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# FUNCTIONS WITH MANY LOCAL EXTREMA

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ABSTRACT. Answering a question addressed by Dirk Werner we show that the set of local extrema of a nowhere constant continuous function $f : [0,1] \to \mathbb{R}$ is always meager but possibly of full measure. The set of local extrema of a nowhere constant $C^{\infty}$-function from $[0,1]$ to $\mathbb{R}$ can be of arbitrarily large measure below 1.

## 1. INTRODUCTION

In [1], Behrends, Natkaniec and the author studied the question whether a continuous function $f$ from a topological space $X$ into the real line can have a local extremum at every point of $X$ without being constant. Among other things it was observed that if $X$ is a connected space of weight $<|\mathbb{R}|$, then every continuous function $f : X \to \mathbb{R}$ that has a local extremum at every point of $X$ is constant. Also, if $X$ is a connected linear order in which every family of pairwise disjoint open intervals is of size $<|\mathbb{R}|$ and $f : X \to \mathbb{R}$ is continuous and has a local extremum at every point of $X$, then $f$ is constant.

The proof of the latter fact given in [1] shows that if $X$ is a connected linear order and $f : X \to \mathbb{R}$ is continuous and has a local extremum at every point of $X$, then $f$ is constant on a nonempty open interval. In fact, the collection of open intervals on which $f$ is constant has a dense union.

Recently, the results mentioned above have been improved by Fedeli and Le Donne (see [2]), who showed that if $X$ is a connected space in which every family of pairwise disjoint open sets is of size $<|\mathbb{R}|$, then every continuous function $f : X \to \mathbb{R}$ that has a local extremum at every point is constant.

In this note we answer a question addressed by Dirk Werner, namely how many local extrema a non-constant continuous function, say from the unit interval, into the reals can actually have.

It is relatively easy to construct a continuous function $f : [0,1] \to \mathbb{R}$ that is not constant and whose set of local minima is open and dense. Just choose a closed nowhere dense set $A \subseteq [0,1]$ of positive measure (see Lemma 1) and let $f(x)$ be the measure of $A \cap [0,x]$. Then clearly, $f$ is continuous, not constant and constant

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on every open interval disjoint from $A$. In particular, $f$ has a local minimum and maximum at every point of $X \setminus A$.

This example shows that we should consider functions that are not constant on any nonempty open interval.

## 2. MEASURE

The following lemma is well known.

**Lemma 1.** *Let $\varepsilon > 0$. Then there is a closed nowhere dense set $A \subseteq [0,1]$ of measure at least $1-\varepsilon$.*

*Proof.* Let $\{(a_n,b_n) : n \in \mathbb{N}\}$ be the collection of all open subintervals of $[0,1]$ with rational endpoints. For each $n \in \mathbb{N}$ let $(c_n,d_n) \subseteq (a_n,b_n)$ be an open interval of length at most $2^{-n}\cdot\varepsilon$. Now $B = \bigcup_{n\in\mathbb{N}}(c_n,d_n)$ is a dense open set of measure at most $\varepsilon$. Hence, the set $A = [0,1]\setminus B$ is closed, nowhere dense and of measure at least $1-\varepsilon$. $\square$

By removing a suitable open interval from $A$ we can actually assume that $A$ is exactly of measure $1-\varepsilon$.

**Lemma 2.** *Let $a,b \in \mathbb{R}$ be such that $a<b$. Let $A \subseteq [a,b]$ be closed and nowhere dense. Then the function $f^A_{a,b} : [a,b] \longrightarrow \mathbb{R}$ that assigns to every point $x$ its distance from $A$ is continuous and has local minima exactly at the points of $A$. Moreover, whenever $I \subseteq [a,b]$ is a maximal open interval disjoint from $A$, then $f^A_{a,b} \restriction \mathrm{cl}(I)$ is piecewise linear and in fact consists of two linear (in the sense of affine linear) pieces, one of slope 1 and one of slope $-1$.*

**Theorem 3.** *There is a continuous function $g : [0,1] \longrightarrow \mathbb{R}$ such that $g$ is not constant on any non-empty open interval and the set of local minima of $g$ is of measure 1. In particular, the set of local minima of $g$ is dense in $[0,1]$.*

*Proof.* Let $a,b \in [0,1]$ be such that $a<b$. Suppose that $f : [a,b] \longrightarrow \mathbb{R}$ is linear (in the sense of affine linear) with $f(a) = c$ and $f(b) = d$. Let $c = a + \frac{1}{8}(b-a)$ and $d = b - \frac{1}{8}(b-a)$. Let $A$ be a closed nowhere dense subset of $[c,d]$ of measure $\frac{1}{2}(b-a)$. We may assume $c,d \in A$.

Now let $f^* : [a,b] \longrightarrow \mathbb{R}$ be defined as follows. For each $x \in [a,b]$ let

$$f(x) = \begin{cases} 4\frac{f(b)-f(a)}{b-a}(x-a) + f(a), & x \leq c \\ f^A_{c,d}(x) + \frac{1}{2}(f(a)+f(b)), & c \leq x \leq d \\ 4\frac{f(b)-f(a)}{b-a}(x-b) + f(b), & x \geq d \end{cases}$$

In other words, $f^*$ is a continuous function whose graph starts and ends at the same points as the graph of $f$, but $f^*$ has local minima at every point of $A$, except possibly the first and last points of $A$, i.e., $c$ and $d$. In particular, the set of local minima of $f^*$ is of measure at least $\frac{1}{2}(b-a)$. We **observe** that

$$\sup\{|f^*(x) - f(x)| : x \in [a,b]\} \leq \max(b-a, |f(b)-f(a)|).$$

Given a function $f : [0,1] \to \mathbb{R}$, we define $f^* : [0,1] \to \mathbb{R}$ as follows. If $I \subseteq [0,1]$ is a maximal open interval such that $f$ is linear in $I$, we let $f^* \upharpoonright \mathrm{cl}\, I = (f \upharpoonright \mathrm{cl}\, I)^*$. If $x \in [0,1]$ is not contained in a maximal open interval on which $f$ is linear, we let $f^*(x) = f(x)$. From our construction it follows that $f^*$ is continuous if $f$ is.

Now choose $A \subseteq [0,1]$ closed, nowhere dense, and of measure $\frac{1}{2}$. Let $f_0 = f^A_{0,1}$. For every $n > 0$ let $f_n = f^*_{n-1}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions. By our observation above, the sequence converges uniformly. It follows that the limit $g$ of this sequence is a continuous function from $[0,1]$ to $\mathbb{R}$.

It is easily checked that $g$ is nowhere constant. Also, the set of local minima of $g$ is the union of the sets of local minima of the $f_n$. By induction it follows that the measure of the set of local minima of $f_n$ is at least $\sum_{k=1}^{n+1} \frac{1}{2^k}$. Hence the measure of the set of local minima of $g$ is 1. $\square$

Clearly, if $f : [0,1] \to \mathbb{R}$ is continuously differentiable and has a dense set of local extrema, then $f$ has to be constant. In particular, a nowhere constant, continuously differentiable function on the unit interval cannot have a set of local extrema of full measure. However, nowhere constant $C^\infty$-functions can have sets of local extrema of large measure.

**Theorem 4.** *For every $\varepsilon > 0$ there is an infinitely often differentiable function $f : [0,1] \to \mathbb{R}$ such that $f$ is not constant on any non-empty open interval and the set of local minima of $f$ is of measure at least $1 - \varepsilon$.*

*Proof.* We start the proof with a preliminary remark.

**Claim 5.** For all $a, b \in [0,1]$ with $a < b$ there is a $C^\infty$-function $h : [0,1] \to \mathbb{R}$ such that $h$ vanishes outside $(a,b)$ and is positive and nowhere constant on $(a,b)$.

For the proof of the claim we define $g : \mathbb{R} \to \mathbb{R}$ as follows: For all $x \in \mathbb{R}$ let

$$g(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1,1) \\ 0 & x \notin (-1,1). \end{cases}$$

It is well known that $g$ is infinitely often differentiable. Clearly, $g$ is nowhere constant and positive on the set $(-1,1)$. The claim is witnessed by translations of scaled versions of $g$.

Now let $A \subseteq [0,1]$ be as in Lemma 1 and choose a maximal family $\mathcal{G}$ of non-negative $C^\infty$-functions on $[0,1]$ with the following properties:

(1) For every $g \in \mathcal{G}$ the set $g^{-1}[(0,\infty)]$ is a non-empty open interval $I_g \subseteq [0,1] \setminus A$.
(2) For $g, h \in \mathcal{G}$ with $g \neq h$ the intervals $I_g$ and $I_h$ are disjoint.

Such a family $\mathcal{G}$ exists by Zorn's Lemma. Since $\{I_g : g \in \mathcal{G}\}$ is a disjoint family of non-empty open intervals, it is countable. It follows that $\mathcal{G}$ is countable.

By the claim, $\bigcup\{I_g : g \in \mathcal{G}\}$ is a dense subset of $[0,1] \setminus A$. Since $A$ is nowhere dense and yet of positive measure, $[0,1] \setminus A$ is not the union of finitely many open intervals and hence $\mathcal{G}$ is infinite. Let $(g_n)_{n\in\omega}$ be an enumeration of $\mathcal{G}$ without repetition.

For every $n \in \omega$ choose $\varepsilon_n > 0$ such that for all $m \leq n$ we have

$$\varepsilon_n \cdot \sup\left\{\left|g_n^{(m)}(x)\right| : x \in [0,1]\right\} < 2^{-n}.$$

Here $g_n^{(m)}$ denotes the $m$-th derivative of $g_n$.

For every $n \in \omega$ let $f_n = \sum_{m=0}^{n} \varepsilon_m g_m$. Since the $I_{g_m}$, $m \in \omega$, are pairwise disjoint and by the choice of the $\varepsilon_m$, the sequence $(f_n)_{n\in\omega}$ converges uniformly in every derivative and hence converges to a $C^\infty$-function $f : [0,1] \to \mathbb{R}$.

Clearly, $B = f^{-1}(0) = [0,1] \setminus \bigcup\{I_g : g \in \mathcal{G}\}$ and $B$ is a closed nowhere dense superset of $A$. Moreover, $f$ is not constant on any open interval disjoint from $B$. Since $B$ is nowhere dense, this implies that $f$ is nowhere constant. Clearly, every point of $B$, and hence of $A$, is a local minimum of $f$. $\square$

Let us point out that the use of Zorn's Lemma in the proof of Lemma 4 can be easily avoided and that for any given $\varepsilon$ a suitable function $f$ can be defined explicitly using a closed, but lengthy, formula.

## 3. Category

We point out that the analog of Theorem 3 for category fails badly.

**Theorem 6.** *If $f : [0,1] \to \mathbb{R}$ is continuous and not constant on any non-empty open interval, then the set of local minima of $f$ is meager.*

The proof of this theorem uses the following lemma.

**Lemma 7.** *The set of local minima of a continuous function $f : [0,1] \to \mathbb{R}$ is $F_\sigma$.*

*Proof.* For $a, b, c, d \in [0,1] \cap \mathbb{Q}$ with $a < b < c < d$ consider the set

$$M_{a,b,c,d} = \{x \in [b,c] : f(x) = \min(f[(a,d)])\}.$$

Clearly, $M_{a,b,c,d}$ is closed and every element of $M_{a,b,c,d}$ is a local minimum of $f$. On the other hand, if $x$ is a local minimum of $f$, then there are $a, b, c, d \in [0,1] \cap \mathbb{Q}$ such that $a < b < c < d$ and $x \in M_{a,b,c,d}$. It follows that the set of local minima of

$f$ is equal to

$$\bigcup\{M_{a,b,c,d} : a, b, c, d \in [0,1] \cap \mathbb{Q} \wedge a < b < c < d\},$$

which is clearly $F_\sigma$. □

*Proof of Theorem 6.* By Lemma 7, the set $M$ of local minima of $f$ can be written as $\bigcup_{n\in\mathbb{N}} M_n$ where each $M_n$ is closed. Assume that $M$ is not meager. Then for some $n \in \mathbb{N}$, $M_n$ is somewhere dense. Since $M_n$ is closed, $M_n$ actually contains a nonempty open interval $(a, b)$. But a continuous function that has a local minimum at each point of a nonempty interval is constant on that interval. A contradiction. □

**Corollary 8.** *If $f : [0,1] \to \mathbb{R}$ is not constant on any non-empty open interval, then the set of local extrema of $f$ is meager. However, even the set of local minima can be of measure 1.*

## References

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