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There are more co-analytic sets than Borel

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Abstract

This paper gives a proof based on large cardinal ideas that there is no injection inside $L(\mathbb{R})$ from $\underset{\sim}{\Pi}^1_1$ to Borel.

1 Introduction

This paper concerns cardinalities inside $L(\mathbb{R})$ under determinacy assumptions, in particular giving another proof of a previous established result that the cardinality of $\underset{\sim}{\Pi}^1_1$ is greater than that of $\underset{\sim}{\Delta}^1_1$ inside $L(\mathbb{R})$.

Definition For $A, B \in L(\mathbb{R})$ we write

$$|A|_{L(\mathbb{R})} \leq |B|_{L(\mathbb{R})}$$

if there is an injection from A to B in $L(\mathbb{R})$. We write

$$|A|_{L(\mathbb{R})} < |B|_{L(\mathbb{R})},$$

and say that —it the $L(\mathbb{R})$-cardinality of A is less than that of B if there is an injection in $L(\mathbb{R})$ from the first set to the second, but not from the second to the first.

From [2]:

Theorem 1.1 *(Hjorth) Assuming $AD^{L(\mathbb{R})}$, for all $\alpha < \beta < \omega_1$*

$$|\underset{\sim}{\Pi}^0_\alpha|_{L(\mathbb{R})} < |\underset{\sim}{\Pi}^0_\beta|_{L(\mathbb{R})}.$$

The exact computations in the Wadge hierarchy were given in [1], which in particular gave:

Theorem 1.2 *(Andretta, Hjorth, Neeman) Assuming $AD^{L(\mathbb{R})}$, for all $n > 1$*

$$|\underset{\sim}{\Delta}^1_1|_{L(\mathbb{R})} < |\underset{\sim}{\Pi}^1_1|_{L(\mathbb{R})} < |\underset{\sim}{\Pi}^1_n|_{L(\mathbb{R})}.$$

The proof given there was exacting a technical. In this short note I will sketch a simpler proof based on large cardinal concepts that

$$|\underset{\sim}{\Delta}^1_1|_{L(\mathbb{R})} < |\underset{\sim}{\Pi}^1_1|_{L(\mathbb{R})}.$$

2 Proof

For conceptual simplicity, let's start by assuming there are enough large cardinals ensure determinacy and absoluteness of the theory of $L(\mathbb{R})$ through all forcing extensions. See for instance [6].

Let

$$U \subset 2^\omega \times 2^\omega$$

be a universal Π^1_1 set. Assume for a contradiction

$$|\underset{\sim}{\Delta}^1_1|_{L(\mathbb{R})} \geq |\underset{\sim}{\Pi}^1_1|_{L(\mathbb{R})}.$$

Then in $L(\mathbb{R})$ we can find a relation

$$R \subset 2^\omega \times 2^\omega \times 2^\omega$$

such that:

(a) $(x, y_1, y_2) \in R \Rightarrow U_{y_1} = 2^\omega \setminus U_{y_2}$;
(b) $\forall x \exists y_1, y_2 R(x, y_1, y_2)$;
(c)if $(x, y_1, y_2) \in R, (x', y_1', y_2') \in R$, and $U_x = U_{x'}$, then $U_{y_1} = U_{y_1'}$.

By Basis($\underset{\sim}{\Sigma}^2_1, \underset{\sim}{\Delta}^1_2$) in $L(\mathbb{R})$ and [3], we can find tree representatives for such a relation R in all generic extensions. By considering homogeneous forcing notions collapsing various cardinals and appealing to the stabilization of the theory we can find some choice of the relation $R \in L(\mathbb{R})$ above, and a measurable cardinal κ, an inaccessible

$$\theta > \kappa$$

which is a limit of measurable cardinals and tree

$$T \subset 2^{<\omega} \times 2^{<\omega} \times 2^{<\omega} \times \delta^{<\omega},$$

on some δ such that in all forcing extensions of size less than $\beth_\omega(\kappa)^+ = |V_{\kappa+\omega}|^+$ we have that T continues to have $p[T] = R$, where R is now interpreted as its canonical extension in $L(\mathbb{R})$ of the generic extension, and thus we continue to have

(a) $(x, y_1, y_2) \in p[T] \Rightarrow U_{y_1} = 2^\omega \setminus U_{y_2}$;
(b) $\forall x \exists y_1, y_2 p[T](x, y_1, y_2)$;
(c)if $(x, y_1, y_2) \in p[T], (x', y_1', y_2') \in p[T]$, and $U_x = U_{x'}$, then $U_{y_1} = U_{y_1'}$.

T will arise from the Scale on Σ^2_1 in $L(\mathbb{R})$ of some suitable massive generic extension. Thus we can assume there is a *function* π uniformly definable over all such $L(\mathbb{R})$'s with

$$\pi(x) = (\pi_0(x), \pi(x_1)),$$

and

$$(x, \pi_0(x), \pi_1(x)) \in R$$

all $x \in 2^\omega$.

For future reference, let us fix now a measure μ on κ.

Definition A countable, transitive structure

$$\mathcal{M} = (M; \in, \kappa_0, \mu_0, T_0)$$

is a *frog* if there exists

$$\rho : \mathcal{M} \rightarrow V_\theta$$

with

$$\kappa_0 \longmapsto \kappa,$$
$$\mu_0 \longmapsto \mu,$$
$$T_0 \longmapsto T.$$

Note that this final clause ensures that any element of $p[T_0]$ is in $p[T]$ and hence R.

A countable transitive structure

$$\mathcal{N} = (N; \in, \kappa_0, \mu_0, A_0)$$

is a *tadpole* if it satisfies powerset, comprehension, and all other axioms of ZFC except possible replacement, and it is iterable against the measure μ_0, and $A_0 \subset \kappa_0$.

Given $\mathcal{M} = (M; \in, \kappa_0, \mu_0, T_0)$ a frog and $A_0 \in \mathcal{P}(\kappa_0)^{\mathcal{M}}$, we let

$$\mathcal{N} = (V_{\kappa_0+\omega}; \in, \kappa_0, \mu_0, A_0)$$

be the *tadpole induced from $\mathcal{M}$ by A.*

Note that any frog has unboundedly many measurables, and it will all generic extensions of the frog will be iterable against the surviving measurables in light of the embedding into a large rank initial segment of V.

Definition For $\mathcal{N} = (N; \in, \kappa_0, \mu_0, A_0)$ a tadpole, we let $V_{\mathcal{N}}$ be the set of codes for ordinals $\alpha < \omega_1$ such that if we take the iteration

$$i_{0,\alpha} : \mathcal{N} \to \mathcal{N}_\alpha$$

of length α against the measure μ_0, then

$$\alpha \in i_{0,\alpha}[A_0].$$

For $x \in 2^\omega$ coding a tadpole $\mathcal{N}$, we let $a(x)$ be chosen canonically, and uniformly recursively in x, with

$$U_{a(x)} = V_{\mathcal{N}}.$$

Note that any two codes for the same tadpole give rise to the same $\underset{\sim}{\Pi}^1_1$ set.
Thus given a tadpole $\mathcal{N}$ there will be a term $\tau_{\mathcal{N}}$ in

$$\mathbb{P}_{\mathcal{N}} = \mathrm{Coll}(\omega, \mathcal{N})$$

such that if $\sigma_{\mathcal{N}}$ is the canonical term for an element of 2^ω coding $\mathcal{N}$ then $\mathbb{P}_{\mathcal{N}}$ forces that $\sigma_{\mathcal{N}}[G]$ is a code for a Borel $B[\dot{G}]$ set of least possible rank with

$$\mathbb{P}_{\mathcal{N}} \Vdash B[\dot{G}] = U_{\pi_0(a(\sigma_{\mathcal{N}}[\dot{G}]))}.$$

Lemma 2.1

$$\mathbb{P}_{\mathcal{N}} \times \mathbb{P}_{\mathcal{N}} \Vdash B[\dot{G}_l] = B[\dot{G}_r].$$

Proof Since

$$\mathbb{P}_{\mathcal{N}} \times \mathbb{P}_{\mathcal{N}} \Vdash U_{\pi_0(a(\sigma_{\mathcal{N}}[\dot{G}_l]))} = U_{\pi_0(a(\sigma_{\mathcal{N}}[\dot{G}_r]))} = V_{\mathcal{N}}.$$

□

So for any$\mathcal{N}$ we can define a coresponding $\alpha_{\mathcal{N}}$ such that

$$\mathbb{P}_{\mathcal{N}} \Vdash B[\dot{G}] \text{ is a Borel set of rank } \alpha_{\mathcal{N}}.$$

By appealing to Wadge determinacy, the calculation of $\alpha_{\mathcal{N}}$ is absolute to inner model containing uncountably many ordinals and satisfying $\underset{\sim}{\Sigma}^1_1$ determinacy. Since every generic extension of a frog can be subject to an iteration of length ω_1, it will continue to correctly calculate $\alpha_{\mathcal{N}}$ for all its tadpoles through all generic extensions.

Lemma 2.2 *Let $\mathcal{M} = (M; \in, \kappa_0, \mu_0, T_0)$ be a frog. $A \in \mathcal{P}(\kappa)^{\mathcal{M}}$, and $\mathcal{N}$ the tadpole induced by A. Then*

$$\mathcal{M} \models \alpha_{\mathcal{N}} < \kappa_0.$$

Proof Take the iteration of $\mathcal{M}$ of length $\alpha_{\mathcal{N}} + 1$ mapping

$$i^{\mathcal{M}}_{0,\alpha_{\mathcal{N}}+1} : \mathcal{M} \to \mathcal{M}_{\alpha_{\mathcal{N}}+1},$$

$$\kappa_0 \mapsto \kappa_{\alpha_{\mathcal{N}}+1}.$$

The important point about this iteration is that it moves κ_0 to an ordinal above $\alpha_{\mathcal{N}}$. $i^{\mathcal{M}}_{0,\alpha_{\mathcal{N}}+1}|\mathcal{N}$ equals the internal iterate of $\mathcal{N}$ along its measure, since $\mathcal{N}$ is closed under power set. Thus $V_{\mathcal{N}} = V_{i^{\mathcal{M}}_{0,\alpha_{\mathcal{N}}+1}(\mathcal{N})}$.
Thus,

$$V_{\mathcal{N}} = V_{i^{\mathcal{M}}_{0,\alpha_{\mathcal{N}}+1}(\mathcal{N})},$$

and hence

$$\alpha_{\mathcal{N}} = \alpha_{i^{\mathcal{M}}_{0,\alpha_{\mathcal{N}}+1}(\mathcal{N})}.$$

and thus by the appeal to Wadge determinacy mentioned above,

$$\mathcal{M}_{\alpha_{\mathcal{N}}+1} \models \alpha_{i^{\mathcal{M}}_{0,\alpha_{\mathcal{N}}+1}(\mathcal{N})} < \kappa_{\alpha_{\mathcal{N}}+1},$$

and hence by elementarity

$$\mathcal{M} \models \alpha_{\mathcal{N}} < \kappa_0.$$

□

Thus by cardinality considerations inside $\mathcal{M}$ we can find a single $\alpha < \kappa_0$ such for some sequence $(A_\beta)_{\beta\in\kappa_0}$ we have that for $\mathcal{N}_\beta$ the tadpole induced from A_β

$$\mathcal{M} \models \mathbb{P}_{\mathcal{N}_\beta} \Vdash B_\beta[\dot{G}] \text{ is a Borel set of rank } \alpha,$$

$$\mathcal{M} \models \mathbb{P}_{\mathcal{N}_\beta} \Vdash B_\beta[\dot{G}] = U_{\pi_0(a(\sigma_{\mathcal{N}}[\dot{G}]))},$$

and hence

$$\mathcal{M} \models \mathbb{P}_{\mathcal{N}_\beta} \times \mathbb{P}_{\mathcal{N}_\beta} \Vdash B_\beta[\dot{G}_l] = B_\beta[\dot{G}_r]$$

and

$$\mathcal{M} \models \mathbb{P}_{\mathcal{N}_\beta} \times \mathbb{P}_{\mathcal{N}_\gamma} \Vdash B_\beta[\dot{G}_l] \neq B_\gamma[\dot{G}_r]$$

for $\beta \neq \gamma$.

Thus we obtain, inside $\mathcal{M}$, more than $\beth_{1+\alpha+1}$ many inequivalent codes for invariant Borel sets - which is exactly the situation ruled out by the paper [5], and hence a contradiction.

So much for the argument under the simplifying assumptions indicated, now for a proof under only $\mathrm{AD}^{L}(\mathbb{R})$.

This part uses some largely unpublished work of Hugh Woodin's, who showed that for any $S \subset Ord$ in $L(\mathbb{R})$ we have that on a cone of $x \in 2^\omega$

$$\mathrm{HOD}_S^{L[x,S]} \models (\omega_2)^L[x,S] \text{ is a Woodin cardinal},$$

where here $\mathrm{HOD}_S^{L[x,S]}$ is the collection of all sets in $L[x,S]$ which (inside $L[x,S]$ are hereditarily definable from S and the ordinals. Working inside such a model where S codes up the tree T for the complete Σ^2_1 set, the argument passes through as above.

References

[1] A. Andretta, G. Hjorth, I. Neeman, *Effective cardinals of boldface pointclasses*, **Journal of Mathematical Logic,** 7 (2007), no. 1, 35–82.

[2] G. Hjorth, *An absoluteness principle for Borel sets,* **Journal of Symbolic Logic,** 63 (1998), no. 2, 663–693.

[3] D.A. Martin, Y.N. Moschovakis, J.R. Steel, *The extent of definable scales,* **Bulletin American Mathematical Society,** (N.S.) 6 (1982), no. 3, 435–440

[4] Y.N. Moschovakis, **Descriptive Set Theory,** Studies in Logic and the Foundations of Mathematics, 100. North-Holland Publishing Co., Amsterdam-New York, 1980.

[5] J. Stern, **Annals of Mathematics**, vol. 120 (1984), pp. 7-37.

[6] W.H. Woodin, *Supercompact cardinals, sets of reals, and weakly homogeneous trees,* **Proceedings of the National Academy of Sciences of the United States of America,** 85 (1988), no. 18, 6587–6591.