Note that

$$(L^*L + \alpha)^{-1}L^*d = (L^*L + \alpha)^{-1}L^*Lf_d = \left(\int_{\mathbf{R}} \frac{\lambda}{\lambda + \alpha}\, dE_\lambda\right) f_d.$$

As a consequence, we obtain

$$\| (L^*L + \alpha)^{-1}L^*d - f_d \,:\, H_K\| = \left\| \left(\int_{\mathbf{R}} \frac{\alpha}{\lambda + \alpha}\, dE_\lambda\right) f_d \right\| = \sqrt{\int_{\mathbf{R}} \frac{\alpha^2}{(\lambda + \alpha)^2}\, d\langle E_\lambda f_d, f_d\rangle},$$

where it will be understood that the integral in the most right-hand side is the Stieltjes integral. Now

$$\int_{\mathbf{R}} d\langle E_\lambda f_d, f_d\rangle = \langle f_d, f_d\rangle < \infty,$$

we are in the position of applying the Lebesgue convergence theorem to have

$$\lim_{\alpha\downarrow 0}\sqrt{\int_{\mathbf{R}} \frac{\alpha^2}{(\lambda+\alpha)^2}\, d\langle E_\lambda f_d, f_d\rangle} = 0,$$

which yields the desired convergence. □

**Proposition 1.9.** *Under the same notation, we have*

$$\| L f_{d,\alpha} : \mathcal{H}\| \le \|d : \mathcal{H}\|.$$

*Proof.* Let us set

$$L = U\sqrt{L^* L},$$

where $U$ is a partial isometry. Then we have

$$\begin{aligned} L\left(\int_{\mathbf{R}} \frac{1}{\lambda+\alpha}\, dE_\lambda\right) L^* &= U\sqrt{L^*L}\left(\int_{\mathbf{R}} \frac{1}{\lambda+\alpha}\, dE_\lambda\right)\sqrt{L^*L}\, U^* \\ &= U\left(\int_{\mathbf{R}} \frac{\lambda}{\lambda+\alpha}\, dE_\lambda\right) U^* \le 1. \end{aligned}$$

Thus, the result is immediate. □

**Proposition 1.10.**

$$\| f_{d,\alpha} : \mathcal{H}\| \le \frac{\|d : \mathcal{H}\|}{2\sqrt{\alpha}}.$$

*Proof.* We shall go through the same argument as above. We now have to consider

$$L\left(\int_{\mathbf{R}} \frac{1}{(\lambda+\alpha)^2}\, d\lambda\right) L^*.$$

Since

$$\begin{aligned} L\left(\int_{\mathbf{R}} \frac{1}{(\lambda+\alpha)^2}\, d\lambda\right) L^* &= U\sqrt{L^*L}\left(\int_{\mathbf{R}} \frac{1}{(\lambda+\alpha)^2}\, d\lambda\right)\sqrt{L^*L}\, U^* \\ &= U\left(\int_{\mathbf{R}} \frac{\lambda}{(\lambda+\alpha)^2}\, d\lambda\right) U^* \\ &\le \frac{1}{4\alpha}, \end{aligned}$$

we obtain the desired result. □

**Theorem 1.11.** *Let $\alpha > 0$. Then the following minimizing problem admits a unique solution*

$$\min_{f \in H_K} \alpha \|f \,:\, H_K\|^2 + \|d - Lf \,:\, \mathcal{H}\|^2.$$

*And the minimum is attaind by*

$$f_{d,\alpha} = \left( \int_{\mathbf{R}} \frac{1}{\lambda + \alpha} dE_\lambda \right) L^* d$$

*Proof.* We complete the square of the formula in question.

$$\begin{aligned}
&\alpha \|f \,:\, H_K\|^2 + \|d - Lf \,:\, \mathcal{H}\|^2 \\
&= \langle (\alpha + L^*L) f, f \rangle - 2\mathrm{Re}\langle L^* d, f \rangle + \|d \,:\, H\|^2 \\
&= \langle (\alpha + L^*L)^{\frac{1}{2}} f, (\alpha + L^*L)^{\frac{1}{2}} f \rangle - 2\mathrm{Re}\langle (\alpha + L^*L)^{-\frac{1}{2}} L^* d, (\alpha + L^*L)^{\frac{1}{2}} f \rangle + \|d \,:\, H\|^2 \\
&= \left\| (\alpha + L^*L)^{\frac{1}{2}} f - (\alpha + L^*L)^{-\frac{1}{2}} L^* d \,:\, H_K \right\|^2 + \|d \,:\, H\|^2 - \|(\alpha + L^*L)^{-\frac{1}{2}} d \,:\, H\|^2.
\end{aligned}$$

As a result, we see that

$$f = (\alpha + L^*L)^{-1} L^* d = \left( \int_{\mathbf{R}} \frac{1}{\alpha + \lambda} dE_\lambda \right) L^* d = f_{d,\alpha}$$

is the unique minimizer. $\square$

**Theorem 1.12.** *Suppose that $\alpha : (0,1) \to (0,\infty)$ is a function of $\delta$ such that*

$$\lim_{\delta \downarrow 0} \left( \alpha(\delta) + \frac{\delta^2}{\alpha(\delta)} \right) = 0.$$

*Let $D : (0,1) \to \mathcal{H}$ be a function such that*

$$\|D(\delta) - d\|_{\mathcal{H}} \leq \delta$$

*for all $\delta \in (0,1)$. If $d \in \mathrm{D}(L^\dagger)$, then we have*

$$\lim_{\delta \downarrow 0} f_{D(\delta),\alpha(\delta)} = f_d = L^\dagger d.$$

*Proof.* Now that we have established $f_d = \lim_{\delta \downarrow 0} f_{d,\alpha(\delta)}$, we have only to show that

$$\lim_{\delta \downarrow 0} \left( f_{D(\delta),\alpha(\delta)} - f_{d,\alpha(\delta)} \right) = 0.$$

However, as we have seen in Proposition 1.10, the function $f_{D(\delta),\alpha(\delta)} - f_{d,\alpha(\delta)} = f_{D(\delta)-d,\alpha(\delta)}$ does not exceed $\dfrac{\delta}{2\sqrt{\alpha(\delta)}}$ in $H_K$-norm. Thus, we obtain the desired result. $\square$

**Theorem 1.13.** *Let $L : H_K \to \mathcal{H}$ be a bounded linear operator. Then define an inner product*

$$\langle f_1, f_2\rangle_{H_{K_\lambda}} = \lambda \langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_{\mathcal{H}}$$

*for $f_1, f_2 \in H_K$. Then $(H_K, \langle\cdot,\cdot\rangle_{H_{K_\lambda}})$ is a reproducing kernel Hilbert space whose reproducing kernel is given by*

$$K_\lambda(p,q) = [(\lambda + L^*L)^{-1}K_q](p).$$

*Proof.* It is easy to check that $K_\lambda(\cdot,q) \in H_K$. Furthermore,

$$\begin{aligned}
\langle K_\lambda(\cdot,q), K_\lambda(\cdot,p)\rangle &= \lambda \langle K_\lambda(\cdot,q), K_\lambda(\cdot,p)\rangle_{H_K} + \langle L^*LK_\lambda(\cdot,q), K_\lambda(\cdot,p)\rangle_{H_K} \\
&= \langle(\lambda + L^*L)K_\lambda(\cdot,q), K_\lambda(\cdot,p)\rangle_{H_K} \\
&= \langle K_q, [(\lambda + L^*L)^{-1}K_p]\rangle_{H_K} \\
&= \langle[(\lambda + L^*L)^{-1}K_q], K_p\rangle_{H_K} \\
&= [(\lambda + L^*L)^{-1}K_q](p) = K_\lambda(p,q).
\end{aligned}$$

Thus, the proof is complete. □

**Corollary 1.14.** *Assume in addition that $\|L\| < \sqrt{\lambda}$, then we have*

$$K_L(p,q;\lambda) = \sum_{n=0}^{\infty}\left(-\frac{L^*L}{\lambda}\right)^n \frac{K(p,q)}{\lambda}.$$

**Theorem 1.15.** *Under the same assumption as Theorem 1.13,*

$$f \in H_K \mapsto \lambda\,\|f\, :\, H_K\|^2 + \|Lf - g\, :\, \mathcal{H}\|^2$$

*attains minimum and the minimum is attained only at $F \in H_K$ such that*

$$F(p) = \langle g, LK_\lambda(\cdot,p)\rangle_{\mathcal{H}}.$$

*Furthermore, $F(p)$ satisfies*

$$|F(p)| \le \|L\|_{H_K\to\mathcal{H}}\sqrt{\frac{K(p,p)}{\lambda}}\|g\|_{\mathcal{H}}. \tag{2}$$

*Proof.* We calculate, keeping to the same notation as Theorem 1.13,

$$\begin{aligned}
\lambda\,\|f\, :\, H_K\|^2 + \|Lf - g\, :\, \mathcal{H}\|^2 &= \lambda \langle f, f\rangle_{H_K} + \langle Lf, Lf\rangle_{\mathcal{H}} - 2\mathrm{Re}\langle Lf, g\rangle_{\mathcal{H}} + \langle g, g\rangle_{\mathcal{H}} \\
&= \langle f, f\rangle_{H_{K_\lambda}} - 2\mathrm{Re}\langle Lf, g\rangle_{\mathcal{H}} + \langle g, g\rangle_{\mathcal{H}}.
\end{aligned}$$

By the Riesz representation theorem there exists $F \in H_{K_\lambda} = H_K$ such that

$$\langle Lf, g\rangle_{\mathcal{H}} = \langle f, F\rangle_{H_{K_\lambda}} \tag{3}$$

for all $f \in H_K$. It is ease to see that the functional in question attains its miminum only at $F$. It is easy to obtain the value of $F(p)$ using (3):

$$F(p) = \langle F, K_\lambda(\cdot,p)\rangle_{H_{K_\lambda}} = \langle g, LK_\lambda(\cdot,p)\rangle_{\mathcal{H}}.$$

Finally let us prove (2). Note that

$$\begin{aligned}\|LK_\lambda(\cdot,p)\ :\ \mathcal{H}\|^2 &= \|L(\lambda+L^*L)^{-1}K_p\ :\ \mathcal{H}\|^2\\ &= \langle(\lambda+L^*L)^{-1}L^*L(\lambda+L^*L)^{-1}K_p, K_p\rangle_{H_K}\\ &\leq \frac{1}{\lambda}\langle K_p, K_p\rangle_{H_K}\\ &\leq \frac{K(p,p)}{\lambda}.\end{aligned}$$

Hence by the Cauchy Schwarz inequality we have (2). □

## Part II

# Application to the real inverse of the Laplace transform

We shall give a numerical real inversion formula of the Laplace transform

$$(\mathcal{L}f)(p) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt, \quad p > 0$$

on a certain function space. This integral transform is fundamental in mathematical science and engineering. The inversion of the Laplace transform is, in general, given by a complex form, however, we are interested in its real inversion, which is a problem to find the original function $f(t)$ from a given image function $F(p), p \geq 0$, and it is required in various practical problems. The real inversion is unstable in usual settings, thus the real inversion is ill-posed in the sense of Hadamard, and numerical real inversion methods have not been established [2, 4]. In other words, the image functions of the Laplace transform are analytic on a half complex plane, and the real inversion will be very complicated. One is lead to thinking that its real inversion is essentially involved, because we need to grasp "analyticity" from the real and discrete data.

In the present paper, we shall propose a new approach to the numerical real inversion of the Laplace transform based on the compactness of the Laplace transform on a certain reproducing kernel Hilbert space [7]. ¿From the inverse analysis view point, there have been several proposals employing stabilisation method such as Tikhonov regularization. On the other hand, singular value decomposition is applicable not only for reconstruction of solutions, but also for Hilbert scales and noise reduction of measurement data. Though the singular value decomposition has various applications, its concrete treatments are hard both mathematically [5] and numerically [3].

The Laplace transform is not compact on usual Lebesgue or Sobolev spaces, and it may has continuous spectrum. We shall discuss in more details the compactness of the modified Laplace transform. In the setting some truncation is required for numerical real inversions [4]. One of our key idea is the use of the reproducing kernel Hilbert space, in which we have a concrete representation for the adjoint operator and this enables us to realize an effective numerical real inversion. The proposed approach by means of numerical singular value decomposition is straightforward, so it is applicable to many inverse problems.

### 1.1 Compactness of the Laplace transform on the reproducing kernel Hilbert space

We shall introduce a simple reproducing kernel Hilbert space $H_K$ comprised of absolutely continuous functions $f$ on the positive real line $\mathbf{R}^+$ with finite norms

$$\|f : H_K\| := \left\{ \int_0^\infty |f'(t)|^2 \frac{e^t}{t}\, dt \right\}^{1/2}$$

and satisfying $f(0) = 0$. This Hilbert space admits the reproducing kernel [6]:

$$K(s,t) = \int_0^{\min(s,t)} \xi e^{-\xi}\, d\xi.$$

Then we see that

$$\int_0^\infty |(\mathcal{L}f)(p)p|^2\, dp \le \frac{1}{2}\|f : H_K\|^2. \tag{4}$$

That is, the linear operator on $H_K$

$$f \mapsto [p \mapsto (\mathcal{L}f)(p)p =: Lf(p)]$$

into $L^2(\mathbf{R}^+, dp) = L^2(\mathbf{R}^+)$ is bounded [7]. We can find some general reproducing kernel Hilbert spaces $H_K$ satisfying (4) in [7]. Furthermore, the following theorem will play a key role in the construction of our real inversion formula.

**Theorem 1.16 (Compactness of $L$).** *Keep to the notation above. Then*

$$f \in H_K \to Lf \in L^2(\mathbf{R}^+)$$

*is an injective and compact linear operator.*

To prove this, it suffices to show that

$$\lim_{R\to\infty} \|L_R : B(H_K, L^2(\mathbf{R}^+))\| = 0$$

where $L_R, R > 0$ is a truncated operator defined by

$$(L_R f)(p) := p \int_R^\infty e^{-pt} f(t)\, dt$$

for $f \in H_K$. Indeed, the difference from $L$ is $H_K$-$L^2(\mathbf{R}^+)$ compact in view of the explicit formula

$$Lf(p) - (L_R f)(p) = p \int_0^R e^{-pt} f(t)\, dt, \quad f \in H_K.$$

Therefore, once we show

$$\lim_{R\to\infty} \|L_R \,:\, B(H_K, L^2(\mathbf{R}^+))\| = 0,$$

then we can conclude that the operator in question itself is compact.

It is not so hard to see that the set of all smooth functions supported on $(0,\infty)$ is dense in $H_K$. Therefore, let $f \in H_K$ and assume that $f$ is a smooth function supported on $(0,\infty)$. Furthermore, as we can see from the proof below the family $\{L_R\}_{R>0}$ forms a uniformly bounded family.

If we carry out integration by parts, then we obtain

$$(L_R f)(p) = \int_R^\infty \frac{d}{dt}(-e^{-pt})\, f(t)\, dt = \int_R^\infty e^{-pt} f'(t)\, dt + e^{-Rp} f(R).$$

As a result,

$$\|L_R f \,:\, L^2(\mathbf{R}^+)\| \le \left\| \int_R^\infty e^{-pt} f'(t)\, dt \,:\, L^2(\mathbf{R}^+) \right\| + \frac{|f(R)|}{\sqrt{2R}}$$

by virtue of the Minkowski inequality.

We estimate the right-hand side by using the Hölder inequality:

$$\begin{aligned} \left| \int_R^\infty e^{-tp} f'(t)\, dt \right| &\le \left( \int_R^\infty |f'(t)|^2 \frac{e^t}{t}\, dt \right)^{\frac{1}{2}} \cdot \left( \int_R^\infty e^{-2tp-t} t\, dt \right)^{\frac{1}{2}} \\ &\le \frac{1}{(2p+1)R} \|f \,:\, H_K\|. \end{aligned}$$

Consequently it follows that

$$\left\| \int_R^\infty e^{-pt} f'(t)\, dt \,:\, L^2(\mathbf{R}^+) \right\| \le \frac{1}{\sqrt{2}R} \|f \,:\, H_K\|.$$

As for the second term, we see

$$|f(R)| \le \int_0^R |f'(t)|\, dt \le \left( \int_0^R |f'(t)|^2 \frac{e^t}{t}\, dt \right)^{\frac{1}{2}} \cdot \left( \int_0^R t e^{-t}\, dt \right)^{\frac{1}{2}}$$

by virtue of the Hölder inequality. Therefore we obtain

$$|f(R)| \le \|f \,:\, H_K\|.$$

In view of these observations, we conclude

$$\|L_R \,:\, B(H_K, L^2(\mathbf{R}^+))\| \le \left(1 + \frac{1}{\sqrt{2}}\right) R^{-\frac{1}{2}},$$

proving that $L_R$ tends to 0 in the norm topology. The proof is now complete.

A simple calculation shows

$$K(s,t) = \begin{cases} -se^{-s} - e^{-s} + 1 & s \le t; \\ -te^{-t} - e^{-t} + 1 & s \ge t, \end{cases}$$

$$(\mathcal{L}K(\cdot,t))(p) = -e^{-tp}e^{-t}\left\{\frac{t}{p(p+1)} + \frac{1}{p(p+1)^2}\right\} + \frac{1}{p(p+1)^2}.$$

Therefore, for the compact operator $L$, its adjoint operator $L^*$ can be written out in full:

$$\begin{aligned}(L^*g)(t) &= (L^*g, K(\cdot,t))_{H_K} \\ &= (g, LK(\cdot,t))_{L^2(\mathbf{R}^+)} \\ &= \int_0^\infty g(\xi)\frac{1}{(\xi+1)^2}\left\{1 - e^{-t(\xi+1)}\Big(t(\xi+1)+1\Big)\right\} d\xi.\end{aligned}$$

Our key theorem assures that the operator $Lf(p) = p(\mathcal{L}f)(p)$ has the singular system. Let $\{\lambda_n\}$ be singular values of the operator $L$, $\{v_n\}$ and $\{u_n\}$ be complete orthonormal systems of $\mathcal{N}(L)^\perp$ (the orthogonal compliment of the null space) and $\overline{\mathcal{R}(L)}$ (the closure of the range space), respectively satisfying

$$Lv_n = \lambda_n u_n, \quad L^*u_n = \lambda_n v_n.$$

Using the singular systems and truncated singular value expansion [4], we obtain the following representation.

**Theorem 1.17 (Real inversion formula of the Laplace transform).** *We consider the Laplace transform $\mathcal{L}f = F$. If the original function $f$ belongs to $H_K$, then the real inversion of the Laplace transform $\mathcal{L}^{-1}$ is*

$$(\mathcal{L}^{-1}F)(t) = \sum_{n=1}^\infty \frac{1}{\lambda_n}\left(\int_0^\infty F(p)u_n(p)p\,dp\right)v_n(t).$$

*And for any $F$ with $F(p)p \in L^2(\mathbf{R}^+)$ and a natural number $M$, spectral cut-off regularization $\mathcal{L}_M^{-1}$ is given as*

$$(\mathcal{L}_M^{-1}F)(t) = \sum_{n=1}^M \frac{1}{\lambda_n}\left(\int_0^\infty F(p)u_n(p)p\,dp\right)v_n(t).$$

# 2 Numerical Singular Value Decomposition and Examples

Now we present a discretization of the singular value decomposition of $L$. The singular system $\lambda_n$ and $v_n$ satisfies an eigenvalue problem $L^*Lv = \lambda^2 v$. Employing a numerical integration scheme with a discretization parameter $N$, quadrature points $x_i$ and weights $w_i, 0 \leq i \leq N$, i.e.,

$$\int_0^\infty f(x)dx \approx \sum_{i=0}^N f(x_i)w_i,$$

we discretize the eigenvalue problem and obtain a linear eigenvalue problem of a matrix whose $(i,j)$-entry is

$$a_{ij} = \sum_{k=0}^N \frac{x_k e^{-x_j x_k}}{(x_k+1)^2} \left\{1 - e^{-x_i(x_k+1)}\{x_i(x_k+1)+1\}\right\} w_j w_k.$$

¿From the compactness of $L$, it can be expected that for sufficiently large $N$ the matrix $(a_{ij})$ has $N+1$ positive eigenvalues $\tilde{\lambda}_n$ and their eigenvectors $(\tilde{v}_{n,0}, \cdots, \tilde{v}_{n,N})$, $1 \leq n \leq N+1$ [1]. Here we suppose that $\tilde{v}_{n,i}$ corresponds to $v_n(x_i)$ and that they satisfy the following normal condition implied by $\|u_n\|_{L^2(\mathbf{R}^+)} = 1$:

$$\sum_{j=0}^N |\tilde{u}_{n,j}|^2 w_j = 1,$$

where $\tilde{u}_{n,j}$ corresponds to $u_n(x_j)$ and is given as

$$\tilde{u}_{n,j} = \frac{1}{\tilde{\lambda}_n} x_j \sum_{k=0}^N \tilde{v}_{n,k} e^{-x_j x_k} w_k.$$

Analogue to the Nyström method, the discretized singular system $\{\tilde{\lambda}_n, \tilde{v}_{n,j}, \tilde{u}_{n,j}\}$ gives approximations of singular functions as

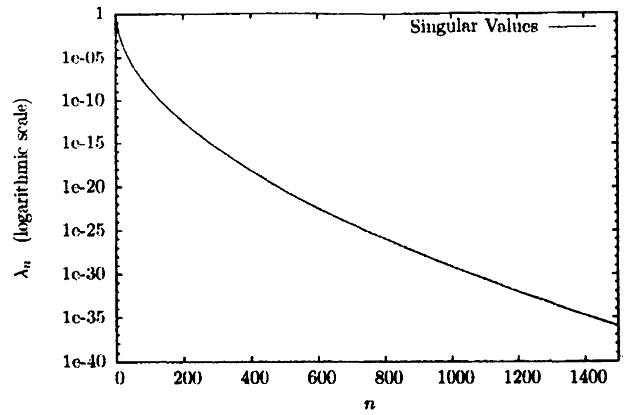
$$\begin{aligned} u_n^{(N)}(p) &= \frac{1}{\tilde{\lambda}_n} p \sum_{k=0}^N \tilde{v}_{n,k} e^{-px_k} w_k, \\ v_n^{(N)}(t) &= \frac{1}{\tilde{\lambda}_n} \sum_{k=0}^N \frac{\tilde{u}_{n,k}}{(x_k+1)^2} \left\{1 - e^{-t(x_k+1)}\{t(x_k+1)+1\}\right\} w_k. \end{aligned}$$

And numerical real inversion formula with spectral cut-off regularization is given by

$$(\mathcal{L}_{M,N}^{-1}F)(t) = \sum_{n=1}^M \frac{1}{\tilde{\lambda}_n} \left( \sum_{k=0}^N F(x_k) x_k \, \tilde{u}_{n,k} \, w_k \right) v_n^{(N)}(t). \tag{5}$$

In the rest of the present paper, we exhibit some numerical examples of the proposed method. Figure 1 shows computed $\tilde{\lambda}_n$, $1 \leq n \leq 50$. Figure 2(a) and Figure 2(b) show computed $v_n^{(N)}$, $1 \leq n \leq 5$, and $u_n^{(N)}$, $1 \leq n \leq 5$, respectively. Here we use the double exponential formula [8], in which we take a discretization parameter $N \in \mathbf{N}$, truncation parameters $L, U \in \mathbf{R}$, and quadrature points $x_i$ as

$$h = \frac{U-L}{N}, \quad \eta_i = L + ih, \quad x_i = \exp\left(\frac{\pi}{2}\sinh \eta_i\right), \quad 0 \leq i \leq N,$$




| $n$ | Computed $\lambda_n$ |
|---|---|
| 1 | 0.63385084 |
| 2 | 0.26410894 |
| 3 | 0.13597863 |
| 4 | 0.07771564 |
| 5 | 0.04743480 |
| 6 | 0.030328393 |
| 7 | 0.020083893 |
| 8 | 0.013674490 |

Figure 1: Numerical results for singular values $\lambda_n$ of $L$

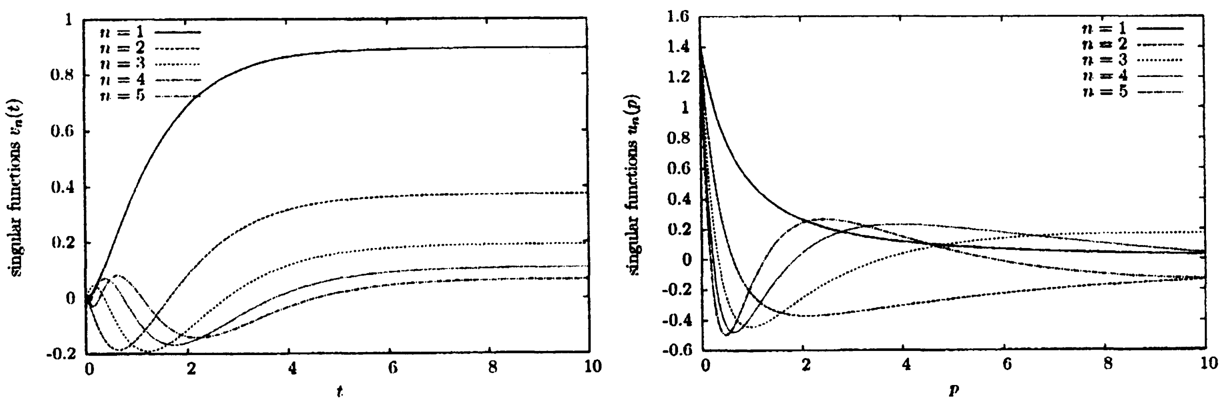



Figure 2: Numerical results for singular functions of $L$

and weights $w_i = (\pi h/2)x_i \cosh \eta_i$. In the actual computation, we adopt $L = -4, U = 4$, and $N = 800$. The dense and non-symmetric eigenvalue problem is solved by LAPACK (AMD Core Math Library) with IEEE754 double precision arithmetic, and computational time is about 90 seconds on Athlon64X2 6000+ (3.0GHz).

Next we apply the proposed method to an example in [4]:

$$f(t) = \begin{cases} t, & 0 \leq t < 1; \\ \frac{1}{2}(3-t), & 1 \leq t < 3; \\ 0, & 3 \leq t, \end{cases}$$

whose Laplace transform is

$$F(p) = \frac{1}{2p^2}\left(2 - 3e^{-p} + e^{-3p}\right). \tag{6}$$

We remark that the original function $f$ belongs to $H_K$ in the example. Figure 3 shows the numerical reconstruction with the spectral cut-off parameter $M = 50$. In the figure, the solid curve shows reconstructed numerical solutions and dotted curve shows the exact solution. ¿From the figure, we can conclude that our method gives

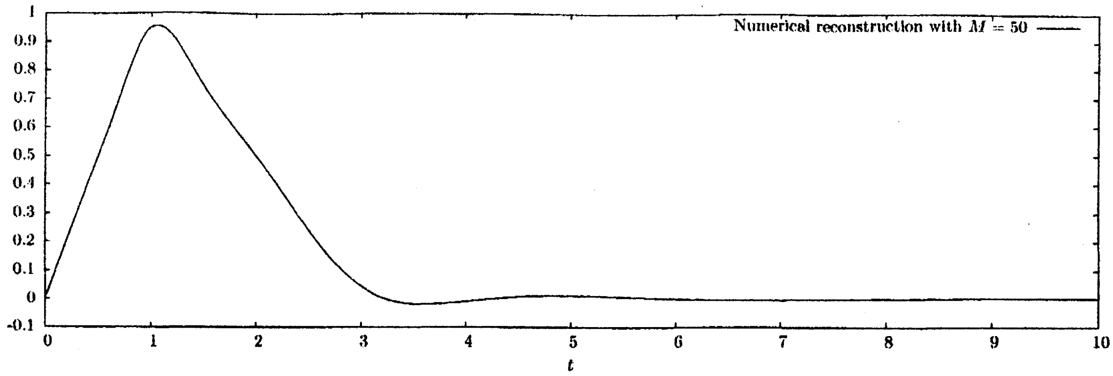



Figure 3: Numerical real inversion for (6)

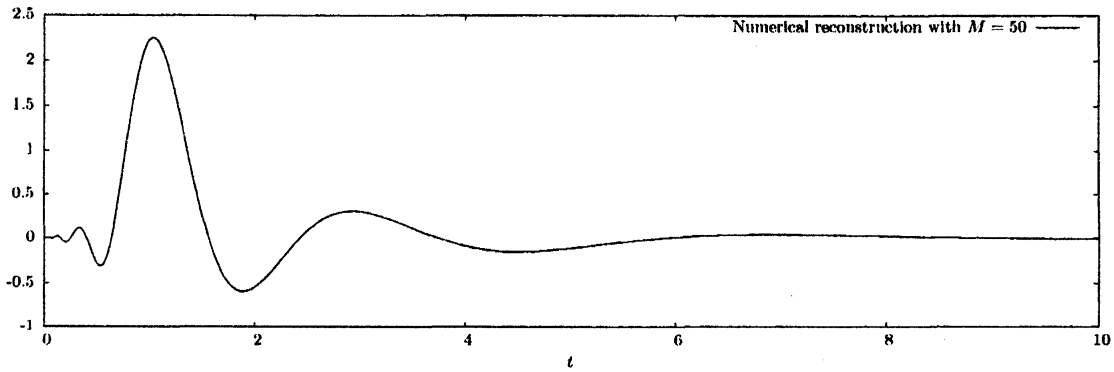



Figure 4: Numerical real inversion for (7)

a good approximation and the computed singular systems is effective in the use of analysis of the Laplace transform.

We show another example for the image function:

$$F(p) = \exp(-p), \tag{7}$$

which is the Laplace transform of the Dirac's delta function $\delta(t-1)$ in the distribution sense. In this example the proposed method gives the numerical results shown in Figure 4, which proves reasonable results.

Finally we propose the following conjectures about the singular system of $L$ from our numerical results:

1. $v_n \in H_K$ converges as $t \to \infty$, and

$$\lim_{t\to\infty} |v_n(t)| = \sqrt{2}\lambda_n, \quad \text{for all } n.$$

2. $u_n \in L^2(\mathbf{R}^+)$ is continuous at $p = 0$, and

$$|u_n(0)| = \sqrt{2}, \quad \text{for all } n.$$

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