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Pair-reaping, finite chromatic ideal and Smirnov compactifications of ω

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Abstract

Kada, Tomoyasu and Yoshinobu [4] investigated the cardinal $\mathfrak{sp}$, the smallest cardinality of a set D of compatible metrics on the countable discrete space ω such that, $\beta\omega$ is approximated by Smirnov compactifications for all metrics in D but any finite subset of D does not suffice. In this article we will observe that the cardinal $\mathsf{non}^*(\mathcal{G}_{FC})$, which was introduced in the context of the investigation of Katětov order among Borel ideals, gives a lower bound for $\mathfrak{sp}'$, a variant of the cardinal $\mathfrak{sp}$.

1 Introduction

We use standard notation and basic facts about set theory. We refer the readers to [1] for undefined set-theoretic notions and symbols for cardinal characteristics of the continuum.

Let X be a non-compact completely regular Hausdorff space. For compactifications αX and γX of X, we write $\alpha X \leq \gamma X$ if there is a continuous surjection $f : \gamma X \to \alpha X$ such that $f \restriction X$ is the identity map on X. If such an f can be chosen to be a homeomorphism, we say αX and γX are *equivalent* and denote this by writing $\alpha X \simeq \gamma X$. It holds that $\alpha X \simeq \gamma X$ if and only if $\alpha X \leq \gamma X$ and $\gamma X \leq \alpha X$ [2, Theorem 3.5.4].

Let $\mathcal{K}(X)$ denote the class of compactifications of X. When we identify equivalent compactifications and regard $\mathcal{K}(X)$ as the collection of equivalence classes, we may regard $\mathcal{K}(X)$ as a set, and then the order structure $(\mathcal{K}(X), \leq)$ is a complete upper semilattice whose largest element is the Stone–Čech compactification βX.

The following lemma is well-known.

Lemma 1.1. *For compactifications $\alpha X, \gamma X$ of a space X, the following conditions are equivalent:*

(1) $\alpha X \leq \gamma X$.
(2) *For $A, B \subseteq X$, if $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$, then $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B = \emptyset$.*

In particular, $\alpha X \simeq \beta X$ if and only if, for any pair of disjoint closed subsets A, B of X we have $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$.

$C^*(X)$ denotes the ring of all bounded continuous functions from X to $\mathbb{R}$, equipped with the uniform norm topology. For a metric space (X, d), $U_d^*(X)$ denotes the set of all bounded uniformly continuous functions from (X, d) to $\mathbb{R}$. $U_d^*(X)$ is a closed subring of $C^*(X)$ which contains all constant functions and generates the topology on X. The *Smirnov compactification* $u_d X$ of a metric space (X, d) is the unique compactification associated with the subring $U_d^*(X)$. More precisely, $u_d X$ is characterized in the following way.

Theorem 1.2. [9, Theorem 2.5] *Let (X, d) be a metric space. For a compactification αX of X, the following conditions are equivalent:*

(1) $\alpha X \simeq u_d X$.
(2) *For $f \in C^*(X)$, f is continuously extended over αX if and only if $f \in U_d^*(X)$.*
(3) *For closed subsets A, B of X, $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$ if and only if $d(A, B) > 0$.*

The following theorem means that the Stone–Čech compactification βX of a metrizable space X is approximated by the collection of Smirnov compactifications for all compatible metrics on X. For a metrizable space X, let $\mathrm{M}(X)$ denote the set of all metrics on X which are compatible with the topology on X.

Theorem 1.3. [9, Theorem 2.11] *For a non-compact metrizable space X, we have* $\beta X \simeq \sup\{u_d X : d \in \mathrm{M}(X)\}$.

Now we set the following general question:

> *How many metrics do we actually need to approximate the Stone-Čech compactification by a collection of Smirnov compactifications?*

This question naturally leads us to the following definition of a cardinal function.

> For a metrizable space X, $\mathfrak{sa}(X)$ is the smallest cardinality of a set $D \subseteq \mathrm{M}(X)$ which satisfies $\beta X \simeq \sup\{u_d X : d \in D\}$.

We have investigated the cardinal $\mathfrak{sa}(X)$ under some reasonable assumption on X (for example, separability or local compactness) [5, 6, 10]. But when we work on the countable discrete space ω, it makes no sense to deal with $\mathfrak{sa}(\omega)$, since $\beta\omega \simeq u_d\omega$ holds for the discrete metric d on ω (that is, $d(x, y) = 1$ whenever $x \neq y$) and hence $\mathfrak{sa}(\omega) = 1$. Here we consider "nontrivial" ways to approximate $\beta\omega$ by Smirnov compactifications of ω.

For a metrizable space X, let $\mathrm{M}'(X)$ be the set of metrics $d \in \mathrm{M}(X)$ for which $\beta X \not\simeq u_d X$.

Definition 1.4. $\mathfrak{sp}$ is the smallest cardinality of a set $D \subseteq \mathrm{M}'(\omega)$ such that, for every finite set $F \subseteq D$ we have $\beta\omega \not\simeq \sup\{u_d\omega : d \in F\}$, and $\beta\omega \simeq \sup\{u_d\omega : d \in D\}$.

For compatible metrics $d_1, d_2 \in \mathrm{M}(X)$ on a metrizable space X, we write $d_1 \preceq d_2$

if $U^*_{d_1}(X) \subseteq U^*_{d_2}(X)$ (or equivalently, $u_{d_1}X \leq u_{d_2}X$). Note that $d_1 \preceq d_2$ if and only if the identity map on X is uniformly continuous as a function from (X, d_2) to (X, d_1).

Definition 1.5. $\mathfrak{sp}'$ is the smallest cardinality of a set $D \subseteq \mathrm{M}'(\omega)$ such that D is directed with respect to $\preceq$ (that is, for any $d_1, d_2 \in D$ there is a $d \in D$ with $d_1 \preceq d$ and $d_2 \preceq d$) and $\beta\omega \simeq \sup\{u_d\omega : d \in D\}$.

It is clear that $\mathfrak{sp} \leq \mathfrak{sp}'$. We do not know whether $\mathfrak{sp} = \mathfrak{sp}'$ holds under ZFC. Note that $\mathfrak{sp}'$ is simply characterized in the following way.

Proposition 1.6. *The following cardinalities are equal:*

(1) $\mathfrak{sp}'$.
(2) *The smallest cardinality of a set $D \subseteq \mathrm{M}'(\omega)$ such that, for any disjoint subsets A, B of ω there is a $d \in D$ such that $d(A, B) > 0$.*
(3) *The smallest cardinality of a set $D \subseteq \mathrm{M}'(\omega)$ such that $C^*(\omega) = \bigcup\{U^*_d(\omega) : d \in D\}$ (that is, for any bounded real-valued function f on ω, there is a $d \in D$ such that f is uniformly continuous with respect to d).*

We have the following relations among $\mathfrak{sp}$, $\mathfrak{sp}'$ and other cardinal characteristics of the continuum [4] (See [4, Definition 1.4] for the definition of $\mathfrak{l}$).

Theorem 1.7. (1) $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{sp}$ *and* $\mathsf{cov}(\mathcal{N}) \leq \mathfrak{sp}$.
(2) $\mathfrak{sp}' \leq \mathfrak{u}$.
(3) $\mathfrak{sp}' \leq \mathfrak{l} \leq \mathsf{cof}(\mathcal{N})$.

2 Pair-reaping and Smirnov compactifications

The cardinal $\mathfrak{r}_{pair}$ was defined independently by Minami, Hrušák and Meza-Alcántara [3, 7, 8]. We deal with subgraphs of the infinite undirected graph $[\omega]^2$. We say a subgraph A of $[\omega]^2$ is *unbounded* if $A \cap [\omega \setminus k]^2 \neq \emptyset$ for all $k < \omega$. For an infinite subset X of ω and an unbounded subgraph A of $[\omega]^2$, we say X *pair-splits* A if X splits infinitely many edges of A, that is, there are infinitely many $a \in A$ such that $|a \cap X| = 1$. We call a collection $\mathcal{R}$ of unbounded subgraphs of $[\omega]^2$ a *pair-reaping family* if for every set $X \in [\omega]^\omega$ there is a member A of $\mathcal{R}$ which is not pair-split by X, that is, for all but finitely many $a \in A$, $a \subset X$ or $a \subset \omega \setminus X$. The *pair-reaping number* $\mathfrak{r}_{pair}$ is the smallest cardinality of a pair-reaping family.

We have the following relations among $\mathfrak{r}_{pair}$ and other cardinal characteristics of the continuum [7].

Theorem 2.1. (1) $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{r}_{pair}$ *and* $\mathsf{cov}(\mathcal{N}) \leq \mathfrak{r}_{pair}$.
(2) $\mathfrak{r}_{pair} \leq \mathfrak{r}$.

We prove that $\mathfrak{r}_{pair}$ is a lower bound for $\mathfrak{sp}'$, which provides a better lower bound than the ones given in Theorem 1.7. Actually, an even better lower bound will be given later, by Theorem 3.1. However, it would be still worth observing the proof of the following proposition for the readers to get the point of the proof of Theorem 3.1.

Proposition 2.2. $\mathfrak{r}_{pair} \leq \mathfrak{sp}'$.

Proof. Let κ be a cardinal with $\kappa < \mathfrak{r}_{pair}$. Fix a subset D of $\mathrm{M}'(\omega)$ which is of size κ and is $\preceq$-directed. We shall find a bounded real-valued function f on ω which is *not* d-uniformly continuous for any $d \in D$.

For each $d \in D$, since $u_d\omega \not\simeq \beta\omega$, there is a pair A, B of disjoint subsets of ω such that $d(A, B) = 0$. Using the sets A, B we can construct an unbounded graph $A_d \in \left[[\omega]^2\right]^\omega$ on ω such that $\lim\{d(x, y) : \{x, y\} \in A_d\} = 0$, that is, for any $\varepsilon > 0$, for all but finitely many edges $\{x, y\}$ of A_d we have $d(x, y) < \varepsilon$.

Since $|D| = \kappa < \mathfrak{r}_{pair}$, we can choose an infinite subset X of ω so that X pair-splits A_d for all $d \in D$ simultaneously. Let f be the characteristic function of X, that is, for $n \in \omega$, $f(n) = 1$ if $n \in X$ and $f(n) = 0$ otherwise. f is a bounded real-valued function on ω, but f is not d-uniformly continuous for any $d \in D$, because, by the choice of A_d and X, for any $\varepsilon > 0$ we can find $x, y \in \omega$ with $d(x, y) < \varepsilon$ and $|f(x) - f(y)| = 1$. □

3 Finite chromatic ideal and Smirnov compactifications

The finite chromatic ideal $\mathcal{G}_{FC}$ was introduced in the context of the investigation of Katětov order among Borel ideals.

For a subgraph A of $[\omega]^2$, a *coloring* of A (or a *node-coloring* of A) is a function f from ω to ω such that $|f''a| = 2$ for every $a \in A$. We say a subgraph A of $[\omega]^2$ is *finitely chromatic* if there is a coloring of A whose range is finite. The collection of all finitely chromatic subgraphs of $[\omega]^2$ is an ideal on $[\omega]^2$, which we call the *finite chromatic ideal* and denote by $\mathcal{G}_{FC}$.

For an ideal $\mathcal{I}$ on a countable set C which contains all singletons, we say $\mathcal{I}$ is *tall* if for each $X \in [C]^\omega$ there is an $I \in \mathcal{I}$ such that $I \cap X$ is infinite. For a tall ideal $\mathcal{I}$ on C, the *uniformity number of* $\mathcal{I}$, denoted by $\mathsf{non}^*(\mathcal{I})$, is defined by the following:

$$\mathsf{non}^*(\mathcal{I}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subset [C]^\omega \text{ and } \forall I \in \mathcal{I}\, \exists A \in \mathcal{A}\ (|A \cap I| < \aleph_0)\right\}.$$

It is known that $\mathfrak{r}_{pair} \leq \mathsf{non}^*(\mathcal{G}_{FC})$ [3], but it is unknown if $\mathfrak{r}_{pair} = \mathsf{non}^*(\mathcal{G}_{FC})$ is proved under ZFC.

The following theorem provides an even better lower bound for $\mathfrak{sp}'$ than the one given by Proposition 2.2.

Theorem 3.1. $\mathsf{non}^*(\mathcal{G}_{FC}) \leq \mathfrak{sp}'$.

Proof. Let κ be a cardinal with $\kappa < \mathsf{non}^*(\mathcal{G}_{FC})$. Fix a subset D of $\mathrm{M}'(\omega)$ which is of size κ and is $\preceq$-directed. We shall find a bounded real-valued function f on ω which is *not* d-uniformly continuous for any $d \in D$.

For each $d \in D$, since $u_d\omega \not\simeq \beta\omega$, there is a pair A, B of disjoint subsets of ω

such that $d(A, B) = 0$. Using the sets A, B we can construct an unbounded graph $A_d \in \left[[\omega]^2\right]^\omega$ on ω such that $\lim\{d(x, y) : \{x, y\} \in A_d\} = 0$, that is, for any $\varepsilon > 0$, for all but finitely many edges $\{x, y\}$ of A_d we have $d(x, y) < \varepsilon$.

Since $|D| = \kappa < \mathsf{non}^*(\mathcal{G}_{FC})$, we can choose a finitely chromatic graph $G \in \mathcal{G}_{FC}$ so that, for every $d \in D$ we have $|A_d \cap G| = \aleph_0$. Let f be a finite coloring of the graph G, that is, the range of f is finite and $|f''e| = 2$ holds for all $e \in G$. Note that f is a bounded real-valued function on ω (which takes only integer values). But f is not d-uniformly continuous for any $d \in D$, because, by the choice of A_d and G, for any $\varepsilon > 0$ we can find $x, y \in \omega$ with $d(x, y) < \varepsilon$ and $|f(x) - f(y)| \geq 1$. □

4 Questions

Question 4.1. $\mathsf{non}^*(\mathcal{G}_{FC}) \leq \mathfrak{sp}$? *Or,* $\mathfrak{r}_{pair} \leq \mathfrak{sp}$?

Question 4.2. $\mathfrak{r} \leq \mathfrak{sp}$? *Or,* $\mathfrak{r} \leq \mathfrak{sp}'$?

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