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Portfolio optimization for piecewise concave criteria functions

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Abstract

In the context of a complete financial model, we study the portfolio optimization problem when the objective function may have a change of concavity at a given positive constant level. This typically includes utility maximization of terminal wealth when the agent modifies her preferences structure from a certain level of wealth. This also allows to consider the portfolio management problem of an investor willing to achieve a given level of performance by penalizing net loss and maximizing net gain. We finally compare some of our results with the classical portfolio choice problem of Merton by doing some numerical experiments.

Key words: Utility maximization, gain/loss criterion, nonconvex optimization, conjugate duality approach, Malliavin calculus.

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1 Introduction

Portfolio optimization problems in finance usually assume concave (or convex) objective functions which are specified independently from the wealth process. The main examples are utility maximization from consumption and terminal wealth and hedging problems (mean-variance, shortfall risk, ...), see Karatzas and Shreve (1998) and references therein.

In this paper, we consider a continuous objective function U on $(0, \infty)$ of the form :

$$U(x) = \begin{cases} U_1(x), & 0 < x \leq h \\ U_2(x), & x > h, \end{cases}$$

where $h > 0$, and U_1, U_2 are C^1, and strictly concave functions. Function U is in general only piecewise C^1 and piecewise concave. We study the problem of maximizing the expected objective function U of terminal wealth in the context of complete Itô processes model. Such a problem typically arises in utility maximization when the preferences structure of the agent changes from U_1 to U_2 at h. The constant h is then interpreted as a level of wealth at which the risk-aversion of the agent is modified. For example, the agent may be less risk-averse when her wealth is large enough and may be more risk-averse when her wealth decreases a lot. Hence, such a piecewise concave utility function allows to take into account an effect of the agent's wealth on her risk behavior. On the other hand, consider an investor who wants to achieve a given level of performance h by adopting the following criterion : She penalizes net loss, i.e. when terminal wealth is below h, and maximizes net gain, i.e. when terminal wealth is above h. We call this criterion the portfolio gain/loss management. This is embedded in our optimization problem by choosing $U_1(x) = -l(h-x)$ and $U_2(x) = g(x-h)$, where l is a convex C^1 loss function and g is a concave C^1 gain function.

We solve our piecewise concave optimization problem by using a martingale duality approach. We consider the conjugate function $\tilde{U}$ of U, i.e. $\tilde{U}(y) = \sup_{x>0}(U(x) - xy)$ and we provide an explicit expression of a function χ that attains the supremum in the definition of $\tilde{U}$. In general, function χ is not continuous. In a second step, we prove continuity of the function $H(y) = E[Z_T^0 \chi(yZ_T^0)]$ under a certain condition, namely that the drift process of the asset is nonzero. Actually, by means of Malliavin calculus, this last condition ensures that the density of the unique martingale measure, Z_T^0, is absolutely continuous with respect to the Lebesgue measure. This continuity result is essential to state the budget constraint and then to adapt the standard martingale approach of Cox and Huang (1989) or Karatzas et al. (1987). In the case where the drift process of the asset is zero, we derive directly, by the dynamic programming methods, the value function of our optimization problem. It appears that when U is not concave, the control problem is singular and there is no optimal portfolio. We also show that duality relation between the value functions of the primal optimization problem and of the dual problem holds, even in the case where the objective function U is not concave. Finally, we analyze the qualitative behavior of the optimal strategy and compare with the classical Merton's portfolio strategy by doing some numerical experiments.

The paper is organized as follows. Section 2 describes the financial model and Section 3 formulates the portfolio optimization problem. In Section 4, we solve the problem by using a martingale duality approach. We also derive the solution by a dynamic programming approach when the drift process of the asset is zero. Section 5 presents some examples and we derive in Section 6 closed-form expressions for the optimal portfolio in the Black-Scholes model. Finally, Section 7 presents some numerical results.

2 The financial model

We consider the standard setup of a complete Itô processes model for a financial market, as described for example in Karatzas and Shreve (1998). There are one bank account, with constant price process S^0, normalized to unity, and d risky assets of price process $S = (S^1, \ldots, S^d)'$ governed by :

$$dS_t = \mu_t dt + \sigma_t dW_t, \quad S_0 = s \in \mathbb{R}^d.$$

Here $W = (W^1, \ldots, W^d)'$ is a standard d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$; this is the P-augmentation of the filtration generated by W. The $\mathbb{R}^d$-valued process μ and the $\mathbb{R}^{d\times d}$-valued process σ are assumed to be progressively measurable with respect to $\mathbb{F}$. We shall also assume that the matrix σ_t is invertible for all $t \in [0, T]$, P a.s.

We define then the 'market price of risk' process :

$$\lambda_t = \sigma_t^{-1}\mu_t, \quad 0 \leq t \leq T,$$

which is assumed to satisfy $\int_0^T |\lambda_t|^2 dt < \infty$. We consider then the exponential P-local martingale :

$$Z_t^0 = \exp\left(-\int_0^t \lambda_u' dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 du\right), \quad 0 \leq t \leq T. \tag{2.1}$$

We shall assume Z^0 is a P-martingale, i.e. $E[Z_T^0] = 1$, so that one can define a probability measure P^0 equivalent to P on $(\Omega, \mathcal{F}_T)$ by :

$$P^0(A) = E[Z_T^0 1_A], \quad A \in \mathcal{F}_T.$$

Recall that a well-known sufficient condition ensuring that $E[Z_T^0] = 1$ is the Novikov criterion : $E\left[\exp\left(\frac{1}{2}\int_0^T |\lambda_t|^2 dt\right)\right] < \infty$, which is obviously satisfied if λ is bounded in (t, ω).

By Girsanov's theorem, the process

$$W_t^0 \;=\; W_t + \int_0^t \lambda_u du, \quad 0 \le t \le T,$$

is a P^0-Brownian motion, and the dynamics of S under the so-called risk-neutral equivalent martingale measure P^0 is :

$$dS_t \;=\; \sigma_t dW_t^0, \quad 0 \le t \le T. \tag{2.2}$$

A portfolio is an $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process $\theta = (\theta^1, \ldots, \theta^d)'$ such that :

$$\int_0^T |\sigma_t' \theta_t|^2 dt \;<\; \infty, \quad a.s.$$

Here, θ_t^i describes the number of shares invested in the i-th risky asset at time t. The (self-financed) wealth process $X^{x,\theta}$ corresponding to an initial capital $x \ge 0$ and a portfolio θ is defined by :

$$\begin{aligned} dX_t^{x,\theta} &= \theta_t' dS_t \\ &= \theta_t' \sigma_t dW_t^0, \quad X_0^{x,\theta} = x. \end{aligned} \tag{2.3}$$

A portfolio θ is called admissible for the initial capital $x \ge 0$, and we write $\theta \in \mathcal{A}(x)$, if :

$$X_t^{x,\theta} \;\ge\; 0, \quad a.s., \; 0 \le t \le T. \tag{2.4}$$

Remark 2.1 From (2.3), the process $X^{x,\theta}$ is a P^0-local martingale, and from (2.4), it is nonnegative, thus also a P^0-supermartingale. We deduce that :

$$E\left[Z_T^0 X_T^{x,\theta}\right] \;\le\; x, \quad \forall \theta \in \mathcal{A}(x). \tag{2.5}$$

3 The portfolio optimization problem

We consider a continuous function $U : (0, \infty) \to \mathbb{R}$ defined by :

$$U(x) \;=\; \begin{cases} U_1(x), & 0 < x \le h \\ U_2(x), & x > h, \end{cases}$$

where $h > 0$, U_1 is strictly concave, of class C^1 on $(0, h]$, and U_2 is strictly concave, of class C^1 on (h, ∞). Notice that continuity of U means that $U_2(h) := \lim_{x \downarrow h} U_2(x) = U_1(h)$. We shall assume

$$U_2'(\infty) := \lim_{x \to \infty} U_2'(x) \;=\; 0. \tag{3.1}$$

This last condition is an Inada type condition on the behaviour at infinity of U'. Notice that we do not impose an Inada type condition on the behaviour at zero of U'. In particular, we can choose an exponential utility function $U_1(x) = -e^{-x}$. We denote $U_2'(h) := \lim_{x \downarrow h} U_2'(x)$. Notice that when $U_1'(h) \neq U_2'(h)$, the function U is not differentiable. Moreover, function U is concave on $(0, \infty)$ iff $U_1'(h) \geq U_2'(h)$. In the general case, the function U is only piecewise C^1 and piecewise concave. In the limiting case, $h = 0$, we recover the usual case of concave and C^1 utility function $U = U_2$.

Our interest is on the optimization problem :

$$J(x) = \sup_{\theta \in \mathcal{A}(x)} E\left[U(X_T^{x,\theta})\right], \quad x > 0. \tag{3.2}$$

Application 1
When U_1 and U_2 are standard utility functions, problem (3.2) is an utility maximization problem from terminal wealth. The constant h is interpreted as a level of wealth at which risk aversion of the agent may change.

Application 2
Consider the case where $U_1(x) = -l(h - x)$ and $U_2(x) = g(x - h)$, with l, a C^1 strictly convex function on $[0, \infty)$ and g, a C^1 strictly concave function on $[0, \infty)$, such that $l(0) = g(0) = 0$. Then problem (3.2) is written equivalently as :

$$J(x) = \sup_{\theta \in \mathcal{A}(x)} E\left[-l(h - X_T^{x,\theta})_+ + g(X_T^{x,\theta} - h)_+\right], \quad x > 0.$$

This is a portfolio management problem for an investor who wishes to achieve a level of performance h, by penalizing net loss and maximizing net gain.

4 Solution to the optimization problem

We define the conjugate function of U :

$$\tilde{U}(y) = \sup_{x>0} \left(U(x) - xy\right), \quad y > 0,$$

which is a nonincreasing and convex function from $(0, \infty)$ into $\mathbb{R} \cup \{\infty\}$. Notice that function $\tilde{U}$ is not necessarily smooth C^1.

We also define the dual value function :

$$\tilde{J}(y) = E\tilde{U}\left(yZ_T^0\right), \quad y > 0.$$

In a first step, we provide an explicit characterization of a function χ that attains the supremum in definition of $\tilde{U}$. We need to introduce some notations. We denote by I_i

the inverse of the derivative of U_i, $i = 1,2$; I_1 is a continuous strictly decreasing function from $[U_1'(h), U_1'(0))$ into $(0, h]$ and is extended by continuity on $[U_1'(h), U_1'(0)]$ when $U_1'(0) := \lim_{x\to 0} U_1'(x) < \infty$, by setting $I_1(U_1'(0)) = 0$. Notice that when $U_1'(0) < \infty$, U_1 is also extended by continuity in 0 by setting $U_1(0) = 0$; I_2 is a continuous strictly decreasing function from $(0, U_2'(h))$ into (h, ∞) and is extended by continuity by h on $[U_2'(h), \infty)$. In the case where $U_1'(h) < U_2'(h)$, we define the function $\phi : [U_1'(h), U_2'(h)] \to \mathbb{R}$ by [1] :

$$\phi(y) = \begin{cases} U_1 \circ I_1(y) - U_2 \circ I_2(y) - y(I_1 - I_2)(y), & U_1'(h) \leq y \leq U_1'(0) \wedge U_2'(h) \\ U_1(0) - U_2 \circ I_2(y) + yI_2(y), & U_1'(0) \wedge U_2'(h) < y \leq U_2'(h). \end{cases} \tag{4.1}$$

Proposition 4.1 *There exists a nonnegative function χ defined on $(0, \infty)$ such that :*

$$\tilde{U}(y) = U(\chi(y)) - y\chi(y), \quad y > 0.$$

Function χ is explicitly characterized as follows :
When $U_1'(h) \geq U_2'(h)$, we get :

$$\chi(y) = \begin{cases} I_2(y), & 0 < y < U_2'(h) \\ h, & U_2'(h) \leq y \leq U_1'(h) \\ I_1(y), & U_1'(h) < y < U_1'(0) \\ 0, & y \geq U_1'(0). \end{cases} \tag{4.2}$$

When $U_1'(h) < U_2'(h)$, we have :

$$\chi(y) = \begin{cases} I_2(y), & 0 < y < y(h) \\ I_1(y), & y(h) \leq y < y(h) \vee U_1'(0) \\ 0, & y \geq y(h) \vee U_1'(0). \end{cases} \tag{4.3}$$

where $y(h)$ is the unique element in $(U_1'(h), U_2'(h))$ such that $\phi(y(h)) = 0$.

Proof. See Appendix. □

Remark 4.1 In the case where $U_1'(h) < U_2'(h)$, function χ of Proposition 4.1 may be not continuous on $(0, \infty)$; from (4.3), there is a discontinuity at point $y = y(h)$ whenever $I_1(y(h))$ $\neq I_2(y(h))$ or $y(h) > U_1'(0)$.

Remark 4.2 By definition of $\tilde{U}$, it is clear that :

$$U(x) \leq \tilde{\tilde{U}}(x) := \inf_{y>0} \left(\tilde{U}(y) + xy\right), \quad x > 0.$$

[1]For any real numbers a and b, we denote by $a \wedge b$ (resp. $a \vee b$) , the minimum (resp. maximum) of a and b.

It is well-known that when U is concave, we have equality $U = \tilde{\tilde{U}}$, see e.g Ekeland and Temam (1976). In our context, in the case where $U_1'(h) < U_2'(h)$, so that U is not concave, one can check that $U \neq \tilde{\tilde{U}}$. For example, when $U_1'(0) \leq U_2'(h)$ and $y(h) \geq U_1'(0)$, i.e. $\phi(U_1'(0)) \leq 0$, a straightforward calculation shows that for $x < I_2(y(h))$:

$$\tilde{\tilde{U}}(x) = \min[U_2 \circ I_2(y(h)) - y(h)I_2(y(h)), U_1(0)] + xy(h),$$

and so $\tilde{\tilde{U}}$ differs from U.

Remark 4.3 By definition of χ and $\tilde{U}$, we have for all $y, z > 0$:

$$\begin{aligned}\tilde{U}(y) - \chi(y)(z-y) &= U(\chi(y)) - \chi(y)z \\ &\leq \tilde{U}(z).\end{aligned}$$

This shows that for all $y > 0$, $-\chi(y) \in \partial\tilde{U}(y)$, the subgradient of the convex function $\tilde{U}$. When U is concave, the converse is true : any element $\hat{x} \in -\partial\tilde{U}(y)$ attains the supremum in $\tilde{U}(y)$, i.e. $\tilde{U}(y) = U(\hat{x}) - \hat{x}y$ (see e.g Ekeland and Temam (1976)). This property is crucial in the dual formulation when the set of martingale measures is not a singleton, see Cvitanić (2000) or Deelstra, Pham and Touzi (2001). This last property is no more valid in our context. We shall give some examples in Section 5.

Remark 4.4 In the case of Example 2, when U_1 and U_2 are on the form $U_1(x) = -l(h-x)$ and $U_2(x) = g(x-h)$, with $l(0) = g(0) = 0$, function ϕ defined in (4.1) reduces to :

$$\phi(y) = \begin{cases} \tilde{l}(y) - \tilde{g}(y), & l'(0) \leq y \leq g'(0) \wedge l'(h) \\ -l(h) - \tilde{g}(y) + yh, & g'(0) \wedge l'(h) < y \leq g'(0), \end{cases}$$

where, $\tilde{l}(y) = \max_{x>0}[-l(x) + xy]$ and $\tilde{g}(y) = \max_{x>0}[g(x) - xy]$.

We shall make the following assumption :

Assumption 4.1

$$E\left[Z_T^0 I_2(yZ_T^0)\right] < \infty, \quad \forall y \in (0,\infty).$$

Remark 4.5 Suppose that there exists $\alpha \in (0,1)$ and $\gamma \in (1,\infty)$ such that $\alpha U_2'(x) \geq U_2'(\gamma x)$, $\forall x \in (h,\infty)$. Then, by similar arguments as in Remark 6.9, p. 107 of Karatzas and Shreve (1998), Assumption 4.1 holds whenever $E\left[Z_T^0 I_2(y_0 Z_T^0)\right] < \infty$ for some y_0 in $(0, U_2'(h))$.

Remark 4.6 Suppose that there exist $C \geq 0$, $m, n > 0$ such that :

$$I_2(y) \leq C(1 + y^m + y^{-n}), \quad \forall y > 0.$$

Then the boundedness of the process λ in (t, ω) is a sufficient condition for Assumption 4.1 to hold.

From expression of function χ in Proposition 4.1 and recalling the nonincreasing feature of I_1 and I_2, we easily see that :

$$\chi(y) \leq I_2(y), \quad \forall y > 0. \tag{4.4}$$

Under Assumption 4.1, one can then define the real-valued function on $(0, \infty)$ by :

$$H(y) = E\left[Z_T^0 \chi(y Z_T^0)\right], \quad y > 0.$$

The second step is to state continuity of function H and then to prove that the budget constraint is satisfied : Given $x > 0$, one can find $\hat{y}(x) > 0$, such that $\hat{X} = \chi(\hat{y}(x) Z_T^0)$ is a terminal wealth satisfying $E^{P^0}[\hat{X}] = H(\hat{y}(x)) = x$.

We need to make some assumptions on the market price of risk λ. We denote by $\mathbb{H}$ the Cameron-Martin space formed by the functions of the form $\psi(t) = \int_0^t \dot{\psi}(s) ds$, $t \in [0, T]$, with $\dot{\psi} \in L^2([0, T], \mathbb{R}^d)$ equipped with the norm $\| \psi \|_{\mathbb{H}} = \left(\int_0^T |\dot{\psi}(s)|^2 ds\right)^{\frac{1}{2}}$. We denote by D the Malliavin derivative operator defined on the domain $\mathbb{D}^{1,2}$ of $L^2(\Omega)$; $D : \mathbb{D}^{1,2} \to L^2(\Omega, \mathbb{H})$. We refer to Nualart (1995) for all unexplained notations. Given a random variable $F \in \mathbb{D}^{1,2}$, $DF(\omega) = (D^1 F(\omega), \ldots, D^d F(\omega))'$ is valued in $\mathbb{H}$ for $\omega \in \Omega$, and $D_t F(\omega) = (D_t^1 F(\omega), \ldots, D_t^d F(\omega))'$, $0 \leq t \leq T$, is defined by :

$$DF = \int_0^{\cdot} D_t F dt, \quad a.s.$$

(CL) For all $t \in [0, T]$, $\lambda_t^i \in \mathbb{D}^{1,2}$, $i = 1, \ldots, d$, and satisfy :

$$E^{P^0}\left[\int_0^T |D_t \lambda_s|^2 ds\right]^{\frac{1}{2}} < \infty, \tag{4.5}$$

where $D_t \lambda_s = (D_t \lambda_s^1, \ldots, D_t \lambda_s^d)$. Moreover,

$$\lambda_t(\omega) \neq 0, \quad dt \times dP \ a.s. \tag{4.6}$$

Remark 4.7 Notice that when λ is a bounded deterministic process, we have $D_t \lambda_s = 0$, and the last condition reduces to $\lambda_t \neq 0$ dt a.e.

Lemma 4.1 *Suppose that condition* **(CL)** *holds. Then, for all $z \geq 0$, $P^0[Z_T^0 = z] = 0$.*

Proof. The case $z = 0$ is obvious since P^0 and P are equivalent. Fix now $z > 0$. The distribution law of $\ln Z_T^0$ under P is given by :

$$\ln Z_T^0 \;=\; -\int_0^T \lambda_s' dW_s - \frac{1}{2}\int_0^T |\lambda_s|^2 ds.$$

From standard calculations on Malliavin derivative (see e.g. Proposition 2.3 in Ocone and Karatzas 1991), we then have for all $0 \leq t \leq T$:

$$\begin{aligned} D_t \ln Z_T^0 &= -\lambda_t - \int_t^T D_t\lambda_s dW_s - \int_t^T D_t\lambda_s\, \lambda_s ds \\ &= -\lambda_t - \int_t^T D_t\lambda_s dW_s^0, \end{aligned} \tag{4.7}$$

The integrability condition (4.5) ensures that for all $t \in [0,T]$, the Itô stochastic integral process $\{\int_0^t D_t\lambda_s dW_s^0,\ 0 \leq t \leq T\}$ is a P^0-martingale, see e.g. Jacod (1979). We deduce from (4.7) that :

$$E^{P^0}[D_t \ln Z_T^0 | \mathcal{F}_t] \;=\; -\lambda_t, \quad 0 \leq t \leq T.$$

Under condition (4.6) and recalling that P^0 is equivalent to P, this implies that :

$$\| D \ln Z_T^0 \|_{\mathcal{H}}^2 \;=\; \int_0^T |D_t \ln Z_T^0|^2 dt \;>\; 0, \quad P^0 \ a.s.$$

From Theorem 3.1.1 in Nualart (1995), we deduce that the distribution law of $\ln Z_T^0$ under P^0 admits a density with respect to the Lebesgue measure on $\mathbb{R}$, and so the required result.

□

Proposition 4.2 *Let Assumption 4.1 hold and suppose that one of the two following conditions holds :*
(i) $U_1'(h) \geq U_2'(h)$,
(ii) $U_1'(h) < U_2'(h)$ and condition **(CL)** *holds.*
Then the function H is continuous on $(0,\infty)$ and for all $x > 0$, there exists $\hat{y}(x) > 0$ (not necessarily unique) such that $H(\hat{y}(x)) = x$.

Proof. First, notice that from (4.4), we have :

$$Z_T^0 \chi(y Z_T^0) \;\leq\; Z_T^0 I_2(y Z_T^0), \quad \forall y > 0. \tag{4.8}$$

1) We suppose that condition (i) holds. Then by Proposition 4.1 (4.2), function χ is continuous on $(0,\infty)$. From Assumption 4.1, (4.8) and the dominated convergence theorem, we obtain the continuity of H on $(0,\infty)$.

2) We now suppose that condition (ii) holds. From Proposition 4.1 (4.3), function χ is right continuous. The right-continuity of H is then stated from Assumption 4.1, (4.8), the nondecreasing feature of function I_2 and dominated convergence theorem. To prove the left-continuity of H in $y > 0$, take a nondecreasing sequence of positive real $(y_n)_n$ such that $y_n \nearrow y$.

We then see from (4.3) that :

$$\chi(y_n Z_T^0) \quad \rightarrow \quad \chi(y Z_T^0) + (I_2 - I_1 1_{y(h) < U_1'(0)})(y(h)) 1_{y Z_T^0 = y(h)}.$$

Observe that for n large enough, $y_n \geq y/2$. Then, by the nondecreasing property of function I_2, we have :

$$Z_T^0 \chi(y_n Z_T^0) \quad \leq \quad Z_T^0 I_2 \left(\frac{y}{2} Z_T^0 \right), \quad \forall y > 0.$$

We deduce by the dominated convergence theorem that :

$$H(y_n) \quad \rightarrow \quad H(y) + (I_2 - I_1 1_{y(h) < U_1'(0)})(y(h)) P^0[y Z_T^0 = y(h)]. \tag{4.9}$$

Using Lemma 4.1, this proves the left-continuity and then the continuity of H in y.

3) In all cases, by noting that $\chi(y) \rightarrow \infty$ when $y \rightarrow 0$, we see, by Fatou's Lemma, that $H(y) \rightarrow \infty$ when $y \rightarrow 0$. By noting that $\chi(y) \rightarrow 0$ when $y \rightarrow \infty$, we obtain, by the dominated convergence theorem that $H(y) \rightarrow 0$ when $y \rightarrow 0$. This property combined with the continuity of H proves the existence of $\hat{y}(x) > 0$ such that $H(\hat{y}(x)) = x$, for all $x > 0$.

□

Remark 4.8 The nonrandomness of the critical value h is only required in this last proposition, see (4.9). Indeed, in this case, $y(h)$ is nonrandom and by Lemma 4.1, $P^0[y Z_T^0 = y(h)]$ $= 0$, for all $y > 0$, which implies the continuity of function H.

Adapting arguments of conjugate duality in complete markets, we characterize the solution to problem (3.2) and prove the duality relation between value functions J and $\tilde{J}$.

Theorem 4.1 *Suppose that conditions of Proposition 4.2 hold. Then for all $x > 0$, there exists an optimal portfolio $\hat{\theta}$ for problem* (3.2) *whose terminal wealth is given by :*

$$\hat{X} \quad = \quad \chi\left(\hat{y}(x) Z_T^0\right), \tag{4.10}$$

where χ is defined in Proposition 4.1 and $\hat{y}(x)$ given by Proposition 4.2. The associated optimal wealth is given by :

$$X_t^{x,\hat{\theta}} \quad = \quad E^{P^0}\left[\hat{X} \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \tag{4.11}$$

Moreover, we have the duality relation :

$$J(x) \;=\; \min_{y>0}\left(\tilde{J}(y)+xy\right), \quad x>0.$$

Proof. From definition of $\hat{y}(x)$, the nonnegative $\mathcal{F}_T$-measurable random variable $\hat{X}$ given in (4.10) lies in $L^1(P^0)$ and

$$E^{P^0}[\hat{X}] \;=\; x. \tag{4.12}$$

Consider then the nonnegative $(P^0,\mathbb{F})$-martingale $M_t = E^{P^0}[\hat{X}|\mathcal{F}_t]$, $0 \le t \le T$. By the martingale representation property under P^0 (see e.g. Lemma 6.7 p.25 in Karatzas and Shreve 1998) and relation (2.2), we obtain the existence of a portfolio $\hat{\theta} \in \mathcal{A}(x)$ such that :

$$M_t \;=\; x+\int_0^t \hat{\theta}'_u dS_u = X_t^{x,\hat{\theta}}, \quad 0\le t\le T. \tag{4.13}$$

Now, by definition of χ in Proposition 4.1, we have for all $\theta \in \mathcal{A}(x)$:

$$U(X_T^{x,\theta})-\hat{y}(x)Z_T^0X_T^{x,\theta} \;\le\; \tilde{U}(\hat{y}(x)Z_T^0) \;=\; U(\hat{X})-\hat{y}(x)Z_T^0\hat{X}.$$

Taking expectation and using (2.5), (4.12), we obtain that :

$$EU(X_T^{x,\theta}) \;\le\; EU(\hat{X}), \quad \forall\theta\in\mathcal{A}(x).$$

Since $\hat{X} = X_T^{x,\hat{\theta}}$ by (4.13), this proves that $\hat{\theta}$ is solution to (3.2) and $J(x) = EU(\hat{X})$. Relation (4.11) is simply relation (4.13). By definition of $\tilde{U}$ and from (2.5), we have for all $x > 0$, $\theta \in \mathcal{A}(x)$, $y > 0$:

$$\begin{aligned} EU\left(X_T^{x,\theta}\right) \;&\le\; E\tilde{U}\left(yZ_T^0\right)+yE\left[Z_T^0X_T^{x,\theta}\right] \\ &\le\; \tilde{J}(y)+xy, \end{aligned}$$

and so $J(x) \le \inf_{y>0}(\tilde{J}(y)+xy)$. On the other hand, given $x > 0$, we have by definition of χ and by (4.10), (4.12) :

$$\begin{aligned} J(x) \;=\; EU\left(\hat{X}\right) \;&=\; E\tilde{U}\left(\hat{y}(x)Z_T^0\right)+\hat{y}(x)E\left[Z_T^0\hat{X}\right] \\ &=\; \tilde{J}(\hat{y}(x))+x\hat{y}(x), \end{aligned}$$

which proves the last assertion of the theorem. □

Conditions of the previous theorem does not include the case where $\lambda \equiv 0$ and $U_1'(h) < U_2'(h)$. In such a context, recall that U is not concave, and function H is equal to χ, which may be discontinuous. One can then not apply the martingale approach. However, it is possible to derive directly the value function of problem (3.2).

Theorem 4.2 *Suppose that $\lambda \equiv 0$ and U is bounded from below. Then the value function J of problem (3.2) is equal to U^{con}, the concave envelope of U (i.e. the least concave majorant function of U) and we have the duality relation :*

$$J(x) \;=\; \inf_{y>0}\left(\tilde{J}(y)+xy\right), \quad x>0. \tag{4.14}$$

Proof. In the case $\lambda \equiv 0$, the dynamics of S is governed by $dS_t = \sigma_t dW_t$. It is convenient to change of control variable by defining $\pi_t = \sigma_t'\theta_t$. We introduce then the dynamic value function associated to problem (3.2) by :

$$\mathcal{J}(t,x) \;=\; \sup_{\pi\in\Pi(t,x)} E\left[U(X_T^{t,x,\pi})\right], \quad t\in[0,T],\ x>0, \tag{4.15}$$

where $\Pi(t,x)$ is the set of adapted processes $(\pi_s)_{t\leq s\leq T}$ satisfying $\int_t^T|\pi_s|^2ds<\infty$ and such that :

$$\begin{aligned} X_s^{t,x,\pi} \;&:=\; x+\int_t^s \pi_u' dW_u \;\geq\; 0, \quad t\leq s\leq T,\\ X_t^{t,x,\pi} \;&=\; x. \end{aligned}$$

Notice that with these notations, we have $J(x)=\mathcal{J}(0,x)$. From dynamic programming principle (see e.g. Fleming and Soner 1993), the value function $\mathcal{J}$ is a lower-semicontinuous viscosity supersolution of :

$$-\frac{\partial w}{\partial t}+\inf_{p\in\mathbb{R}}\left(-\frac{1}{2}p^2\frac{\partial^2 w}{\partial x^2}\right) \;=\; 0. \tag{4.16}$$

By using similar arguments as in Lemma 5.1 in Cvitanic, Pham and Touzi (1999), we deduce from this last relation that function $\mathcal{J}$ is concave in x and nonincreasing in t (this is formally proved by sending p respectively to infinity and zero in (4.16)). Moreover, we clearly have from (4.15) and Fatou's lemma (recall that U is bounded from below) that $\mathcal{J}(T^-,x)\geq U(x)$.

By definition of the concave envelope, this implies that :

$$\mathcal{J}(t,x) \;\geq\; U^{con}(x), \quad t\in[0,T),\ x>0.$$

On the other hand, since $U(\infty)>-\infty$, it is clear that U^{con} is nondecreasing. We then have for all $t\in[0,T)$, $x>0$, $\pi\in\Pi(t,x)$:

$$\begin{aligned} E\left[U(X_T^{t,x,\pi})\right] \;&\leq\; E\left[U^{con}(X_T^{t,x,\pi})\right]\\ &\leq\; U^{con}\left(E[X_T^{t,x,\pi}]\right)\\ &\leq\; U^{con}(x), \end{aligned}$$

where the second relation follows from Jensen's inequality and the third from the fact that $E[X_T^{t,x,\pi}] \leq x$ and the nondecreasing feature of U^{con}. This proves that $\mathcal{J}(t,x) \leq U^{con}(x)$ and so the required equality. Finally, by noting that $\tilde{J}(y) = \tilde{U}(y)$ (since $Z_T^0 = 1$), the duality relation (4.14) follows from Proposition I.4.1. in Ekeland and Temam (1976). □

Remark 4.9 In the case where U is concave and so $J \equiv U$, the optimal control is given by $\pi \equiv 0$. This means that the optimal portfolio is to invest nothing in the stocks. When U is not concave, the control problem (4.15) is singular : there is no optimal control in the class $\mathcal{A}(x)$ (an optimal one would be obtained for a process θ taking only values 0 and infinity).

Remark 4.10 Theorems 4.1 and 4.2 show that, although duality relation between U and $\tilde{U}$ does not hold (see Remark 4.2), we have duality relation between value functions J and $\tilde{J}$.

5 Examples

5.1 Power-Power utility function

We consider the example where U_i, $i = 1, 2$, are power utility function with constant relative risk aversion $1 - \alpha_i$, $\alpha_i \in (0, 1)$:

$$\begin{aligned} U_1(x) &= \frac{x^{\alpha_1}}{\alpha_1}, \\ U_2(x) &= \frac{x^{\alpha_2}}{\alpha_2} + C, \end{aligned}$$

where $C = \frac{h^{\alpha_1}}{\alpha_1} - \frac{h^{\alpha_2}}{\alpha_2}$ is a constant added in order to ensure continuity of the utility function U, i.e. $U_1(h) = U_2(h)$. Notice that $U_1'(0) = \infty$ and $U_1'(h) \geq U_2'(h)$ iff $h^{\alpha_1} \geq h^{\alpha_2}$.
Case : $h^{\alpha_1} \geq h^{\alpha_2}$
We have :

$$\chi(y) = \begin{cases} y^{-\frac{1}{1-\alpha_2}}, & 0 < y < h^{\alpha_2 - 1} \\ h, & h^{\alpha_2-1} \leq y \leq h^{\alpha_1 - 1} \\ y^{-\frac{1}{1-\alpha_1}}, & y > h^{\alpha_1 - 1} \end{cases}$$

Case : $h^{\alpha_1} < h^{\alpha_2}$
We have :

$$\phi(y) = \frac{y^{-\beta_1}}{\beta_1} - \frac{y^{-\beta_2}}{\beta_2} - C, \quad y \in [h^{\alpha_1 - 1}, h^{\alpha_2 - 1}],$$

where $\beta_i = \alpha_i/(1-\alpha_i)$. Then $y(h)$ is the unique solution in $(h^{\alpha_1-1}, h^{\alpha_2-1})$ of

$$\frac{y^{-\beta_1}}{\beta_1} - \frac{h^{\alpha_1}}{\alpha_1} = \frac{y^{-\beta_2}}{\beta_2} - \frac{h^{\alpha_2}}{\alpha_2}.$$

Function χ is explicitly expressed in :

$$\chi(y) = \begin{cases} y^{-\frac{1}{1-\alpha_2}}, & 0 < y < y(h) \\ y^{-\frac{1}{1-\alpha_1}}, & y \geq y(h). \end{cases}$$

Notice that χ is discontinuous in $y(h)$.
A straightforward computation leads to :

$$\tilde{U}(y) = \begin{cases} \frac{y^{-\beta_2}}{\beta_2} + C, & 0 < y < y(h) \\ \frac{y^{-\beta_1}}{\beta_1}, & y \geq y(h). \end{cases}$$

Function $\tilde{U}$ is differentiable on $(0, \infty)$ except in $y(h)$. For $y \neq y(h)$, we have $\tilde{U}'(y) = -\chi(y)$. For $y = y(h)$, the subgradient of $\tilde{U}$ is given by

$$-\partial\tilde{U}(y(h)) = \left[y(h)^{-\frac{1}{1-\alpha_1}}, y(h)^{-\frac{1}{1-\alpha_2}}\right].$$

We easily check that any element $\hat{x}$ in the interior of $-\partial\tilde{U}(y(h))$ does not attain the maximum in $\tilde{U}(y(h))$, i.e. $U(\hat{x}) - \hat{x}y(h) < \tilde{U}(y(h))$.

5.2 Exponential-Logarithm utility function

We consider the example where U_1 is an exponential utility function with absolute risk aversion η and U_2 is a logarithm utility function :

$$\begin{aligned} U_1(x) &= -\exp(-\eta x), \\ U_2(x) &= \ln x + C, \end{aligned}$$

where $C = -\exp(-\eta h) - \ln h$ is a constant added in order to ensure continuity of the utility function U, i.e. $U_1(h) = U_2(h)$.
We see that $U_1'(h) = \eta e^{-\eta h} < U_2'(h) = \frac{1}{h}$ and function U is non concave. We have $U_1'(0) = \eta$ and

$$\phi(y) = \begin{cases} -\frac{y}{\eta} + \ln y - \frac{y}{\eta}\ln\frac{y}{\eta} + 1 - C, & \eta e^{-\eta h} < y \leq \eta \wedge \frac{1}{h} \\ \ln y - C, \quad \eta \wedge \frac{1}{h} < y \leq \frac{1}{h}. \end{cases}$$

We have to distinguish two cases depending on the sign of $\phi(U_1'(0)) = \ln \eta h + e^{-\eta h}$.

Case : $\ln \eta h + e^{-\eta h} \geq 0$

Then $y(h)$ is the unique solution in $(\eta e^{-\eta h}, \eta \wedge \frac{1}{h})$ of $-\frac{y}{\eta} + \ln y - \frac{y}{\eta}\ln\frac{y}{\eta} = C - 1$ and

$$\chi(y) = \begin{cases} \frac{1}{y}, & 0 < y < y(h) \\ -\frac{1}{\eta}\ln\frac{y}{\eta}, & y(h) \leq y < \eta \\ 0, & y \geq \eta. \end{cases}$$

Case : $\ln \eta h + e^{-\eta h} < 0$

Notice that this implies $\eta h < 1$. We then have $y(h) = e^C$ and

$$\chi(y) = \begin{cases} \frac{1}{y}, & 0 < y < e^C \\ 0, & y \geq e^C \end{cases}$$

Notice that χ is discontinuous in e^C.

A straightforward computation leads to :

$$\tilde{U}(y) = \begin{cases} -\ln y + C - 1, & 0 < y < e^C \\ -1, & y \geq e^C. \end{cases}$$

Function $\tilde{U}$ is differentiable on $(0, \infty)$ except in e^C. For $y \neq e^C$, we have $\tilde{U}'(y) = -\chi(y)$. For $y = e^C$, the subgradient of $\tilde{U}$ is given by

$$-\partial\tilde{U}(e^C) = [0, e^{-C}].$$

We easily see that any element $\hat{x}$ in the interior of $-\partial\tilde{U}(e^C)$ does not attain the maximum in $\tilde{U}(e^C)$.

5.3 Power Loss function-Power Utility function

We consider the case where :

$$U_1(x) = -\frac{(h-x)^p}{p}, \quad 0 < x \leq h,$$
$$U_2(x) = \frac{(x-h)^\alpha}{\alpha}, \quad x > h,$$

where $p > 1$ and $0 < \alpha < 1$. Then, $U_1'(h) = 0$ and $U_2'(h) = \infty$, and so $U_1'(h) \leq U_1'(0) < U_2'(h)$. We easily see that $\tilde{l}(y) = \frac{y^q}{q}$, where $q = \frac{p}{p-1}$ and $\tilde{g}(y) = \frac{y^{-\beta}}{\beta}$, where $\beta = \frac{\alpha}{1-\alpha}$. Moreover,

$$\phi(y) = \begin{cases} \frac{y^q}{q} - \frac{y^{-\beta}}{\beta}, & 0 \leq y \leq h^{p-1} \\ -\frac{h^p}{p} - \frac{y^{-\beta}}{\beta} + yh, & y > h^{p-1}. \end{cases}$$

Then $\phi(U_1'(0)) = \frac{h^p}{q} - \frac{h^{-(p-1)\beta}}{\beta}$ and we have to distinguish two cases depending on the sign of $\phi(U_1'(0))$.

Case : $h^{p+(p-1)\beta} \leq \frac{q}{\beta}$

Then $y(h)$ is the solution of $-\frac{h^p}{p} - \frac{y^{-\beta}}{\beta} + hy = 0$ in (h^{p-1}, ∞), and we get :

$$\chi(y) = \begin{cases} h + y^{-1-\beta}, & 0 < y < y(h) \\ 0, & y \geq y(h) \end{cases}$$

Case : $h^{p+(p-1)\beta} > \frac{q}{\beta}$

Then $y(h) \in (0, h^{p-1})$ and is equal to $y(h) = \left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}$, and we have :

$$\chi(y) = \begin{cases} h + y^{-1-\beta}, & 0 < y < \left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}} \\ h - y^{q-1}, & \left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}} \leq y < h^{p-1} \\ 0, & y \geq h^{p-1} \end{cases}$$

A straightforward computation leads to :

$$\tilde{U}(y) = \begin{cases} \frac{y^{-\beta}}{\beta} - yh, & 0 < y < \left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}} \\ \frac{y^q}{q} - yh, & \left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}} \leq y < h^{p-1} \\ -\frac{h^p}{p}, & y \geq h^{p-1} \end{cases}$$

Function $\tilde{U}$ is differentiable on $(0, \infty)$ except in $\left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}$. For $y \neq \left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}$, we have $\tilde{U}'(y)$ $= -\chi(y)$. For $y = \left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}$, the subgradient of $\tilde{U}$ is given by

$$-\partial\tilde{U}\left(\left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}\right) = \left[h - \left(\frac{q}{\beta}\right)^{\frac{q-1}{q+\beta}}, h - \left(\frac{q}{\beta}\right)^{\frac{\alpha-1}{q+\beta}}\right].$$

We easily check that any element $\hat{x}$ in the interior of $-\partial\tilde{U}\left(\left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}\right)$ does not attain the maximum in $\tilde{U}\left(\left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}\right)$.

Notice that in both cases, $h^{p+(p-1)\beta} > \frac{q}{\beta}$ and $h^{p+(p-1)\beta} \leq \frac{q}{\beta}$, function χ is discontinuous in $y(h)$.

6 Case of constant market price of risk

In this section, we consider the case where the market price of risk λ is a nonzero constant; this is essentially the Black-Scholes model. The density of the risk-neutral martingale measure is then given by :

$$Z_t^0 \;=\; \exp\left(-\lambda' W_t^0 + \frac{1}{2}|\lambda|^2 t\right), \quad 0 \le t \le T. \tag{6.1}$$

It follows that Z_T^0/Z_t^0 is independent of $\mathcal{F}_t$ and has same distribution law under P^0 as Z_{T-t}^0. We deduce from (4.11) that the optimal wealth process is given by :

$$X_t^{x,\hat{\theta}} \;=\; \mathcal{H}(t, \hat{y}(x) Z_t^0), \quad 0 \le t \le T, \tag{6.2}$$

where

$$\mathcal{H}(t,y) \;=\; E^{P^0}\left[\chi(y Z_{T-t}^0)\right], \quad (t,y) \in [0,T] \times (0,\infty),$$

and $\hat{y}(x) > 0$ is solution of $\mathcal{H}(0, \hat{y}(x)) = H(\hat{y}(x)) = x$. Moreover, when function $\mathcal{H}$ is smooth $C^{1,2}$, the optimal portfolio is simply obtained by applying Itô's formula on (6.2) and identifying diffusion terms :

$$\hat{\theta}_t \;=\; -\frac{\partial \mathcal{H}}{\partial y}(t, \hat{y}(x) Z_t^0) \hat{y}(x) Z_t^0 (\sigma_t')^{-1} \lambda, \quad 0 \le t \le T. \tag{6.3}$$

In the sequel, we provide some explicit examples where we compute function $\mathcal{H}$. We introduce the following notations : for all $\tau \in (0,T]$, $c \in (0,\infty)$, $\gamma \in \mathbb{R}$, we denote

$$d(\tau, c, \gamma) \;=\; \frac{\ln c - |\lambda|^2 \tau (\gamma + \frac{1}{2})}{|\lambda| \sqrt{\tau}}.$$

We also denote by Φ the distribution function of the standard normal law :

$$\Phi(d) \;=\; \int_{-\infty}^{d} \varphi(z) dz, \quad d \in \mathbb{R},$$

where $\varphi(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$.

Lemma 6.1 *For all $\tau \in (0,T]$, $c \in (0,\infty)$, $\gamma \in \mathbb{R}$, we have :*

$$E^{P^0}\left[(Z_\tau^0)^\gamma 1_{Z_\tau^0 \le c}\right] \;=\; \exp\left(\frac{\gamma(\gamma+1)|\lambda|^2 \tau}{2}\right) \Phi(d(\tau, c, \gamma)).$$

$$E^{P^0}\left[\ln\left(Z_\tau^0\right) 1_{Z_\tau^0 \le c}\right] \;=\; \frac{|\lambda|^2 \tau}{2} \Phi(d(\tau, c, 0)) - \varphi\left(d\left(\tau, c, \frac{1}{2}\right)\right)$$

Proof. See Appendix. □

We now provide explicit expressions of function $\mathcal{H}$ for the examples of the previous section.

Power-power utility function

We consider the example of paragraph 5.1. By using Lemma 6.1, a straightforward calculation leads to :

- For $h^{\alpha_1} \geq h^{\alpha_2}$:

$$
\begin{aligned}
\mathcal{H}(t,y) &= y^{-\frac{1}{1-\alpha_2}} \exp\left(\frac{\beta_2}{1-\alpha_2}\frac{|\lambda|^2(T-t)}{2}\right)\Phi\left(d\left(T-t,\frac{h^{\alpha_2-1}}{y},-\frac{1}{1-\alpha_2}\right)\right) \\
&+ h\left[\Phi\left(d\left(T-t,\frac{h^{\alpha_1-1}}{y},0\right)\right)-\Phi\left(d\left(T-t,\frac{h^{\alpha_2-1}}{y},0\right)\right)\right] \\
&+ y^{-\frac{1}{1-\alpha_1}} \exp\left(\frac{\beta_1}{1-\alpha_1}\frac{|\lambda|^2(T-t)}{2}\right)\left[1-\Phi\left(d\left(T-t,\frac{h^{\alpha_1-1}}{y},-\frac{1}{1-\alpha_1}\right)\right)\right].
\end{aligned}
$$

- For $h^{\alpha_1} < h^{\alpha_2}$:

$$
\begin{aligned}
\mathcal{H}(t,y) &= y^{-\frac{1}{1-\alpha_2}} \exp\left(\frac{\beta_2}{1-\alpha_2}\frac{|\lambda|^2(T-t)}{2}\right)\Phi\left(d\left(T-t,\frac{y(h)}{y},-\frac{1}{1-\alpha_2}\right)\right) \\
&+ y^{-\frac{1}{1-\alpha_1}} \exp\left(\frac{\beta_1}{1-\alpha_1}\frac{|\lambda|^2(T-t)}{2}\right)\left[1-\Phi\left(d\left(T-t,\frac{y(h)}{y},-\frac{1}{1-\alpha_1}\right)\right)\right].
\end{aligned}
$$

Exponential-logarithm utility function

We consider the example of paragraph 5.2. By using Lemma 6.1, we get :

- For $\ln \eta h + e^{-\eta h} \geq 0$:

$$
\begin{aligned}
\mathcal{H}(t,y) &= \frac{1}{y}\Phi\left(d\left(T-t,\frac{y(h)}{y},-1\right)\right) \\
&- \frac{1}{\eta}\left(\ln\frac{y}{\eta}+\frac{|\lambda|^2(T-t)}{2}\right)\left[\Phi\left(d\left(T-t,\frac{\eta}{y},0\right)\right)-\Phi\left(d\left(T-t,\frac{y(h)}{y},0\right)\right)\right] \\
&+ \frac{1}{\eta}\left[\varphi\left(d\left(T-t,\frac{\eta}{y},\frac{1}{2}\right)\right)-\varphi\left(d\left(T-t,\frac{y(h)}{y},\frac{1}{2}\right)\right)\right].
\end{aligned}
$$

- For $\ln \eta h + e^{-\eta h} < 0$:

$$
\mathcal{H}(t,y) = \frac{1}{y}\Phi\left(d\left(T-t,\frac{e^C}{y},-1\right)\right).
$$

Power loss function-power utility function

We consider the example of paragraph 5.3. Again, by using Lemma 6.1, we obtain :

- For $h^{p+(p-1)\beta} \leq \frac{q}{\beta}$:

$$
\begin{aligned}
\mathcal{H}(t,y) &= h\Phi\left(d\left(T-t,\frac{y(h)}{y},0\right)\right) \\
&+ y^{-1-\beta}\exp\left(\beta(1+\beta)\frac{|\lambda|^2(T-t)}{2}\right)\Phi\left(d\left(T-t,\frac{y(h)}{y},-1-\beta\right)\right).
\end{aligned}
$$

• For $h^{p+(p-1)\beta} > \frac{q}{\beta}$:

$$
\begin{aligned}
\mathcal{H}(t,y) \;=\;& h\Phi\left(d\left(T-t,\frac{\left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}}{y},0\right)\right) \\
&+ y^{-1-\beta}\exp\left(\beta(1+\beta)\frac{|\lambda|^2(T-t)}{2}\right)\Phi\left(d\left(T-t,\frac{\left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}}{y},-1-\beta\right)\right) \\
&+ h\left[\Phi\left(d\left(T-t,\frac{h^{p-1}}{y},0\right)\right)-\Phi\left(d\left(T-t,\frac{\left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}}{y},0\right)\right)\right] \\
&- y^{q-1}\exp\left(q(q-1)\frac{|\lambda|^2(T-t)}{2}\right)\left[\Phi\left(d\left(T-t,\frac{h^{p-1}}{y},q-1\right)\right)\right. \\
&\left.- \Phi\left(d\left(T-t,\frac{\left(\frac{q}{\beta}\right)^{\frac{1}{q+\beta}}}{y},q-1\right)\right)\right].
\end{aligned}
$$

In all those examples, function $\mathcal{H}$ is smooth $C^{1,2}$ and the optimal portfolio is given by (6.3).

7 Numerical results

In this section, we consider an agent in a Black-Scholes-Merton model :

$$S_t \;=\; S_0\exp\left((\mu-\frac{\sigma^2}{2})t+\sigma W_t\right), \tag{7.1}$$

with power nonconcave utility functions, who modifies her risk aversion from $1-\alpha_1$ to $1-\alpha_2$ at a level of wealth h. This is the example of paragraph 5.1 with $h^{\alpha_1} < h^{\alpha_2}$.

We provide numerical results for the optimal wealth-proportion invested in the risky asset S starting from initial wealth x, and which is given by :

$$\hat{\pi}_t \;:=\; \frac{\theta_t S_t}{X_t^{x,\hat{\theta}}} \;=\; \hat{p}(t,S_t),$$

where :

$$\hat{p}(t,s) \;:=\; \frac{\mu}{\sigma^2}\left(\frac{-y\frac{\partial\mathcal{H}}{\partial y}}{\mathcal{H}}\right)\left(t,\hat{y}(x)\left(\frac{s}{S_0}\right)^{-\frac{\mu}{\sigma^2}}e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}-\mu)t}\right),$$

from (2.1), (6.3) and (7.1). We compare our results with the constant optimal wealth-proportion in the Merton model for power utility function with risk aversion $1-\alpha_i$:

$$\pi_{merton}(\alpha_i) \;=\; \frac{\mu}{\sigma^2}\frac{1}{1-\alpha_i}.$$

We focus first on an agent whose risk aversion decreases when she reaches a high level of wealth : Graphicss 1 illustrate this case for the values $\alpha_1 = 0.2$, $\alpha_2 = 0.7$ and $h = 10x = 900$. We have also set $\mu = 0.15$, $\sigma = 0.1$ and $S_0 = 90$. Given a trajectory of the asset price S_t (Figure 1b), Figure 1.a gives the evolution of the optimal portfolio $t \mapsto \hat{\pi}_t = \hat{p}(t, S_t)$. Graphics 1.c and 1.d. provide the graph of the optimal proportion function $\hat{p}(t, .)$ for a long and short maturity.

Graphics 1 show that the agent starts with a strategy close to the Merton's optimal strategy $\pi_{merton}(\alpha_2)$ corresponding to the lower risk aversion $1 - \alpha_2$. When the time to maturity decreases, she switchs to the Merton's optimal strategy $\pi_{merton}(\alpha_1)$ corresponding to the higher risk aversion $1 - \alpha_1$. Notice that the size of this switch of strategy is more important as the range between α_1 and α_2 is large. For large time to maturity, the agent adopts the behavior of the Merton's agent with the lower risk aversion $1 - \alpha_2$ since she expects a higher objective value function. However, for short time to maturity, she must take into account her actual wealth which shall remain with large probability under the level h, and so she adopts the behavior of the Merton's agent with risk aversion $1 - \alpha_1$.

Similarly, Graphics 2 illustrate the case of an agent whose risk aversion increases when her wealth decreases largely. We choose the values $\alpha_1 = 0.7$, $\alpha_2 = 0.2$ and $h = x/100 = 0.9$. We have also set $\mu = -0.15$, $\sigma = 0.1$ and $S_0 = 90$. Again, for large time to maturity, the agent follows the strategy of the Merton's agent with lower risk aversion $1 - \alpha_1$. For short time to maturity, she must take into account her actual wealth which shall remain with large probability above the level h, and so she follows the strategy of the Merton's agent wity risk aversion $1 - \alpha_2$.

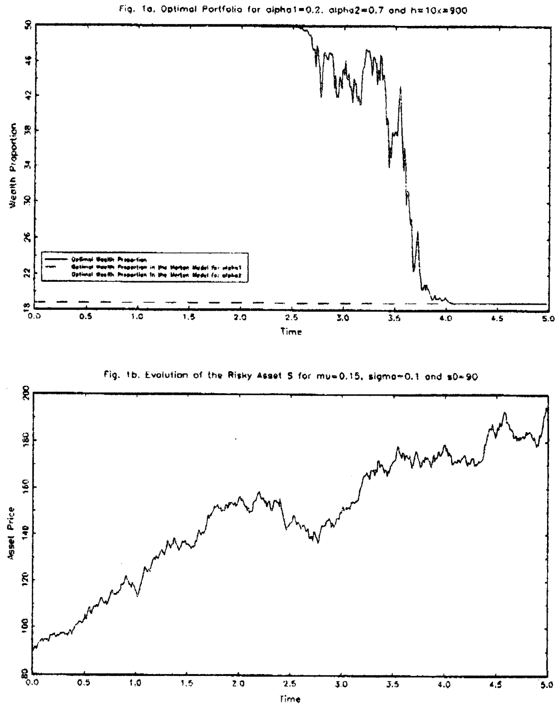

Figure 1: Graphics 1ab

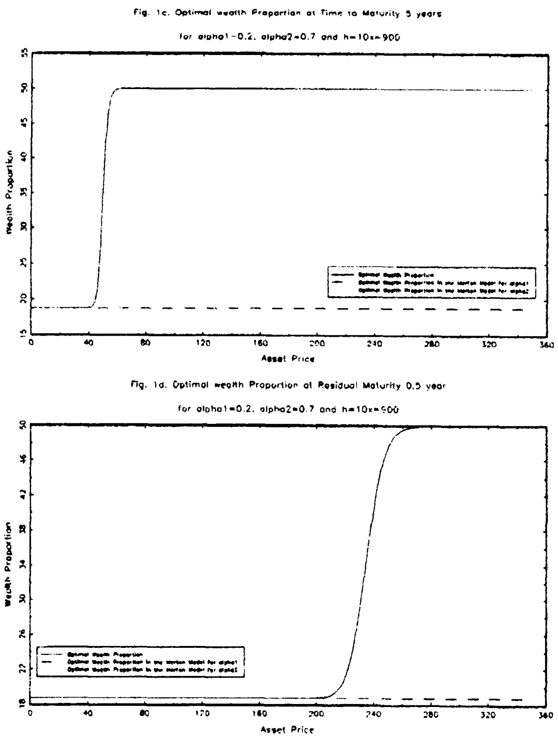

Figure 2: Graphics 1cd

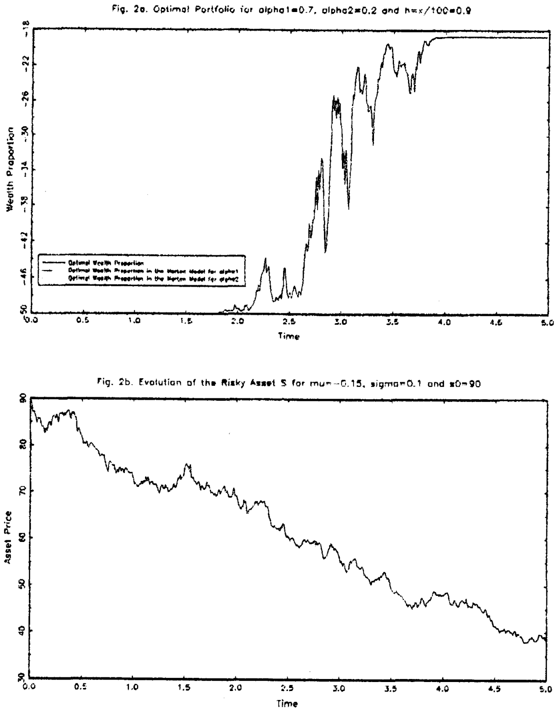

Figure 3: Graphics 2ab

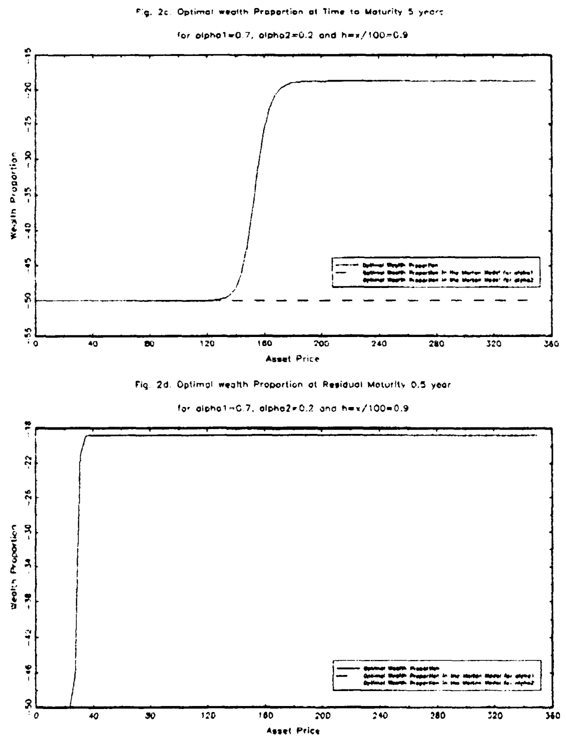

Figure 4: Graphics 2cd

8 Appendix

8.1 Proof of Proposition 4.1

We introduce the Fenchel-Legendre transform $\tilde{U}_i$ of the convex function $-U_i(-)$, for $i = 1, 2$: $\tilde{U}_1(y) = \sup_{0<x\leq h}[U_1(x) - xy]$ and $\tilde{U}_2(y) = \sup_{x\geq h}[U_2(x) - xy]$ for $y > 0$, so that :

$$\tilde{U}(y) = \max[\tilde{U}_1(y), \tilde{U}_2(y)], \quad y > 0.$$

The functions $\tilde{U}_i$ are convex and we easily see that $\tilde{U}_i(y) = U_i(\chi_i(y)) - \chi_i(y)y$, $i = 1, 2$, where χ_1 is a continuous nonincreasing function valued in $(0, h]$, defined by :

$$\chi_1(y) = \begin{cases} h, & 0 < y < U_1'(h) \\ I_1(y), & U_1'(h) \leq y < U_1'(0) \\ 0, & y \geq U_1'(0) \end{cases} \tag{8.1}$$

(where the third domain is empty when $U_1'(0) = \infty$) and χ_2 is a continuous nonincreasing function valued in $[h, \infty)$ and defined by

$$\chi_2(y) = \begin{cases} I_2(y), & 0 < y < U_2'(h) \\ h, & y \geq U_2'(h). \end{cases} \tag{8.2}$$

We then have $\tilde{U}(y) = U(\chi(y)) - \chi(y)y$, where :

$$\chi(y) = \chi_1(y) 1_{\tilde{U}_1(y) \geq \tilde{U}_2(y)} + \chi_2(y) 1_{\tilde{U}_1(y) < \tilde{U}_2(y)} \tag{8.3}$$

In order to compute explicitly χ, we have to characterize the domain $\{y > 0 \ : \tilde{U}_1(y) < \tilde{U}_2(y)\}$.

Let us define the following functions :

$$\begin{aligned} \bar{U}_1(y) &= U_1 \circ I_1(y) - y\left(I_1(y) - h\right), \quad \text{for } U_1'(h) \leq y < U_1'(0), \\ \bar{U}_2(y) &= U_2 \circ I_2(y) - y\left(I_2(y) - h\right), \quad \text{for } 0 < y \leq U_2'(h). \end{aligned}$$

These two functions are continuously differentiable and we have :

$$\begin{aligned} \bar{U}_1'(y) &= h - I_1(y) > 0, \quad \text{for } U_1'(h) < y < U_1'(0) \\ \bar{U}_2'(y) &= h - I_2(y) < 0, \quad \text{for } 0 < y < U_2'(h). \end{aligned}$$

Therefore, function $\bar{U}_1$ is strictly increasing and function $\bar{U}_2$ is strictly decreasing. Noting that for $i = 1, 2$, we have $\bar{U}_i(U_i'(h)) = U_i(h)$, we deduce that :

$$\bar{U}_1(y) - U_1(h) > 0, \quad \text{for } U_1'(h) < y < U_1'(0) \tag{8.4}$$

$$\bar{U}_2(y) - U_2(h) > 0, \quad \text{for } 0 < y < U_2'(h). \tag{8.5}$$

We first compute χ on the two following domains :

a) For $y \in (0, U_1'(h) \wedge U_2'(h))$, we have $\chi_1(y) = h$ and $\chi_2(y) = I_2(y)$. Hence, $\tilde{U}_1(y) - \tilde{U}_2(y) = U_1(h) - \bar{U}_2(y) = U_2(h) - \bar{U}_2(y) < 0$, by (8.5). Therefore, by (8.3), $\chi(y) = I_2(y)$.

d) For $y \in [U_1'(0) \vee U_2'(h), \infty)$, (notice that when $U_1'(h) \geq U_2'(h)$, $U_1'(0) \vee U_2'(h) = U_1'(0)$), we have $\chi_1(y) = 0$ and $\chi_2(y) = h$. If $U_1'(0) = \infty$, this case is vacuous. Otherwise, $\tilde{U}_1(y) - \tilde{U}_2(y) = U_1(0) - U_2(h) + hy = U_1(0) - U_1(h) + hy \geq U_1(0) - U_1(h) + hU_1'(0) > 0$, since U_1 is strictly concave on $(0, h]$. Therefore, by (8.3), $\chi(y) = 0$.

For the other domains, we now distinguish two cases :

First case : $U_1'(h) \geq U_2'(h)$.

b1) For $y \in [U_2'(h), U_1'(h)]$, we have $\chi_1(y) = \chi_2(y) = h$. Hence, by (8.3), $\chi(y) = h$.

c1) For $y \in (U_1'(h), U_1'(0))$, we have $\chi_1(y) = I_1(y)$ and $\chi_2(y) = h$. Hence, $\tilde{U}_1(y) - \tilde{U}_2(y) = \bar{U}_1(y) - U_2(h) = \bar{U}_1(y) - U_1(h) > 0$, by (8.4). Therefore, by (8.3), $\chi(y) = I_1(y)$.

Relation (4.2) is then stated by combining *a)*, *b1)*, *c1)* and *d)*.

Second case : $U_1'(h) < U_2'(h)$.
We shall see below that $\tilde{U}_1 - \tilde{U}_2$ is actually equal to the function ϕ introduced in (4.1) on $[U_1'(h), U_2'(h)]$. We first prove that there exists a unique $y(h) \in (U_1'(h), U_2'(h))$ such that $\phi(y(h)) = 0$. For all $y \in [U_1'(h), U_2'(h)]$, we have :

$$\phi(y) \quad = \quad \begin{cases} \bar{U}_1(y) - \bar{U}_2(y), & \text{for } U_1'(h) \leq y \leq U_1'(0) \wedge U_2'(h), \\ U_1(0) - \bar{U}_2(y) + hy, & \text{for } U_1'(0) \wedge U_2'(h) < y \leq U_2'(h). \end{cases}$$

Function ϕ is continuous on $[U_1'(h), U_2'(h)]$. Since ϕ is differentiable for $y \in (U_1'(h), U_1'(0) \wedge U_2'(h)) \cup (U_1'(0) \wedge U_2'(h), U_2'(h))$, and $\phi'(y) > 0$, we get that ϕ is a continuous strictly increasing function on $[U_1'(h), U_2'(h)]$. Recalling that $U_1(h) = U_2(h)$, we have :

$$\begin{aligned} \phi(U_1'(h)) &= U_2(h) - \bar{U}_2(U_1'(h)) = \bar{U}_2(U_2'(h)) - \bar{U}_2(U_1'(h)), \\ \phi(U_2'(h)) &= \begin{cases} U_1(0) - \bar{U}_2(U_2'(h)) + hU_2'(h) > U_1(0) - U_1(h) + hU_1'(0), & \text{for } U_1'(0) < U_2'(h) \\ \bar{U}_1(U_2'(h)) - \bar{U}_1(U_1'(h)), & \text{for } U_1'(0) \geq U_2'(h). \end{cases} \end{aligned}$$

From the strictly monotonicity of $\bar{U}_2$ and $\bar{U}_1$, and the strict concavity of U_1, we obtain that, $\phi(U_1'(h)) < 0$ and $\phi(U_2'(h)) > 0$. We deduce the existence of an unique $y(h) \in (U_1'(h), U_2'(h))$ such that $\phi(y(h)) = 0$, and :

$$\phi(y) \;<\; 0, \quad \text{for } U_1'(h) \leq y < y(h) \tag{8.6}$$

$$\phi(y) \;>\; 0, \quad \text{for } y(h) < y \leq U_2'(h). \tag{8.7}$$

Notice that $y(h) < U_1'(0)$ iff $\phi(U_1'(0) \wedge U_2'(h)) > 0$.
To compute χ, we have to distinguish several cases.

b2) For $y \in [U_1'(h), U_1'(0) \wedge U_2'(h))$, we have $\chi_1(y) = I_1(y)$ and $\chi_2(y) = I_2(y)$. Hence, $\tilde{U}_1(y) - \tilde{U}_2(y) = \phi(y)$, and from (8.6)-(8.7) and (8.3), $\chi(y) = I_2(y)1_{y<y(h)} + I_1(y)1_{y \geq y(h)}$.

c2) For $y \in [U_1'(0) \wedge U_2'(h), U_1'(0) \vee U_2'(h))$,

- If $U_1'(0) \leq U_2'(h)$, we have $\chi_1(y) = 0$ and $\chi_2(y) = I_2(y)$. Hence, $\tilde{U}_1(y) - \tilde{U}_2(y) = \phi(y)$, and again from (8.6)-(8.7) and (8.3), $\chi(y) = I_2(y)1_{y<y(h)}$.
- If $U_1'(0) > U_2'(h)$, we have $\chi_1(y) = I_1(y)$ and $\chi_2(y) = h$. Hence, by same arguments as in *c1)*, we have $\chi(y) = I_1(y)$.

Finally, we obtain the expression (4.3) of χ by noting the following points. When $U_1'(0) > U_2'(h)$, we have $\phi(U_2'(h) \wedge U_1'(0)) > 0$ and so $y(h) < U_1'(0)$. When $U_1'(0) \leq U_2'(h)$, we have either $[y(h), y(h) \vee U_1'(0)) = \emptyset$ and $\chi(y) = I_2(y)$ on $[U_1'(h), U_1'(0))$, whenever $\phi(U_1'(0)) \leq 0$; or $[y(h), y(h) \vee U_1'(0)) = [y(h), U_1'(0))$ and $\chi(y) = 0$ on $[U_1'(0), U_2'(h))$, whenever $\phi(U_1'(0)) > 0$. The proof is ended by combining *a*), *b2*), *c2*) and *d*).

8.2 Proof of Lemma 6.1

Consider the probability measure Q^γ with density with respect to P^0 given by :

$$\frac{dQ^\gamma}{dP^0} = \exp\left(-\gamma\lambda' W_T^0 - \frac{\gamma^2|\lambda|^2}{2}T\right).$$

Then, from (6.1) and Bayes formula, we have :

$$E^{P^0}\left[(Z_\tau^0)^\gamma 1_{Z_\tau^0 \leq c}\right] = \exp\left(\frac{\gamma(\gamma+1)|\lambda|^2\tau}{2}\right) Q^\gamma[Z_\tau^0 \leq c].$$

By noting that

$$Z_\tau^0 = \exp\left(-\lambda' W_\tau^{Q^\gamma} + |\lambda|^2\tau(\gamma + \frac{1}{2})\right),$$

where $W_t^{Q^\gamma} = W_t^0 + \gamma\lambda t$ is a Q^γ-brownian motion by Girsanov's theorem, we obtain the first relation of the Lemma.

On the other hand, from (6.1), we have :

$$E^{P^0}\left[\ln(Z_\tau^0)\, 1_{Z_\tau^0 \leq c}\right] = E^{P^0}\left[N 1_{N \leq d(c,\tau,\frac{1}{2})}\right] + \frac{|\lambda|^2\tau}{2} P^0[Z_\tau^0 \leq c],$$

where $N = -\lambda' W_\tau^0/(|\lambda|\sqrt{\tau})$ is a standard normal random variable under P^0. Finally, using the first relation of the Lemma for $\gamma = 0$, we obtain the required result.

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