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Left-separated topological spaces under Fodor-type Reflection Principle

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Abstract

Assuming Fodor-type Reflection Principle, we prove that every T_1-space with a point countable base is left-separated if all of its subspaces of cardinality $\leq \aleph_1$ are left-separated. This result improves a theorem by Fleissner [4] who proved the same assertion under Axiom R.

1 Introduction

Axiom R introduced in Fleissner [4] is often used to show that some property of certain topological space reflects down to a subspace of small cardinality. Let us mention the following two well-known results:

Theorem 1.1. (1) (Balogh [1, Theorem 2.2]) *Assume* Axiom R. *Suppose that X is locally countably compact. If X is not metrizable then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not metrizable.*

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(2) (Fleissner [4, Theorem 4.2]) *Assume* Axiom R. *Suppose that X is a T_1-space with a point countable base. If X is not left-separated then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not left-separated.*

Both of the assertions cited in Theorem 1.1 are known to be independent from ZFC. For example, the existence of non-reflecting stationary subset of $E^\kappa_\omega = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ for some regular $\kappa > \aleph_1$ implies the negation of both of (1) and (2) in Theorem 1.1 (see [7] and [4], for the independence of the assertion of (2) see also Proposition 2.4 below). Thus we do need some assumption like Axiom R in these results.

In Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [7], it is shown that Axiom R in Theorem 1.1, (1) can be replaced by Fodor-type Reflection Principle (FRP, see Section 3 for the definition of this principle) which is a consequence of Axiom R.

One of the advantages of replacing Axiom R with FRP is that it is shown that FRP is compatible with arbitrary size of the continuum (see [7]) while Axiom R implies that the continuum has size $\leq \aleph_2$. Actually, it is shown in [7] that FRP is preserved by any generic extension by a c.c.c. poset. Hence conclusions of FRP are compatible with any property which can be forced to be true by a c.c.c. poset starting from a model of ZFC + FRP.

Let P be a property of topological spaces and κ a cardinal. We shall say that a topological space X is $\leq \kappa$-P ($< \kappa$-P, respectively) if every subspace Y of X of cardinality $\leq \kappa$ ($< \kappa$, respectively) has the property P. In this notation, we shall always put '$\leq$' or '$<$' to the cardinal κ since very often "κ P" or "κ-P" is already used for some other notions (this is e.g. the case with "$\aleph_1$ meta-Lindelöf"). X is said to be *almost* P if X is $< |X|$- P, that is, if every subspace of X of cardinality $< |X|$ has the property P.

Using this terminology, Theorem 1.1 can be reformulated as follows:

Theorem 1.2. (a reformulation of Theorem 1.1)

(1) (Balogh [1, Theorem 2.2]) *Assume* Axiom R. *Suppose that X is locally countably compact. If X is $\leq \aleph_1$-metrizable, then X is metrizable.*

(2) (Fleissner [4, Theorem 4.2]) *Assume* Axiom R. *Suppose that X is a T_1-space with a point countable base. If X is $\leq \aleph_1$-left-separated, then X is left-separated.*

In this paper, we show that Axiom R in Theorem 1.1 (2) (or Theorem 1.2 (2)) can be also replaced by FRP (Theorem 4.1).

2 Preliminaries

Let us first review the topological notions appeared in Theorem 1.1 (2) (or Theorem 1.2 (2)).

A family $\mathcal{F}$ of subsets of X is said to be *point countable* if $\{a \in \mathcal{F} : p \in a\}$ is countable for all $p \in X$. By Bing-Nagata-Smirnov theorem, metrizable spaces are examples of topological spaces with a point countable base. If a space X has a point countable base, then X is *countably tight*, i.e. for any $p \in X$ and $Y \subseteq X$, $p \in \overline{Y}$ if and only if there is some $a \in [Y]^{\aleph_0}$ such that $p \in \overline{a}$.

A topological space X is *left-separated* if there is a well-ordering $<$ of X such that every initial segment with respect to $<$ is a closed subset of X. For a left-separated space X with a well-ordering $<$ as above, we say that X is *left-separated in order type* κ if $otp(X, <) = \kappa$.

Left-separated T_1-spaces with a point countable base enjoy a nice characterization (Theorem 2.1). Let us first review some more notions used in the characterization.

A topological space X is said to be *weakly separated* if there is a family $\{U_p : p \in X\}$ such that, for each $p \in X$, U_p is a neighborhood of p and, for distinct p, $q \in X$, at least one of $p \notin U_q$ or $q \notin U_p$ holds. A left-separated space X is weakly separated since, for a well ordering $<$ of X witnessing the left-separatedness of X, the family $\{U_p : p \in X\}$ with $U_p = \{q \in X : q = p \text{ or } p < q\}$ for $p \in X$ has the property above. X is σ *weakly separated* if X is a union of countably many weakly separated subspaces.

A family $\mathcal{F}$ of closed subsets of X is said to be *closure preserving* if $\bigcup \mathcal{G}$ is closed for any $\mathcal{G} \subseteq \mathcal{F}$.

Theorem 2.1. (Fleissner [4, Theorem 2.2]) *For a T_1-space X with a point-countable base, the following are equivalent:*

(a) *X is left-separated in order type $|X|$;*

(b) *X is σ-weakly separated;*

(c) *X has a closure preserving cover consisting of countable closed sets.* □

Corollary 2.2. *A T_1-space X with a point-countable base is left-separated if and only if it is left separated in order type $|\kappa|$.*

Proof. If X is left-separated in order type $|\kappa|$ then it is surely left-separated.

If X is left-separated then it is weakly separated. By Theorem 2.1, (b) $\Rightarrow$ (a), it follows that X is left-separated in order type $|\kappa|$. □ (Corollary 2.2)

Lemma 2.3. (a) *Suppose that X is a T_1-space and $X = \bigcup_{\xi<\delta} X_\xi$ where $\langle X_\xi : \xi < \delta\rangle$ is a continuously increasing sequence of subspaces of X. If X_ξ is left separated and closed in X for all $\xi < \delta$ then X is also left-separated.*

(b) *Suppose that X is an almost left-separated T_1-space with a point countable base. Then X is left-separated if and only if X has a filtration consisting of closed subsets of X.*

Proof. (a): We may assume that δ is a limit ordinal. For each $\xi < \delta$, let $\leq_\xi$ be a well-ordering of X_ξ witnessing the left-separatedness of X_ξ. Let $<$ be the well-ordering of X defined by

$$x < y \Leftrightarrow x \in X_\xi \text{ and } y \notin X_\xi \text{ for some } \xi < \delta$$
$$\text{or } x, y \in X_{\xi+1} \setminus X_\xi \text{ for some } \xi \text{ and } x <_{\xi+1} y$$

Since each initial segment with respect to $<$ is either X_ξ or $X_\xi \cup$ an initial segment of $X_{\xi+1}$ with respect to $<_{\xi+1}$ for some $\xi < \delta$, it follows that all initial segments with respect to $<$ are closed in X. Thus $<$ witnesses the left-separatedness of X.

(b): If X is left-separated then, by Corollary 2.2, X is left-separated by order type $|X|$. Let $\kappa = |X|$ and let $f : \kappa \to X$ be a bijection such that $f''\alpha$ is closed subset for all $\alpha < \kappa$. Then $\langle f''\alpha : \alpha < \kappa\rangle$ is a filtration of X consisting of closed subsets of X.

Suppose now that X has a filtration $\langle X_\alpha : \alpha < \kappa\rangle$ such that all X_α, $\alpha < \kappa$ are closed in X. Since X is almost left-separated, all X_α, $\alpha < \kappa$ are left-separated. Hence, by (a), it follows that X is also left-separated. □ (Lemma 2.3)

The following proposition shows that the assertion of Theorem 1.1, (2) (or Theorem 1.2,(2)) is independent even if the condition "T_1-space with a point countable base" is replaced by "metrizable space". The proof of the next Proposition of Fleissner given here is perhaps less elegant than the one given in Fleissner [4]. Nevertheless we included our proof since it fits Lemma 2.3 and its proof.

Proposition 2.4. (Fleissner [4]) *Suppose that κ is a regular uncountable cardinal and there is a non-reflecting stationary set $S \subseteq E^\kappa_\omega$. Then there is a metrizable space X of cardinality κ which is almost left-separated but not left-separated.*

Proof. Let $S \subseteq E^\kappa_\omega$ be a non-reflecting stationary set. That is, S itself is stationary in κ but $S \cap \alpha$ is not stationary in α for all $\alpha < \kappa$. Let $\bar{a}$ be a ladder system on S. That is, $\bar{a} : S \times \omega \to \kappa$ and, for all $\alpha \in S$, $\langle \bar{a}(\alpha, n) : n \in \omega\rangle$ is a strictly increasing sequence of ordinals $< \alpha$ such that $\lim_{n\to\infty} \bar{a}(\alpha, n) = \alpha$.

For α, $\beta \in S$, let

(2.1) $d(\alpha, \beta) = 2^{-\mu n(\bar{a}(\alpha,n)\neq\bar{a}(\beta,n))}$

if $\alpha \neq \beta$ and $d(\alpha, \beta) = 0$ if $\alpha = \beta$.

Claim 2.4.1. *d is a metric on S.*

$\vdash$ We only show that d satisfies the triangle inequality since it is easy to see that the other properties of a metric are satisfied by d.

Suppose α, β, $\gamma \in S$. We show that $d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma)$. Without loss of generality, we may assume that α, β, γ are pairwise distinct. Let

$$n_{\alpha,\beta} = \mu n(\bar{a}(\alpha, n) \neq \bar{a}(\beta, n));$$
$$n_{\beta,\gamma} = \mu n(\bar{a}(\beta, n) \neq \bar{a}(\gamma, n));$$
$$n_{\alpha,\gamma} = \mu n(\bar{a}(\alpha, n) \neq \bar{a}(\gamma, n)).$$

Then, there are the following three cases:

Case 1. $n_{\alpha,\beta} < n_{\beta,\gamma}$. In this case, we have $n_{\alpha,\gamma} = n_{\alpha,\beta}$.
Case 2. $n_{\alpha,\beta} > n_{\beta,\gamma}$. In this case, we have $n_{\alpha,\gamma} = n_{\beta,\gamma}$.
Case 3. $n_{\alpha,\beta} = n_{\beta,\gamma}$. In this case, we have $n_{\alpha,\gamma} \geq n_{\alpha,\beta}$, $n_{\beta,\gamma}$.

In all of these cases it is easy to see that we have

$$d(\alpha, \gamma) = 2^{-n_{\alpha,\gamma}} \leq 2^{-n_{\alpha,\beta}} + 2^{-n_{\beta,\gamma}} = d(\alpha, \beta) + d(\beta, \gamma).$$

$\dashv$ (Claim 2.4.1)

Let τ be the topology induced from the metric d and let us consider S as the topological space (S, τ). Clearly $|S| = \kappa$. We show that S is a topological space as desired.

Let θ be a sufficiently large regular cardinal and let $M \prec \mathcal{H}(\theta)$ be such that S, $\bar{a} \in M$ and $\kappa \cap M \in S$. Let $\alpha = \kappa \cap M$. Then, it is easy to check that $\alpha \in \overline{\alpha}$. Hence α is not closed in S. Since there are stationarily many α representable as $\kappa \cap M$ for some M as above, it follows from Lemma 2.3, (b) that S is not left-separated.

Now, we are done showing that S is almost left-separated. To prove this, it is enough to show that $S \cap \alpha$ for all $\alpha < \kappa$ is left-separated. We prove this by induction on $\alpha < \kappa$. If $S \cap \alpha$ is finite, this is clear. So suppose that we have shown that all $S \cap \beta$, $\beta < \alpha$ are left-separated.

If α is a successor of some $\delta \in \kappa \setminus S$, then $S \cap \alpha = S \cap \delta$. Since $S \cap \delta$ is left-separated by the induction hypothesis, so is also $S \cap \alpha$.

If α is a successor of some $\delta \in S$ then $S \cap \alpha = (S \cap \delta) \cup \{\delta\}$. By the induction hypothesis, there is a well-ordering $\sqsubset$ of $S \cap \delta$ witnessing the left-separatedness of $S \cap \delta$. Let $\tilde{\sqsubset}$ be the well-ordering of $S \cap \alpha$ defined by

$$\begin{aligned} \beta \mathrel{\tilde{\sqsubset}} \beta' \;\Leftrightarrow\; & \beta' = \delta \text{ or} \\ & \beta \leq \bar{a}(\delta, n) < \beta' < \delta \text{ for some } n \in \omega \text{ or} \\ & (\beta, \beta' < \bar{a}(\delta, 0) \text{ and } \beta \sqsubset \beta') \text{ or} \\ & (\bar{a}(\delta, n) < \beta, \beta' \leq \bar{a}(\delta, n+1) \text{ for some } n \in \omega \text{ and } \beta \sqsubset \beta') \end{aligned}$$

Since $S \cap (\bar{a}(\delta, n) + 1)$ is closed in $S \cap \alpha$ for all n, it follows that $\tilde{\sqsubset}$ witnesses the left-separatedness of $S \cap \alpha$.

Finally suppose that α is a limit. Since $S \cap \alpha$ is non-stationary, there is a club $C \subseteq \alpha$ disjoint from S. Let $\langle \alpha_\xi : \xi < \delta \rangle$ be an increasing enumeration of C. By $\alpha_\delta \notin S$, we have that $S \cap \alpha_\xi$ is closed in S for all $\xi < \delta$. Also, $S \cap \alpha_\xi$ is left-separated for all $\xi < \delta$ by the induction hypothesis. Hence it follows by Lemma 2.3, (a) that $S \cap \alpha = \bigcup_{\xi < \delta} S \cap \alpha_\xi$ is left-separated. □ (Proposition 2.4)

Lemma 2.5. (Fleissner [4, Lemma 4.1]) *Suppose that X is a $\leq \aleph_1$-left-separated T_1-space with a point countable base. Then, for all $Y \in [X]^{\leq \aleph_1}$, $|\overline{Y}| = |Y|$.* □

3 Fodor-type Reflection Principle

In this section, we summarize the definitions and basic results in connection with Fodor-type Reflection Principle. For the omitted proofs, the reader may consult Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [7]. More results on Fodor-type Reflection Principle will appear in Fuchino, Sakai, Soukup and Usuba [8].

Definition 3.1. Let κ be a cardinal of cofinality $\geq \omega_1$. *The Fodor-type Reflection Principle for κ* ($\mathsf{FRP}(\kappa)$) is the following statement:

$\mathsf{FRP}(\kappa)$: For any stationary $S \subseteq E^\kappa_\omega$ and mapping $g : S \to [\kappa]^{\leq \aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(3.1) $\mathrm{cf}(I) = \omega_1$;

(3.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(3.3) for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in *sup*(I).

Note that, for S and I as above, $S \cap I$ is stationary in $\sup(I)$. In particular, if $S \cap I$ were empty, then $\emptyset : S \cap I \to \kappa$ is a/the regressive function for which there is no ξ^* as in (3.3). Note also that $\mathsf{FRP}(\omega_1)$ holds in ZFC: indeed, if we take $I = \omega_1$ then the statement follows immediately from the Fodor Lemma.

Lemma 3.2. ([7]) FRP(κ) *fails for a singular* κ. □

Definition 3.3. Fodor-type Reflection Principle (FRP) is the assertion:

FRP: FRP(κ) *holds for all regular* $\kappa \geq \aleph_1$.

Recall that Axiom R is the principle asserting that the following AR($[\kappa]^{\aleph_0}$) holds for all cardinals $\kappa \geq \aleph_2$:

AR($[\kappa]^{\aleph_0}$): For any stationary $S \subseteq [\kappa]^{\aleph_0}$ and ω_1-club $T \subseteq [\kappa]^{\aleph_1}$, there is $I \in T$ such that $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $T \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1*-club* (or *tight and unbounded* in Fleissner's terminology in Fleissner [4]) if

(3.4) T is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and

(3.5) for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in T of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in T$.

For regular $\kappa \geq \aleph_2$, FRP(κ) is not provable in ZFC since, for example, the existence of a non-reflecting subset of E^κ_ω would refute FRP(κ).

However, we have:

Theorem 3.4. ([7]) *For any regular cardinal* $\kappa > \aleph_1$, RP($[\kappa]^{\aleph_0}$) *implies* FRP(κ). □

Here, for a cardinal $\kappa \geq \aleph_2$, RP($[\kappa]^{\aleph_0}$) is the following principle:

RP($[\kappa]^{\aleph_0}$): For any stationary $S \subseteq [\kappa]^{\aleph_0}$, there is an $I \in [\kappa]^{\aleph_1}$ such that

(3.6) $\omega_1 \subseteq I$;

(3.7) $\mathrm{cf}(I) = \omega_1$;

(3.8) $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

AR($[\kappa]^{\aleph_0}$) implies RP($[\kappa]^{\aleph_0}$) for a cardinal κ of cofinality $\geq \omega_1$ since $T = \{I \in [\kappa]^{\aleph_0} : \omega_1 \subseteq I \text{ and } \mathrm{cf}(I) = \omega_1\}$ is ω_1-club. Jech [9] called a weakening of RP($[\kappa]^{\aleph_0}$) "*Reflection Principle*" which is obtained by dropping the condition (3.7) from the definition of RP($[\kappa]^{\aleph_0}$). Jech's reflection principle is sometimes also called "*Weak Reflection Principle*" in the literature (see, e.g. König, Larson and Yoshinobu [10]) and so we denote this principle by WRP($[\kappa]^{\aleph_0}$).

Axiom R follows from $\mathrm{MA}^+(\sigma\text{-closed})$ (see Beaudoin [2]) which in turn is a consequence of Martin's Maximum (see Foreman, Magidor and Shelah [5]). In more modern terminology of Foreman and Todorcevic [6], Axiom R is equivalent to the stationary reflection to a internally unbounded structure (this fact is stated essentially in Dow [3] under the definition of Axiom R which is slightly stronger than the one we use here). Since $\mathrm{MA}^+(\sigma\text{-closed})$ is consistent with CH (modulo some large cardinal), all the reflection principles we treat here are compatible with CH.

It is still open if $\mathsf{WRP}([\kappa]^{\aleph_0})$, $\mathsf{RP}([\kappa]^{\aleph_0})$ and $\mathsf{AR}([\kappa]^{\aleph_0})$ can be separated. This seems to be a quite difficult problem if these principles should be ever separated: it is known that $\mathsf{RP}([\omega_2]^{\aleph_0})$ and $\mathsf{AR}([\omega_2]^{\aleph_0})$ are equivalent; under $2^{\aleph_1} = \aleph_2$, $\mathsf{WRP}([\omega_2]^{\aleph_0})$ and $\mathsf{RP}([\omega_2]^{\aleph_0})$ are equivalent and, e.g. under GCH, $\mathsf{WRP}([\omega_n]^{\aleph_0})$ and $\mathsf{RP}([\omega_n]^{\aleph_0})$ for all $n \in \omega$ are equivalent (see König, Larson and Yoshinobu [10]).

Nevertheless, our Fodor-type Reflection Principle can be easily separated from these reflection principles:

Theorem 3.5. ([7]) *Suppose that* $\mathsf{FRP}(\kappa)$ *holds and* $\mathbb{P}$ *is a c.c.c. poset. Then* $\Vdash_{\mathbb{P}}$ "$\mathsf{FRP}(\kappa)$ *holds*".

Starting form a model of ZFC + FRP, we can add more than $\aleph_2$ reals by a c.c.c. poset. Since $\mathsf{WRP}([\aleph_2]^{\aleph_0})$ implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević, see [9] for a proof), $\mathsf{WRP}([\aleph_2]^{\aleph_0})$ does not hold in the generic extension while FRP is still valid in the extension by Theorem 3.5.

In the application of FRP in the next section, we use the following characterization of the principle:

Lemma 3.6. ([7]) *For a regular cardinal* $\kappa \geq \aleph_2$, $\mathsf{FRP}(\kappa)$ *is equivalent to the following* $\mathsf{FRP}^\bullet(\kappa)$*:*

$\mathsf{FRP}^\bullet(\kappa)$ *: For any stationary* $S \subseteq E^\kappa_\omega$ *and mapping* $g : S \to [\kappa]^{\leq\aleph_0}$ *there is a continuously increasing sequence* $\langle I_\xi : \xi < \omega\rangle$ *of countable subsets of* κ *such that*

(3.9) $\langle \sup(I_\xi) : \xi < \omega_1\rangle$ *is strictly increasing;*

(3.10) *each* I_ξ *is closed with respect to* g *and*

(3.11) $\{\xi < \omega_1 : \sup(I_\xi) \in S$ *and* $g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$ *is stationary in* ω_1.

4 Left-separated spaces under FRP

As announced in the introduction, we prove the following theorem:

Theorem 4.1. (FRP) *Suppose that* X *is a* T_1*-space with a point countable base. If* X *is* $\leq\aleph_1$*-left-separated, then* X *is left-separated.*

Let us begin with the following lemma:

Lemma 4.2. (FRP) *Suppose that $\kappa \geq \aleph_1$, X is a $\leq \kappa$-left-separated T_1-space with a point countable base. Then for any $Y \in [X]^{\leq\kappa}$ we have $|\overline{Y}| = |Y|$.*

Proof. We prove the lemma by induction on κ. For $\kappa = \aleph_1$, this is just Lemma 2.5.

Assume that $\kappa > \aleph_1$ and the assertion of the lemma holds with κ replaced by any λ such that $\aleph_1 \leq \lambda < \kappa$. Suppose that X is a $\leq \kappa$-left-separated T_1-space with a point countable base and $Y \in [X]^\kappa$. It is enough to show that $|\overline{Y}| = \kappa$.

Case I. $\mathrm{cf}(\kappa) > \omega$. Let $\lambda = \mathrm{cf}(\kappa)$ and let $\langle Y_\alpha : \alpha < \lambda\rangle$ be a filtration of Y. By the induction hypothesis we have $|\overline{Y_\alpha}| < \kappa$ for all $\alpha < \lambda$. Since X is countably tight and $\lambda > \omega$ is regular it follows that $\overline{Y} = \bigcup_{\alpha<\lambda} \overline{Y_\alpha}$ and thus $|\overline{Y}| = |\bigcup_{\alpha<\lambda} \overline{Y_\alpha}| = \kappa$.

Case II. $\mathrm{cf}(\kappa) = \omega$. Assume toward a contradiction that there is a $Y \in [X]^\kappa$ such that $|\overline{Y}| > \kappa$. Let $Z \subseteq \overline{Y}$ be such that $Y \subseteq Z$ and $|Z| = \kappa^+$. Let $\langle Z_\alpha : \alpha < \kappa^+\rangle$ be a filtration of Z with $Z_0 = Y$. For $\alpha < \kappa^+$, let $x_\alpha \in Z_{\alpha+1} \setminus Z_\alpha$ and let $a_\alpha \in [Y]^{\aleph_0}$ be such that $x_\alpha \in \overline{a_\alpha}$. By identifying Z with κ^+ is such a way that each Z_α corresponds to an ordinal $< \kappa^+$, we may apply $\mathsf{FRP}^\bullet(\kappa^+)$ to this situation to obtain a continuously and strictly increasing sequence $\langle U_\xi : \xi < \omega_1\rangle$ of countable subsets of Z and a continuously and strictly increasing sequence $\langle \alpha_\xi : \xi < \omega_1\rangle$ of ordinals $< \kappa^+$ such that

(4.1) $U_\xi \subseteq Z_{\alpha_\xi}$ and $x_{\alpha_\xi} \in U_{\xi+1}$ for all $\xi < \omega_1$;

(4.2) $\{\xi < \omega_1 : a_{\alpha_\xi} \subseteq U_\xi\}$ is stationary in ω_1.

Let $U = \bigcup_{\xi<\omega_1} U_\xi$. By (4.1) and (4.2), $\{\xi < \omega_1 : U_\xi \text{ is not closed in } U\}$ is stationary. Hence, by Lemma 2.3 (b), U is not left-separated. But this is a contradiction to the $\leq \kappa$-left-separatedness of X. □ (Lemma 4.2)

Proof of Theorem 4.1: Assume for contradiction that there are counter-examples to the theorem. Let X be such a counter-example with minimal possible cardinality. Thus X is T_1-space with a point countable base and, by minimality of $\kappa = |X|$, we have

(4.3) X is almost left-separated; while

(4.4) X is not left-separated.

Case I. $\mathrm{cf}(\kappa) = \omega$. Let $\langle X_n : n \in \omega\rangle$ be a filtration of X. By Lemma 4.2, we may choose X_n's such that they are all closed subsets of X. Since all of X_n's are left-separated by (4.3), it follows by Lemma 2.3, (b) that X is left-separated which is a contradiction to (4.4).

Case II. κ is a singular cardinal with $\mathrm{cf}(\kappa) > \omega$. Let $\lambda = \mathrm{cf}(\kappa)$. By Lemma 4.2, we can construct a (not necessarily continuously) increasing sequence $\langle X_\xi : \xi < \lambda \rangle$ of closed subsets of X such that

(4.5) $\lambda < |X_\xi| < \kappa$ for all $\xi < \lambda$;

(4.6) $X = \bigcup_{\xi<\lambda} X_\xi$.

By (4.3) each X_ξ is left-separated. Hence, by Lemma 2.1, (c), there is a closure preserving cover C_ξ of X_ξ consisting of countable closed sets of X_ξ.

Now let $\langle Z_\delta : \delta < \lambda \rangle$ be a filtration of X such that, for all $\xi < \delta$,

(4.7) if $x \in X_\xi \cap Z_\delta$ then there is some $c \in C_\xi$ such that $x \in c \subseteq Z_\delta$ for all $\xi < \lambda$.

Claim 4.2.1. *Z_δ is a closed subset of X for all $\delta < \lambda$.*

$\vdash$ Suppose that $x \in \overline{Z_\delta}$. We show $x \in Z_\delta$. By the countable tightness of X, there is an $a \in [Z_\delta]^{\aleph_0}$ such that $x \in \overline{a}$. Since λ is regular and $> \omega$, there is $\xi^* < \lambda$ such that $a \subseteq X_{\xi^*}$. By (4.7) $Z_\delta \cap Z_{\xi^*}$ is the union of a subset of C_{ξ^*} and hence, by the closure preservation of C_ξ, $Z_\delta \cap Z_{\xi^*}$ is closed. It follows that $x \in \overline{a} \subseteq Z_\delta \cap Z_{\xi^*} \subseteq Z_\delta$.
$\dashv$ (Claim 4.2.1)

By (4.3), Z_δ's are all left-separated. Hence, by the Claim above and Lemma 2.3, X is left-separated. This is a contradiction to (4.4).

Case III. κ is regular. Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a filtration of X. By Lemma 4.2, we may choose X_α, $\alpha < \kappa$ such that $X_{\alpha+1}$ is a closed subset of X for all $\alpha < \kappa$. By the countable tightness of X, it follows that

(4.8) X_α is a closed subset of X for all $\alpha \in \kappa \setminus E^\kappa_\omega$.

By (4.3), each X_α is left-separated. Hence, by (4.4) and Lemma 2.3,

$$S = \{\alpha < \kappa : X_\alpha \text{ is not a closed subset of } X\}$$

is stationary. By (4.8), we have $S \subseteq E^\kappa_\omega$. For each $\alpha \in S$, let $x_\alpha \in X$ be such that $x_\alpha \in \overline{X_\alpha} \setminus X_\alpha$ and let $a_\alpha \in [X_\alpha]^{\aleph_0}$ be such that $x \in \overline{a_\alpha}$.

By the same argument as in the Case II of the proof of Lemma 4.2, we can apply FRP to obtain continuously and strictly increasing sequence $\langle Y_\alpha : \alpha < \omega_1 \rangle$ of countable subsets of X and a continuously and strictly increasing sequence $\langle \xi_\alpha : \alpha < \omega_1 \rangle$ of ordinals $< \kappa$ such that

(4.9) $Y_\alpha \subseteq X_{\xi_\alpha}$ for all $\alpha < \omega_1$;

(4.10) $x_{\xi_\beta} \in Y_\alpha$ for all $\beta < \alpha$ with $\xi_\beta \in S$;

(4.11) $S \cap \{\xi_\alpha : \alpha < \omega_1, a_{\xi_\alpha} \subseteq Y_\alpha\}$ is stationary in $\sup_{\alpha<\omega_1} \xi_\alpha$.

Let $Y = \bigcup_{\alpha<\omega_1} Y_\alpha$. Then, by (4.10) and (4.11), and since $\langle \xi_\alpha : \alpha < \omega_1 \rangle$ is continuously and strictly increasing, we have

(4.12) $\{\alpha < \omega_1 : Y_\alpha$ is not closed in $Y\}$ is stationary.

By Lemma 2.3, it follows that Y is not left-separated. But, since $|Y| = \aleph_1$, this is a contradiction to (4.3). □ (Theorem 4.1)

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