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# Quasiconformal extension of univalent functions and Becker's theorem

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**Abstract**

This is a reserch for a subclass of univalent holomorphic functions on the unit disc normalized by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which can be extended to $k$-quasiconformal mappings on the disc $\{z \mid |z| < R\}$ where $R > 1$. Such a subclass is denoted by $\mathcal{S}(k, R)$. In this note, the class $\mathcal{S}(k, R)$ is introduced through the observation of Becker's theorem which ensures a $k$-quasiconformal extendibility of univalent holomorphic functions on the disc to the Riemann sphere with Löwner chains.

## 1 Motivation

Let $\mathbf{D} = \{z \mid |z| < 1\}$ and

$$\mathcal{S} = \{f \mid f \text{ is holomorphic and univalent on } \mathbf{D},\ f(0) = f'(0) - 1 = 0\},$$

$$\mathcal{S}(k) = \{f \mid f \in \mathcal{S},\ f \text{ can be extended to a } k\text{-quasiconformal mapping on } \widehat{\mathbf{C}}\},$$

$$\mathcal{S}_0(k) = \{f \mid f \in \mathcal{S}(k), \text{the extended mappings fix } \infty\},$$

respectively, where $k \in [0, 1)$. The class $\mathcal{S}(k)$ has been studied by numerous authors in connection with the theory of Teichmüller spaces. In those investigations, an interesting method for quasiconformal extension of univalent functions was obtained by Becker ([1], see also [5]) which relies on the *Löwner chains* described by the *Löwner equation*

$$\frac{\partial f(z,t)}{\partial t} = zp(z,t)\frac{\partial f(z,t)}{\partial z} \tag{1}$$

for $z \in \mathbf{D}$ and $t \in [0, \infty)$. This equation determines an expanding flow. Here, the function $f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$ is holomorphic in $|z| < 1$ for each $t \in [0, \infty)$, absolutely continuous in $t \in [0, \infty)$ for each $|z| < r_0$ and satisfies the inequality $|f(z,t)| \le K_0 e^t$ $(|z| < r_0, t \le 0)$ for some positive constants $K_0$ and $r_0$. Also a function $p(z,t)$ is measurable on $\mathbf{D} \times [0, \infty)$, holomorphic in $|z| < 1$, and satisfies $\operatorname{Re} p(z,t) > 0$ and the partial differential equation (1) for a.e. $t$.

**Theorem 1** ([1]). *If $f(z,t)$ is a univalent solution to* (1) *with $p(z,t)$ satisfying the condition*

$$\left|\frac{p(z,t)-1}{p(z,t)+1}\right| \le k < 1 \tag{2}$$

*then, for each $t \ge 0$, the function $f_t(z) = f(z,t)$ maps $\mathbf{D}$ onto a Jordan domain bounded by a $k$-quasiconformal image of $\partial\mathbf{D}$, and the map $\hat{f}(z)$ defined by*

$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta},0) & r \le 1 \\ f(e^{i\theta}, \log r) & r > 1 \end{cases}$$

*is a $k$-quasiconformal extension of $f(z,0)$ onto $\widehat{\mathbf{C}}$ with $\hat{f}(\infty) = \infty$ (thus $\hat{f}(z) \in S_0(k)$).*

Observe that $p(\mathbf{D},t)$ must be contained in the disc $|z - (1+k^2)/(1-k^2)| \le 2k/(1-k^2)$ for *all* $t \in [0,\infty)$ so that we can apply Theorem 1 to the Löwner chains (Fig.1). This strong assumption can be weakened by restricting the range of the parameter $t$. In fact, the following is true;

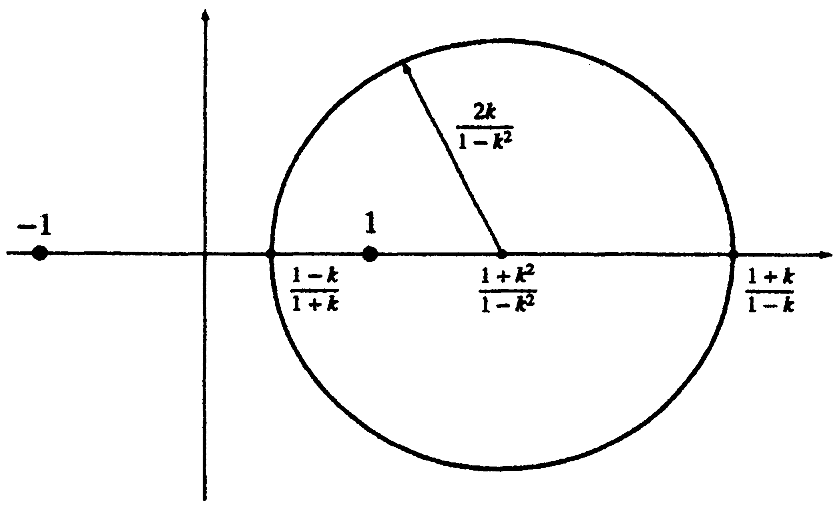


Figure 1 : $p(z,t)$ must be in this circle for all $z \in \mathbf{D}$ and $t \in [0,\infty)$.

**Corollary 2.** *If $f(z,t)$ is a univalent solution to* (1) *and there exists $t_0 > 0$ such that for all $t \in [0, t_0]$ $p(z,t)$ satisfies the condition*

$$\left|\frac{p(z,t)-1}{p(z,t)+1}\right| \le k < 1, \tag{3}$$

*then the map $\hat{f}(z)$ is a $k$-quasiconformal extension of $f(z,0)$ defined on $\{z \mid |z| < e^{t_0}\}$ with $\hat{f} \neq \infty$.*

Now we shall introduce the classes $\mathcal{S}(k,R)$ and $\mathcal{S}_0(k,R)$; namely

$$\mathcal{S}(k,R) = \{f \mid f \in \mathcal{S}, f \text{ can be extended to a } k\text{-quasiconformal mapping } \hat{f} \text{ on } \{|z| < R\}\}$$

and

$$\mathcal{S}_0(k,R) = \{f \mid f \in \mathcal{S}(k,R), \text{ the extended mapping } \hat{f} \text{ doesn't take } \infty \text{ on } \{|z| < R\}\}$$

respectively, where $R > 1$.

## 2 Properties of the class $\mathcal{S}(k,R)$

The class $\mathcal{S}(k,R)$ was studied by some authors in another context. We shall give some known results for the classes $\mathcal{S}(k,R)$ and $\Sigma(k,r)$, where $\Sigma(k,r)$ is a family of univalent holomorphic functions on $\{z \in \widehat{\mathbf{C}} - \overline{\mathbf{D}}\}$ which can be extended to a $k$-quasiconformal mapping on $\{|z| > r\}, r < 1$.

McLeavey [8] (see also [9]) first considered the subclass of $\Sigma$ with $K(|z|)$-quasiconformal extensions into the interior of $\mathbf{D}$ where $K(|z|)$ is a piecewise continuous function of bounded variation on $[r, 1]$, $0 \le r < 1$. She obtained for this class the analogs of the classical Grunsky and Goluzin inequalities and sharp estimates for the coefficients $b_0$ and $b_1$ of $\Sigma(k,r)$ and $a_2$ of $\mathcal{S}_0(k,R)$ with extremal function as follow;

**Theorem 3** ([8]). *If $g \in \Sigma(k,r)$ and $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ for $|z| > 1$, then*

$$|b_1| \le \frac{k+r^2}{1+kr^2}.$$

*Equality occurs if and only if*

$$g(z) = \begin{cases} z + b_0 + \left(\dfrac{k+r^2}{1+kr^2}\right)\dfrac{e^{i\alpha}}{z} & |z| \geq 1 \\ \left(\dfrac{1}{1+kr^2}\right)\left(z + \dfrac{r^2 e^{i\alpha}}{z} + ke^{i\alpha}\bar{z} + \dfrac{kr^2}{\bar{z}}\right) & r < |z| \leq 1. \end{cases}$$

**Corollary 4** ([8]). *Suppose* $f \in \mathcal{S}(k, R)$ *and* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *for* $z \in \mathbf{D}$. *Then*

$$|a_3 - a_2^2| \leq \frac{1+kR^2}{k+R^2}.$$

*If, in addition, extend mappings do not take* $\infty$ *on* $\{|z| < R\}$, *then*

$$|a_2| \leq 2\frac{1+kR}{k+R}. \tag{4}$$

Kühnau [6] also proved similar results of those through introducing the class $\Sigma(Q_1, \cdots, Q_n)$ of $K(|z|)$-quasiconformal mapping of the plane which are conformal on $\{z; |z| > 1\}$ with a development $f(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ and which have piecewise bounded dilatation in $\mathbf{D}$ ; $K(|z|) \leq Q_i$ $(Q_i \geq 1)$ in $R_i < |z| < R_{i-1}$ $(i = 1, \cdots, n)$, with $R_0 = 1$, $R_n = 0$. Schober [9] mentioned above results in his book, Chap.14. He also gave some more results, for instance, generalized Gronwall's area theorem for $\Sigma(k, r)$;

**Theorem 5** ([9]). *If* $g \in \Sigma(k, r)$ *and* $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ *for* $|z| > 1$, *then*

$$\sum_{m=1}^{\infty} m|b_m|^2 \leq \left(\frac{k+r^2}{1+kr^2}\right)^2. \tag{5}$$

Under the more general case, Lehto [7] showed a majorant principle for a holomorphic functional as follow;

Let $A$ be a domain in $\widehat{\mathbf{C}}$ which is bounded by a quasicircle, $B$ be a domain whose closure $\overline{B} \subset A$, and $\mathcal{F}_k$, $0 \leq k < 1$, be a family of functions which are $k$-quasiconformal on $A$ and conformal on $\overline{B}$. Denote by $\mathcal{F}_1$ the family of all conformal mappings on $B$.

We introduce four different normalizations to cover a large number of cases appearing in applications. Let $z_1, z_2, z_3$ be distinct points of $B$ and $\alpha_1, \alpha_2, \alpha_3, \beta$ are complex numbers, the

$\alpha$'s are different from each other and $\beta \neq 0$. The families $\mathcal{F}_k$ and $\mathcal{F}_1$ are called normalized if all the functions $f$ of $A$ contained in $\mathcal{F}_k$ or $\mathcal{F}_1$ have one of the following conditions;

1. $f(z_i) = \alpha_i$, $i = 1, 2, 3$,
2. $f(z_i) = \alpha_i$, $i = 1, 2$, and $f(z) \neq \infty$ in $A$,
3. $f(z_1) = \alpha_1$, $f'(z_1) = \beta$ and $f(z) \neq \infty$ in $A$,
4. If $\infty \in B$, then $f(z) - z \to 0$ as $z \to \infty$.

We shall suppose here $\mathcal{F}_k$ and $\mathcal{F}_1$ are normalized. Remark that normalized $\mathcal{F}_k$ and $\mathcal{F}_1$ are closed normal families.

Let $\Psi$ be a holomorphic functional defined on the family $F_k$ or $F_1$, i.e. $\Psi(f) = \omega(f(z_0), f'(z_1), \cdots, f^{(n)}(z_n))$, where $\omega$ is a complex-valued holomorphic function of the variables $f^{(i)}(z_i), i = 1, 2, \cdots$, each $f^{(i)}(z_i)$ being the value at fixed point $z_i \in B$.

Set

$$M(k) = \sup_{f \in \mathcal{F}_k} |\Psi(f)|, \qquad 0 \leq k \leq 1.$$

Since $\mathcal{F}_k$ is a closed normal family, there exists an extremal function maximizing $|\Psi(f)|$ in $\mathcal{F}_k$.

**Theorem 6** ([7]). *For a holomorphic functional in $\mathcal{F}_k$,*

$$M(k) \leq M(1) \frac{k + \frac{M(1)}{M(0)}}{1 + k\frac{M(1)}{M(0)}}. \tag{6}$$

This result contains some coefficient estimates as corollaries; for the class $\Sigma(k, r)$

$$\max_{\Sigma(k,r)} |b_n| \leq \frac{k + r^{n+1}}{1 + kr^{n+1}} \max_{\Sigma} |b_n|, \qquad n = 1, 2, \cdots,$$

which imply

$$|b_1| \leq \frac{k + r^2}{1 + kr^2} \qquad \text{and} \qquad |b_2| \leq \frac{2}{3} \frac{k + r^3}{1 + kr^3}.$$

The Grunsky type inequalities for $\Sigma(k, r)$ also follow easily from the general inequality (6). For $f \in \Sigma$, let $A_{mn}$, $m, n = 1, 2, \cdots$, be the numbers determined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} A_{mn} z^{-m} \zeta^{-n}.$$

Then for any complex numbers $x_1, x_2, \cdots, x_N$,

$$\left|\sum_{m,n}^{N} A_{mn}x_m x_n\right| \le \frac{k+r^2}{1+kr^2}\sum_{n}^{N}\frac{|x_n|^2}{n} \tag{7}$$

and

$$\sum_{n}^{N} n\left|\sum_{m}^{N} A_{mn}x_m\right|^2 \le \left(\frac{k+r^2}{1+kr^2}\right)^2\sum_{n=1}^{N}\frac{|x_2|^2}{n}. \tag{8}$$

Remark that (7) and (8) is not sharp (see [8]).

Deiermann treats several similar problems of those in [2] and [3] with the method of extremal length. Recently, Krushkal gives a short mention for those reserches in his survey [4], Chap.6.3.

# 3 Main Results

Now the more applications of Theorem 6 are given to $\mathcal{S}_0(k,R)$ and $\Sigma(k,r)$ (again remark that these results are not sharp because (7) and (8) is not sharp) ;

**Theorem 7.**

$$\sup_{\mathcal{S}_0(k,R)} |a_n| \le n\frac{1+kR^{n-1}}{k+R^{n-1}}.$$

***Proof.*** Let us take $\mathcal{F}_k = \mathcal{S}_0(k,R)$, then $\mathcal{F}_1$ is the well-known class $\mathcal{S}$. Choose $\Psi(f) = a_n$. Then $M(1) = n$, and $M(0) = n/R^{n-1}$ because $Rf(z/R) \in \mathcal{S}$ for arbitrary $f \in \mathcal{F}_0$. Hence the inequality (6) follow the theorem. □

**Theorem 8** (Generalized Goluzin inequality). *If* $g \in \Sigma(k,r)$ *and* $z_\nu \in \widehat{\mathbf{C}} - \mathbf{D}$, $\gamma_\nu \in \mathbf{C}$ $(\nu = 1,2,\cdots,n)$, $n = 1,2,\cdots$, *then*

$$\left|\sum_{\mu}\sum_{\nu}\gamma_\mu\gamma_\nu \log\frac{g(z_\mu)-g(z_\nu)}{z_\mu - z_\nu}\right| \le \frac{k+r^2}{1+kr^2}\sum_{\mu}\sum_{\nu}\gamma_\mu\overline{\gamma_\nu}\log\frac{1}{1-(z_\mu\overline{z_\nu})^{-1}}. \tag{9}$$

***Proof.*** We shall apply the inequality (7) with $x_m = \sum_{\nu=1}^{N}\gamma_\nu z_\nu^{-m}$, $m = 1,2,\cdots$. In fact, we have

$$\sum_{\mu}\sum_{\nu}\gamma_\mu\gamma_\nu \log \frac{g(z_\mu)-g(z_\nu)}{z_\mu - z_\nu} = -\sum_{m,n}\sum_{\mu,\nu} A_{mn}\gamma_\mu\gamma_\nu z_\mu^{-m} z_\nu^{-n}$$

$$= -\sum_{m,n} A_{mn}x_m x_n.$$

Hence (7) shows that the left-hand side of (9) is

$$\leq \frac{1+kr^2}{k+r^2}\sum_n \frac{1}{n}|x_n|^2 = \frac{1+kr^2}{k+r^2}\sum_n \frac{1}{n}\sum_{\mu,\nu}\gamma_\mu\overline{\gamma_\nu} z_\mu^{-k}\overline{z_\nu}^{-k}$$

$$= \frac{1+kr^2}{k+r^2}\sum_{\mu,\nu}\gamma_\mu\overline{\gamma_\nu}\log\frac{1}{1-(z_\mu\overline{z_\nu})^{-1}}.$$

This completes the proof of the theorem. □

**Theorem 9.** *For $f \in \mathcal{S}_0(k,R)$ and $z \in \mathbf{D}$,*

$$\left|\log\frac{zf'(z)}{f(z)}\right| \leq \frac{1+kR}{k+R}\log\frac{1+|z|}{1-|z|}.$$

***Proof.*** In (9) let $n = 2$, $\gamma_1 = 1$, $\gamma_2 = -1$, then

$$\left|\log\frac{g'(z)g'(\zeta)(z-\zeta)^2}{(g(z)-g(\zeta))^2}\right| \leq \frac{k+r^2}{1+kr^2}\log\frac{|z\overline{\zeta}-1|^2}{(|z|^2-1)(|\zeta|^2-1)} \quad (z,\zeta \in \widehat{\mathbb{C}} - \mathbf{D}). \tag{10}$$

We want to appply (9) to the function $f \in \mathcal{S}_0(k,R)$. If we put

$$g(\zeta) = 1/\sqrt{f(\zeta^{-2})}, \tag{11}$$

then $g \in \Sigma(k, 1/\sqrt{R})$. Since $g$ is odd function, it follows from (10) with $z = -\zeta$ that

$$\left|2\log\frac{\zeta g'(\zeta)}{g(\zeta)}\right| \leq \frac{k+(1/R)}{1+k(1/R)}2\log\frac{|\zeta|^2+1}{|\zeta|^2-1} \quad (|\zeta|>1).$$

If we choose $z = \zeta^{-2}$ and use (11) we obtain the desire inequality. □

**Corollary 10.** *For $f \in \mathcal{S}_0(k,R)$ and $z \in \mathbf{D}$,*

$$\left(\frac{1-|z|}{1+|z|}\right)^{(1+kR)/(k+R)} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \left(\frac{1+|z|}{1-|z|}\right)^{(1+kR)/(k+R)}.$$

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