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On the strongly starlikeness of starlike functions

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Abstract

It is the purpose of the present paper to prove that if $f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$ is analytic in $|z|<1$ and if

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}>\frac{1+\alpha}{2}\quad \text{in } |z|<1$$

where $0\leqq\alpha<1$.
Then $f(z)$ is strongly starlike of order $(1-\alpha)$ or

$$\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi}{2}(1-\alpha)\quad \text{in } |z|<1.$$

1 Introduction

Let $f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$ be convex in $\mathbb{E}=\{z\mid |z|<1\}$, that is, $f(z)$ is analytic in $\mathbb{E}$ and maps $\mathbb{E}$ univalently onto a convex domain. It is well known that the necessary and sufficient condition for the convexity of $f(z)$ is

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}>0\quad \text{in } \mathbb{E}.$$

On the other hand, let $f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$ be starlike in $\mathbb{E}$, that is, $f(z)$ is analytic in $\mathbb{E}$ and maps $\mathbb{E}$ univalently onto a starlike domain which is starshaped with respect to the origin. It is well known that the necessary and sufficient condition for the starlikeness of $f(z)$ is

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0\quad \text{in } \mathbb{E}.$$

Sheil-Small [2] proved the following theorem.

Theorem A Let $f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$ be starlike in $\mathbb{E}$, let $C(r,\theta)=\{f(te^{i\theta})\mid 0\leqq t\leqq r\}$ and let $T(r,\theta)$ be the total variation of $\arg f(te^{i\theta})$ on $C(r,\theta)$, so that

$$T(r,\theta)=\int_{0}^{r}\left|\frac{\partial}{\partial t}\arg f(te^{i\theta})\right|dt.$$

Then we have

$$T(r,\theta)<\pi.$$

2 Theorem

Theorem 1 Let $f(z)$ be convex in $\mathbb{E}$ and suppose that

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)} > \frac{1+\alpha}{2} \quad \text{in } \mathbb{E},$$

where $0 \leqq \alpha < 1$.
Then $f(z)$ is starlike in $\mathbb{E}$ and strongly starlike of order $(1-\alpha)$ or

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}(1-\alpha) \quad \text{in } \mathbb{E}.$$

Proof. Let us put

(1)
$$1+\frac{zf''(z)}{f'(z)} = \left(\frac{1+\alpha}{2}\right) + \left(1-\frac{1+\alpha}{2}\right)\frac{zg'(z)}{g(z)}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic in $\mathbb{E}$.
Then, from the hypothesis of the theorem, $g(z)$ is starlike in $\mathbb{E}$. From (1), we have

(2)
$$\frac{f''(z)}{f'(z)} = \left(\frac{1-\alpha}{2}\right)\left(\frac{g'(z)}{g(z)} - \frac{1}{z}\right)$$

and integrating (2) along the straight line from the origin to z, then it follows that

$$\int_0^z \frac{f''(\zeta)}{f'(\zeta)}d\zeta = \int_0^z \left(\frac{1-\alpha}{2}\right)\left(\frac{g'(\zeta)}{g(\zeta)} - \frac{1}{\zeta}\right)d\zeta$$

and so on

$$\log f'(z) = \left(\frac{1-\alpha}{2}\right)\log\frac{g(z)}{z}$$

or

$$f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1-\alpha}{2}}.$$

Then we have

(3)
$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z\left(\frac{g(z)}{z}\right)^{\frac{1-\alpha}{2}}}{\int_0^z \left(\frac{g(\zeta)}{\zeta}\right)^{\frac{1-\alpha}{2}} d\zeta} \\ &= \frac{1}{\int_0^z \left(\frac{z}{\zeta}\right)^{\frac{1-\alpha}{2}}\left(\frac{g(\zeta)}{g(z)}\right)^{\frac{1-\alpha}{2}}\frac{d\zeta}{z}}\end{aligned}$$

where $z = re^{i\theta}$, $\zeta = te^{i\theta}$ and $0 \leqq t \leqq r$.
From (3), it follows that

(4)
$$\frac{zf'(z)}{f(z)} = \left(\int_0^1 t^{\frac{\alpha-1}{2}}\left(\frac{g(tz)}{g(z)}\right)^{\frac{1-\alpha}{2}} dt\right)^{-1}$$

Then, from Theorem A, we have

$$-\pi < \arg\left(\frac{g(tz)}{g(z)}\right) < \pi \tag{5}$$

where $0 \leqq t \leqq 1$.
Putting

$$s = t^{\frac{\alpha-1}{2}}\left(\frac{g(tz)}{g(z)}\right)^{\frac{1-\alpha}{2}},$$

then we have

$$\arg s = -\left(\frac{1-\alpha}{2}\right)\arg\left(\frac{g(tz)}{g(z)}\right) \tag{6}$$

and from (4), we have

$$\arg\frac{zf'(z)}{f(z)} = -\arg\left(\int_0^1 s\,dt\right). \tag{7}$$

Then form (5) and (6), we have

$$|\arg s| < \frac{\pi}{2}(1-\alpha) \quad \text{in } \mathbb{E}. \tag{8}$$

From the property of integral mean (see e.g. [1, Lemma 1]) and from (8), we have

$$\left|\arg\left(\int_0^1 s\,dt\right)\right| < \frac{\pi}{2}(1-\alpha) \quad \text{in } \mathbb{E}. \tag{9}$$

Then, form (7) and (9), we have

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}(1-\alpha) \quad \text{in } \mathbb{E}.$$

This completes the proof of Theorem 1. □

References

[1] Ch. Pommerenke, *On close-to-convex functions*, Trans. Amer. Math. Soc., **114** (1965), 176-186.

[2] T. Sheil-Small, *Some conformal mapping inequalities for starlike and convex functions*, J. London Math. Soc., **1** (1969), 577-587.

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