

N- Fractional Calculus and $n(\in \mathbb{Z}^+)$ th Derivatives of Some Logarithmic Functions

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Abstract

In this article N- fractional calculus and n -th derivatives of logarithmic functions

$$\log((z-b)^m - c), \quad (m \in \mathbb{Z}_0^+) \quad \text{and} \quad \log(z^2 + 2az + d)$$

are reported.

That is, we have the following, for example.

$$(i) \quad (\log((z-b)^m - c))_\gamma = -e^{-i\pi\gamma} m(z-b)^{-\gamma} \Gamma(\gamma)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(mk + \gamma)}{\Gamma(\gamma)\Gamma(mk + 1)} \left(\frac{c}{(z-b)^m} \right)^k$$

$$(|\Gamma(\gamma)|, |\Gamma(mk + \gamma)| < \infty),$$

and

$$(ii) \quad (\log((z-b)^m - c))_n = (-1)^{n+1} m(z-b)^{-n} \Gamma(n)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(mk + n)}{\Gamma(n)\Gamma(mk + 1)} \left(\frac{c}{(z-b)^m} \right)^k$$

$$(n \in \mathbb{Z}^+),$$

where

$$(z-b)^m - c \neq 0, 1, \quad \text{and} \quad |c/(z-b)^m| < 1.$$

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi - z) \leq \pi$ for C_- , $0 \leq \arg(\xi - z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in \mathbb{C}$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{d\xi}{(\xi - z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

with
$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation \circ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{ N^\nu \} = \{ N^\nu | \nu \in \mathbb{R} \} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the

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function f such that $f \in F = \{f; 0 \neq |f_v| < \infty, v \in \mathbb{R}\}$, where $f = f(z)$ and $z \in \mathbb{C}$.
(vis. $-\infty < v < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group) [3]

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) **Lemma.** We have [1]

$$(i) \quad ((z-c)^b)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-b)}{\Gamma(-b)} (z-c)^{b-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-b)}{\Gamma(-b)} \right| < \infty \right),$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ for (i) and $z-c \neq 0, 1$ for (ii), (iii),

§ 1. Preliminary

Theorem D. below for the fractional calculus of a logarithmic function is reported by K. Nishimoto (cf. J. Frac. Calc. Vol. 29, May (2006), p. 40.).

Theorem D. We have

$$(i) \quad (\log((z-b)^\beta - c))_\gamma = -e^{-i\pi\gamma} \beta (z-b)^{-\gamma} \Gamma(\gamma)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + \gamma)}{\Gamma(\gamma) \Gamma(\beta k + 1)} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (|\Gamma(\gamma)|, \left| \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k + 1)} \right| < \infty), \quad (1)$$

and

$$(ii) \quad (\log((z-b)^\beta - c))_m = (-1)^{m+1} \beta (z-b)^{-m} \Gamma(m)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + m)}{\Gamma(m) \Gamma(\beta k + 1)} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (m \in \mathbb{Z}^+), \quad (2)$$

where

$$(z-b)^\beta - c \neq 0, 1, \quad \text{and} \quad |c/(z-b)^\beta| < 1.$$

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§ 2. N-Fractional Calculus of Functions

$$\log((z-b)^m - c)$$

Theorem 1. *We have*

$$(i) \quad (\log((z-b)^m - c))_\gamma = -e^{-i\pi\gamma} m(z-b)^{-\gamma} \Gamma(\gamma) \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(mk + \gamma)}{\Gamma(\gamma)\Gamma(mk + 1)} \left(\frac{c}{(z-b)^m} \right)^k \quad (1) \\ (|\Gamma(\gamma)|, |\Gamma(mk + \gamma)| < \infty),$$

and

$$(ii) \quad (\log((z-b)^m - c))_n = (-1)^{n+1} m(z-b)^{-n} \Gamma(n) \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(mk + n)}{\Gamma(n)\Gamma(mk + 1)} \left(\frac{c}{(z-b)^m} \right)^k \quad (n \in \mathbf{Z}^+), \quad (2)$$

where

$$m \in \mathbf{Z}_0^+, \quad (z-b)^m - c \neq 0, 1, \quad \text{and} \quad |c/(z-b)^m| < 1.$$

Proof of (i). Set $\beta = m$ in Theorem D. (i) in Preliminary we have (1) clearly, under the conditions stated before.

Proof of (ii). Set $\gamma = n$ in (1), we have then (2).

Corollary 1. *We have*

$$(i) \quad (\log((z-b)^2 - c))_\gamma = -e^{-i\pi\gamma} 2(z-b)^{-\gamma} \Gamma(\gamma) \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(2k + \gamma)}{\Gamma(\gamma)\Gamma(2k + 1)} \left(\frac{c}{(z-b)^2} \right)^k \quad (3) \\ (|\Gamma(\gamma)|, |\Gamma(2k + \gamma)| < \infty),$$

and

$$(ii) \quad (\log((z-b)^2 - c))_n = (-1)^{n+1} 2(z-b)^{-n} \Gamma(n) \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(2k + n)}{\Gamma(n)\Gamma(2k + 1)} \left(\frac{c}{(z-b)^2} \right)^k \quad (n \in \mathbf{Z}^+), \quad (4)$$

where

$$(z-b)^2 - c \neq 0, 1, \quad \text{and} \quad |c/(z-b)^2| < 1.$$

Proof. Set $m = 2$ in Theorem 1.

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Theorem 2. We have the following identity.

$$(i) \quad 2 \sum_{k=0}^{\infty} \frac{\Gamma(2mk + \gamma)}{\Gamma(\gamma)\Gamma(2mk + 1)} T^k = \sum_{k=0}^{\infty} \frac{\Gamma(mk + \gamma)}{\Gamma(\gamma)\Gamma(mk + 1)} \{(T^{1/2})^k + (-T^{1/2})^k\} \quad (5)$$

$$(|\Gamma(mk + \gamma)| < \infty),$$

and

$$(ii) \quad 2 \sum_{k=0}^{\infty} \frac{[2mk + 1]_{n-1}}{\Gamma(n)} T^k = \sum_{k=0}^{\infty} \frac{[mk + 1]_{n-1}}{\Gamma(n)} \{(T^{1/2})^k + (-T^{1/2})^k\} \quad (6)$$

$$(n \in \mathbb{Z}^+),$$

where

$$T = \frac{c}{(z-b)^{2m}}, \quad |T| < 1, \quad m \in \mathbb{Z}^+, \quad (7)$$

$$[\lambda]_k = \lambda(\lambda + 1)\cdots(\lambda + k - 1) = \Gamma(\lambda + k)/\Gamma(\lambda) \text{ with } [\lambda]_0 = 1.$$

(Notation of Pochhammer).

Proof of (i). We have

$$\log((z-b)^{2m} - c) = \log((z-b)^m - \sqrt{c}) + \log((z-b)^m + \sqrt{c}). \quad (8)$$

$$((z-b)^m \pm \sqrt{c} \neq 0, 1).$$

Operate N^γ to the both sides of (8), we have then

$$(\log((z-b)^{2m} - c))_\gamma = (\log((z-b)^m - \sqrt{c}))_\gamma + (\log((z-b)^m + \sqrt{c}))_\gamma. \quad (9)$$

Now we have

$$(\log((z-b)^{2m} - c))_\gamma = -e^{-i\pi\gamma} 2m(z-b)^{-\gamma} \Gamma(\gamma) \sum_{k=0}^{\infty} \frac{\Gamma(2mk + \gamma)}{\Gamma(\gamma)\Gamma(2mk + 1)} T^k \quad (10)$$

$$(|\Gamma(2mk + \gamma)| < \infty),$$

and

$$(\log((z-b)^m - \sqrt{c}))_\gamma = -e^{-i\pi\gamma} m(z-b)^{-\gamma} \Gamma(\gamma) \sum_{k=0}^{\infty} \frac{\Gamma(mk + \gamma)}{\Gamma(\gamma)\Gamma(mk + 1)} (T^{1/2})^k \quad (11)$$

$$(|\Gamma(mk + \gamma)| < \infty),$$

by Theorem D. (i), respectively.

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Then we obtain (i) from (9), applying (10) and (11), under the conditions.

Proof of (ii). Set $\gamma = n$ in (i).

Corollary 2. We have the following identity.

$$(i) \quad 2(z-b)^{-\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(2k+\gamma)}{\Gamma(\gamma)\Gamma(2k+1)} T^k = \frac{1}{(z-b-\sqrt{c})^\gamma} + \frac{1}{(z-b+\sqrt{c})^\gamma} \quad (12)$$

$$(\ |\Gamma(2k+\gamma)| < \infty),$$

and

$$(ii) \quad 2(z-b)^{-n} \sum_{k=0}^{\infty} \frac{[2k+1]_{n-1}}{\Gamma(n)} T^k = \frac{1}{(z-b-\sqrt{c})^n} + \frac{1}{(z-b+\sqrt{c})^n} \quad (13)$$

$$(n \in \mathbb{Z}^+).$$

Proof of (i). Set $m = 1$ in Theorem 2. (i), we have then

$$2 \sum_{k=0}^{\infty} \frac{\Gamma(2k+\gamma)}{\Gamma(\gamma)\Gamma(2k+1)} T^k = \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)\Gamma(k+1)} \{(T^{1/2})^k + (-T^{1/2})^k\}. \quad (14)$$

Now we have

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)\Gamma(k+1)} (T^{1/2})^k = \sum_{k=0}^{\infty} \frac{[\gamma]_k}{k!} (T^{1/2})^k \quad (15)$$

$$= \frac{(z-b)^\gamma}{(z-b-\sqrt{c})^\gamma}. \quad (16)$$

Therefore, we obtain (12) from (14) and (16).

Proof of (ii). Set $\gamma = n$ in (i).

Note. Other proof of (i). We have

$$(\log((z-b)^2 - c))_\gamma = (\log((z-b) - \sqrt{c}))_\gamma + (\log((z-b) + \sqrt{c}))_\gamma, \quad (17)$$

from (9), setting $m = 1$.

Next we have

$$(\log((z-b) - \sqrt{c}))_\gamma = -e^{-i\pi\gamma} \Gamma(\gamma) (z-b-\sqrt{c})^{-\gamma}, \quad (|\Gamma(\gamma)| < \infty) \quad (18)$$

by Lemma (ii).

Therefore, we obtain (12) from (17), applying (3) and (18).

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Theorem 3. We have

$$(i) \quad (\log(z^2 + 2az + d))_\gamma = -e^{-i\pi\gamma} 2(z+a)^{-\gamma} \Gamma(\gamma) \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(2k+\gamma)}{\Gamma(\gamma)\Gamma(2k+1)} \left(\frac{a^2-d}{(z+a)^2} \right)^k \quad (19)$$

$$(|\Gamma(\gamma)|, |\Gamma(2k+\gamma)| < \infty),$$

and

$$(ii) \quad (\log(z^2 + 2az + d))_n = (-1)^{n+1} 2(z+a)^{-n} \Gamma(n) \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(2k+n)}{\Gamma(n)\Gamma(2k+1)} \left(\frac{a^2-d}{(z+a)^2} \right)^k \quad (n \in \mathbb{Z}^+), \quad (20)$$

where

$$z^2 + 2az + d \neq 0, 1, \quad \text{and} \quad |(a^2-d)/(z+a)^2| < 1.$$

Proof of (i). We have

$$z^2 + 2az + d = (z+a)^2 - c, \quad (c = a^2 - d). \quad (21)$$

hence

$$\log(z^2 + 2az + d) = \log((z+a)^2 - c). \quad (22)$$

Operate N^γ to the both sides of (22), we have then

$$(\log(z^2 + 2az + d))_\gamma = (\log((z+a)^2 - c))_\gamma, \quad (23)$$

therefore, we obtain (19) clearly, setting $b = -a$ and $c = a^2 - d$ in Corollary 1.

(i), under the conditions stated before.

Proof of (ii). Set $\gamma = n$ in (19).

§ 3. Semi Derivatives and Integrals

Corollary 3. We have

$$(i) \quad (\log((z-b)^m - c))_{1/2} = im(z-b)^{-1/2} \sqrt{\pi} \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(mk + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(mk+1)} \left(\frac{c}{(z-b)^m} \right)^k \quad (1)$$

$$(\text{semi derivatives}),$$

and

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$$(ii) \quad (\log((z-b)^m - c))_{-1/2} = im(z-b)^{1/2} 2\sqrt{\pi} \times \sum_{k=0}^{\infty} \frac{\Gamma(mk - \frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(mk+1)} \left(\frac{c}{(z-b)^m} \right)^k \quad (2)$$

(semi integrals),

where

$$m \in \mathbf{Z}_0^+, \quad (z-b)^m - c \neq 0, 1, \quad \text{and} \quad |c/(z-b)^m| < 1.$$

Proof. Set $\gamma = 1/2$ and $-1/2$ in Theorem 1. (i), we have then (1) and (2) respectively

Corollary 4. We have

$$(i) \quad (\log((z-b)^2 - c))_{1/2} = i2\sqrt{\pi}(z-b)^{-1/2} \sum_{k=0}^{\infty} \frac{\Gamma(2k + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2k+1)} \left(\frac{c}{(z-b)^2} \right)^k \quad (3)$$

(semi derivatives),

and

$$(ii) \quad (\log((z-b)^2 - c))_{-1/2} = i4\sqrt{\pi}(z-b)^{1/2} \sum_{k=0}^{\infty} \frac{\Gamma(2k - \frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(2k+1)} \left(\frac{c}{(z-b)^2} \right)^k \quad (4)$$

(semi integrals),

where

$$(z-b)^2 - c \neq 0, 1, \quad \text{and} \quad |c/(z-b)^2| < 1.$$

Proof. Set $m = 2$ in Corollary 3.

Corollary 5. We have

$$(i) \quad (\log((z-b)^2 - c))_{1/2} = i\sqrt{\pi} \left(\frac{1}{\sqrt{z-b-\sqrt{c}}} + \frac{1}{\sqrt{z-b+\sqrt{c}}} \right) \quad (5)$$

(semi derivatives)

and

$$(i) \quad (\log((z-b)^2 - c))_{-1/2} = i2\sqrt{\pi} \left(\sqrt{z-b-\sqrt{c}} + \sqrt{z-b+\sqrt{c}} \right) \quad (6)$$

(semi integrals)

where

$$(z-b)^2 - c \neq 0, 1.$$

Proof. Set $\gamma = 1/2$ and $-1/2$ in Corollary 2, we have then (5) and (6) respectively, under the conditions.

§ 4. Some Special Cases

[I] When $n = 1$, we have

$$(\log((z-b)^2 - c))_1 = 2(z-b)^{-1} \sum_{k=0}^{\infty} \left(\frac{c}{(z-b)^2} \right)^k \quad (1)$$

$$= 2(z-b)^{-1} \sum_{k=0}^{\infty} \frac{[1]_k}{k!} \left(\frac{c}{(z-b)^2} \right)^k = 2(z-b)^{-1} \left(1 - \frac{c}{(z-b)^2} \right)^{-1} \quad (2)$$

$$= 2((z-b)^2 - c)^{-1} (z-b) \quad , \quad (3)$$

from Corollary 1 (ii).

[II] When $n = 2$, we have

$$(\log((z-b)^2 - c))_2 = -2(z-b)^{-2} \sum_{k=0}^{\infty} \frac{\Gamma(2k+2)}{\Gamma(2k+1)} \left(\frac{c}{(z-b)^2} \right)^k \quad (4)$$

$$= -2(z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1]_k (2k+1)}{k!} T^k \quad \left(T = \frac{c}{(z-b)^2} \right) \quad (5)$$

$$= -2(z-b)^{-2} \left\{ \sum_{k=0}^{\infty} \frac{[1]_k 2k}{k!} T^k + \sum_{k=0}^{\infty} \frac{[1]_k}{k!} T^k \right\} \quad (6)$$

$$= 2(z-b)^{-2} \{ 2T(1-T)^{-2} + (1-T)^{-1} \} \quad (7)$$

$$= -2((z-b)^2 + c) ((z-b)^2 - c)^{-2} \quad (8)$$

from Corollary 1 (ii).

This result coincides with the one obtained by the classical calculus ;

$$\frac{d^2}{dz^2} \log((z-b)^2 - c) .$$

Note.

$$\sum_{k=0}^{\infty} \frac{[1]_k 2k}{k!} T^k = 2 \sum_{k=1}^{\infty} \frac{[1]_k}{(k-1)!} T^k = 2T \sum_{k=0}^{\infty} \frac{[1]_{k+1}}{k!} T^k$$

$$= 2T \sum_{k=0}^{\infty} \frac{[2]_k}{k!} T^k = 2T(1-T)^{-2} .$$

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