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On Nunokawa's Lemma

Mamoru Nunokawa

Abstract

I. S. Jack [1] proved the following lemma :
Let $w(z)$ be regular in the unit disc, $w(0)=0$. Then if $w(z)$ attains its maximum value on the circle $|z|=r$ at a point z_1, then we can write

$$z_1w'(z_1)=kw(z_1)$$

where k is real and $1\leqq k$. Many and many mathematicians applied the above lemma and obtained numerous interesting results. In this paper, we will obtain a lemma which may be connected intimately to the Jack's lemma.

1 Basic geometrical property

Property 1 *Let $\varphi(z)$ be analytic in $|z|<1$, $\varphi(z)\neq 0$ in $|z|<1$ and suppose that*

$$\min_{|z|\leqq r}|\varphi(z)|=|\varphi(z_0)|$$

and

$$\max_{|z|\leqq r}|\varphi(z)|=|\varphi(z_1)|$$

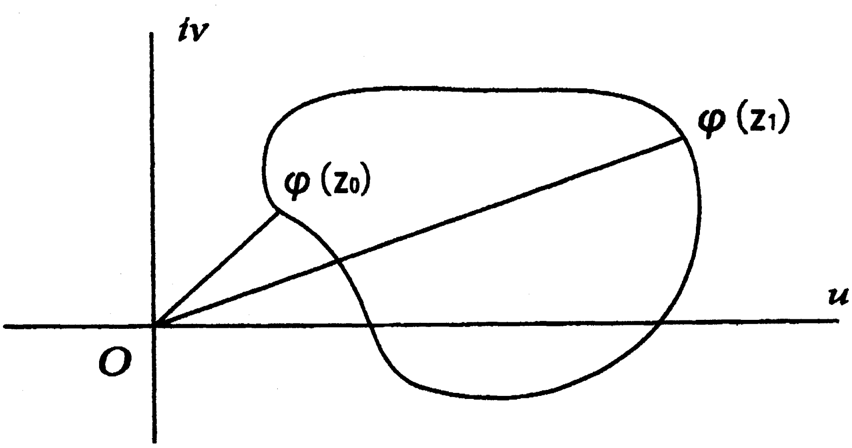
where $0<r<1$ and $|z_0|=|z_1|=r$. Then we have

$$\frac{z_0\varphi'(z_0)}{\varphi(z_0)}=\mathrm{Re}\frac{z_0\varphi'(z_0)}{\varphi(z_0)}=\left(\frac{d\arg\ \varphi(z)}{d\theta}\right)_{\theta=\theta_0}<0$$

and

$$\frac{z_1\varphi'(z_1)}{\varphi(z_1)}=\mathrm{Re}\frac{z_1\varphi'(z_1)}{\varphi(z_1)}=\left(\frac{d\arg\ \varphi(z)}{d\theta}\right)_{\theta=\theta_1}>0$$

where $z=re^{i\theta}$, $0\leqq\theta<2\pi$, $z_0=re^{i\theta_0}$ and $z_1=re^{i\theta_1}$.



A proof of Property 1 is trivial by considering geometrical property.

2 Nunokawa's lemma

Lemma 1 *Let $\varphi(z)$ be analytic in $|z| < 1$, $1 < \varphi(0)$ and suppose that there exists a point z_0, $|z_0| < 1$ such that*

$$1 < |\varphi(z)| \quad \text{for } |z| < |z_0|$$

$$1 = |\varphi(z_0)| \quad \text{and} \quad \varphi(z_0) \neq -1$$

or

$$\min_{|z| \leqq |z_0|} |\varphi(z)| = |\varphi(z_0)| = 1 \neq -\varphi(z_0).$$

Then we have

$$\frac{z_0\varphi'(z_0)}{\varphi(z_0)} = \mathrm{Re}\frac{z_0\varphi'(z_0)}{\varphi(z_0)} \leqq -\frac{\varphi(0)-1}{\varphi(0)+1}.$$

Proof. Let us put

(1)
$$\varphi(z) = \frac{1+p(z)}{1-p(z)} \quad \text{for } |z| < |z_0|.$$

Then it follows that

$$p(z) = \frac{\varphi(z)-1}{\varphi(z)+1}$$

and

$$0 < p(0) = \frac{\varphi(0)-1}{\varphi(0)+1} = \mathrm{Re}\frac{\varphi(0)-1}{\varphi(0)+1} < 1.$$

From the hypothesis of the Lemma 1 and (1), we have

$$0 < \mathrm{Re}\, p(z) \quad \text{for } |z| < |z_0|$$

and

$$0 = \mathrm{Re}\, p(z_0).$$

Putting

$$\Phi(z) = \frac{p(0)-p(z)}{p(0)+p(z)}, \quad \Phi(0) = 0$$

and applying the same method as the proof of [2], we have

$$|\Phi(z)| < 1 \quad \text{for } |z| < |z_0|$$

and

$$|\Phi(z_0)| = 1$$

and therefore

$$\frac{z_0\Phi'(z_0)}{\Phi(z_0)} = \frac{-2p(0)z_0p'(z_0)}{p(0)^2 - p(z_0)^2} = \frac{-2p(0)z_0p'(z_0)}{p(0)^2 + |p(z_0)|^2} \geqq 1.$$

It shows that

$$-z_0p'(z_0) \geqq \frac{1}{2}\left(\frac{p(0)^2 + |p(z_0)|^2}{p(0)}\right).$$

From (1) and Lemma 1, we have

$$
\begin{aligned}
\frac{z_0\varphi'(z_0)}{\varphi(z_0)} &= \mathrm{Re}\frac{z_0\varphi'(z_0)}{\varphi(z_0)} \\
&= \frac{2z_0p'(z_0)}{1-p(z_0)^2} \\
&= \frac{2z_0p'(z_0)}{1+|p(z_0)|^2} \\
&\leqq -\frac{1}{p(0)}\left(\frac{p(0)^2+|p(z_0)|^2}{1+|p(z_0)|^2}\right) \\
&= -\frac{p(0)}{p(0)^2}\left(\frac{p(0)^2+|p(z_0)|^2}{1+|p(z_0)|^2}\right) \\
&= -p(0)\left(\frac{p(0)^2+|p(z_0)|^2}{p(0)^2+p(0)^2|p(z_0)|^2}\right) \\
&< -p(0) \\
&= -\left(\frac{\varphi(0)-1}{\varphi(0)+1}\right).
\end{aligned}
$$

It completes the proof. □

Applications of this lemma will be obtained by me and my friends.

References

[1] I. S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc. (2), **3** (1971), 469-474.

[2] M. Nunokawa, *On properties of Non-Carathéodory functions*, Proc. Japan Acad. Vol. 68, Ser. A, No.6 (1992), 152-153.

Mamoru Nunokawa
Emeritus Professor of University of Gunma
Hoshikuki-cho 798-8, Chuou-Ward,
Chiba city 260-0808
Japan
e-mail: mamoru_nuno@doctor.nifty.jp