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On geometric properties of certain multivalent functions with real coefficients

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Abstract

Let $\mathcal{T}(p)$ be the class of analytic functions with real coefficients in the open unit disk $\mathbb{U}$. For $f(z)$ belonging to the class $\mathcal{T}(p)$, some sufficient conditions for p-valently starlikeness and p-valently convexity are discussed.

1 Introduction

Let $\mathcal{A}(p)$ be the class of functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

We denote by $\mathcal{S}^*(p)$ and $\mathcal{K}(p)$ the subclasses of $\mathcal{A}(p)$ whose members map $\mathbb{U}$ onto domain which are p-valently starlike and p-valently convex.

A function $f(z) \in \mathcal{A}(p)$ is said to be p-valently starlike in $\mathbb{U}$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \mathbb{U}). \tag{1.2}$$

Similarly, $f(z) \in \mathcal{A}(p)$ is said to be p-valently convex in $\mathbb{U}$ if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathbb{U}). \tag{1.3}$$

Let us define $\mathcal{T}(p)$ the class of analytic functions with real coefficients, that is,

$$\mathcal{T}(p) = \left\{ f(z) \in \mathcal{A}(p) \,\middle|\, f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \ \ a_{n+p} \in \mathbb{R} \right\} \tag{1.4}$$

where $\mathbb{R}$ is the set of real numbers. Then it follows that $\mathcal{T}(p) \subset \mathcal{A}(p)$.
Furthermore, let us define $\mathcal{P}$ the class of analytic functions in $\mathbb{U}$, that is,

$$\mathcal{P} = \left\{ p(z) \,\middle|\, p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \ \ \mathrm{Re}\, p(z) > 0 \right\}. \tag{1.5}$$

$p(z) \in \mathcal{P}$ is called Caraéodory function.

2 Preliminaries

For our results, we prepare the next lemmas.

Lemma 1 (Nunokawa [3]) *Let $p(z) \in \mathcal{P}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\begin{aligned} &\operatorname{Re} p(z) > 0 \quad \textit{for} \quad |z| < |z_0| \\ &\operatorname{Re} p(z_0) = 0 \quad \textit{and} \quad p(z_0) \neq 0. \end{aligned} \tag{2.1}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik \tag{2.2}$$

where k is real and $|k| \geqq 1$.

Lemma 2 (Saitoh [5]) *Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in $\mathbb{U}$ and all coefficients p_i are real numbers.*
Suppose that

$$\operatorname{Re}\{p(z) + \alpha z p'(z)\} > 0 \quad \textit{in} \quad \mathbb{U} \tag{2.3}$$

where $\alpha \geqq 1$. Then we have

$$1 + \operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} > 0 \quad \textit{in} \quad \mathbb{U}. \tag{2.4}$$

Lemma 3 (Nunokawa [2]) *Let $f(z) \in \mathcal{A}(p)$ and suppose*

$$p + \operatorname{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)} > 0 \quad \textit{in} \quad \mathbb{U}. \tag{2.5}$$

Then $f(z)$ is p-valent in $\mathbb{U}$ and

$$k + \operatorname{Re}\frac{zf^{(k+1)}(z)}{f^{(k)}(z)} > 0 \quad \textit{in} \quad \mathbb{U}, \tag{2.6}$$

for $k = 0, 1, 2, \cdots, p-1$. This shows that $f(z) \in \mathcal{K}(p)$ and $f(z) \in \mathcal{S}^*(p)$.

Lemma 4 (Owa-Nunokawa [4]) *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. If*

$$\operatorname{Re}\{p(z) + \alpha z p'(z)\} > \beta \quad \textit{in} \quad \mathbb{U}, \tag{2.7}$$

then

$$\operatorname{Re}\{p(z)\} > \beta + (1-\beta)\left\{2\int_0^1 \frac{1}{1+\rho^{n\operatorname{Re}(\alpha)}} d\rho - 1\right\} \quad \textit{in} \quad \mathbb{U}, \tag{2.8}$$

where $\alpha \neq 0$, $\operatorname{Re}(\alpha) \geqq 0$ and $\beta < 1$.

3 Main results

First, we prove

Theorem 1 *Let $f(z) \in \mathcal{A}(p)$ and suppose that*

$$\operatorname{Re}\{f^{(p)}(z) + \alpha z f^{(p+1)}(z)\} > -\frac{p!}{2}\alpha \qquad (z \in \mathbb{U}) \tag{3.1}$$

for some α $(\alpha > 0)$. *Then we have*

$$\mathrm{Re}\{f^{(p)}(z)\} > 0 \qquad (z \in \mathbb{U}). \tag{3.2}$$

Proof. If there exists a point $z_0 \in \mathbb{U}$ such that

$$\mathrm{Re}\frac{f^{(p)}(z)}{p!} > 0 \quad \textit{for} \quad |z| < |z_0|$$

and

$$\mathrm{Re}\frac{f^{(p)}(z_0)}{p!} = 0 \quad \textit{and} \quad \frac{f^{(p)}(z_0)}{p!} \neq 0,$$

then from Lemma 1, we have

$$z_0 f^{(p+1)}(z_0) \leqq -\frac{p!}{2}\left(1 + \left|\frac{f^{(p)}(z_0)}{p!}\right|^2\right).$$

This contradicts the assumption (3.1) and completes the proof. □

Now, we prove

Theorem 2 *Let* $f(z) \in \mathcal{T}(p)$ *be analytic in* $\mathbb{U}$.
Suppose that

$$\mathrm{Re}\left\{\frac{(1 - \alpha p + \alpha j)f^{(j)}(z) + \alpha z f^{(j+1)}(z)}{z^{p-j}}\right\} > 0 \qquad (z \in \mathbb{U}) \tag{3.3}$$

where $\alpha \geqq 1$. *Then we have*

$$j + \mathrm{Re}\frac{zf^{(j+1)}(z)}{f^{(j)}(z)} > 0 \qquad (z \in \mathbb{U}) \tag{3.4}$$

for $j = 0, 1, 2, \cdots, p$.

Proof. Let $p(z) = \dfrac{(p-j)!f^{(j)}(z)}{p!z^{p-j}}$. Applying Lemma 2,

$$1 + \alpha\mathrm{Re}\frac{zf^{(j+1)}(z) - (p-j)f^{(j)}(z)}{f^{(j)}(z)} > 0 \qquad (z \in \mathbb{U}).$$

Therefore, we obtain

$$j + \mathrm{Re}\frac{zf^{(j+1)}(z)}{f^{(j)}(z)} > p - \frac{1}{\alpha} \geqq p - 1 > 0 \qquad (z \in \mathbb{U}).$$

□

Putting $j = 0$ in Theorem 2, we have

Corollary 1 *Let* $f(z) \in \mathcal{T}(p)$ *be analytic in* $\mathbb{U}$.
Suppose that

$$\mathrm{Re}\left\{\frac{(1 - \alpha p)f(z) + \alpha z f'(z)}{z^p}\right\} > 0 \qquad (z \in \mathbb{U}) \tag{3.5}$$

where $\alpha \geqq 1$. *Then we have*

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \qquad (z \in \mathbb{U}),$$

that is $f(z) \in \mathcal{S}^*(p)$.

Letting $j = 1$ in Theorem 2, we have

Corollary 2 *Let* $f(z) \in \mathcal{T}(p)$ *be analytic in* $\mathbb{U}$.
Suppose that

$$\mathrm{Re}\left\{\frac{(1-\alpha p+\alpha)f'(z)+\alpha zf''(z)}{z^{p-1}}\right\} > 0 \qquad (z \in \mathbb{U}) \tag{3.6}$$

where $\alpha \geqq 1$. *Then we have*

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)} > 0 \qquad (z \in \mathbb{U}),$$

that is $f(z) \in \mathcal{K}(p)$.

Next we prove

Theorem 3 *Let* $f(z) \in \mathcal{T}(p)$ *be analytic in* $\mathbb{U}$.
Suppose that

$$\mathrm{Re}\left\{\frac{(1-\alpha p+\alpha j)f^{(j)}(z)+\alpha zf^{(j+1)}(z)}{z^{p-j}}\right\} > 0 \qquad (z \in \mathbb{U}) \tag{3.7}$$

for $j = 2, 3, \cdots, p$, *where* $\alpha \geqq 1$. *Then we have*

$$k+\mathrm{Re}\frac{zf^{(k+1)}(z)}{f^{(k)}(z)} > 0$$

for $k = 0, 1, 2, \cdots, j-1$. *Therefore, we have* $f(z) \in \mathcal{S}^*(p)$ *and* $f(z) \in \mathcal{K}(p)$.

Proof. From Theorem 2,

$$j+\mathrm{Re}\frac{zf^{(j+1)}(z)}{f^{(j)}(z)} > 0 \qquad (z \in \mathbb{U})$$

for $j = 0, 1, 2, \cdots, p$. If $j \geqq 2$, using Lemma 3, we show that

$$k+\mathrm{Re}\frac{zf^{(k+1)}(z)}{f^{(k)}(z)} > 0 \qquad (z \in \mathbb{U})$$

for $k = 0, 1, 2, \cdots, j-1$. In the case of $k = 0$ and $k = 1$, we have $f(z) \in \mathcal{S}^*(p)$ and $f(z) \in \mathcal{K}(p)$. □

Putting $j = p$ in Theorem 3, we obtain

Corollary 3 *Let* $f(z) \in \mathcal{T}(p)$ *be analytic in* $\mathbb{U}$.
Suppose that

$$\mathrm{Re}\{f^{(p)}(z)+\alpha zf^{(p+1)}(z)\} > 0 \qquad (z \in \mathbb{U}) \tag{3.8}$$

where $\alpha \geqq 1$. *Then we have* $f(z) \in \mathcal{S}^*(p)$ *and* $f(z) \in \mathcal{K}(p)$.

Let us define generalized Libera-Bernardi integral operator

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (c > -p) \tag{3.9}$$

for $f(z) \in \mathcal{A}(p)$.

Next, we prove the following theorem.

Theorem 4 *Let $f(z) \in \mathcal{T}(p)$ be analytic in $\mathbb{U}$ and satisfies $\operatorname{Re} f^{(p)}(z) > 0 \quad (z \in \mathbb{U})$, then the function*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \qquad (c > -p)$$

belongs to $\mathcal{S}^*(p)$ and $\mathcal{K}(p)$ for all c $(p-1 \leqq -c < p)$.

Proof. By differentiating (3.9), we have

$$F^{(p)}(z) + \frac{1}{c+p} z F^{(p+1)}(z) = f^{(p)}(z).$$

Therefore,

$$\operatorname{Re}\left\{F^{(p)}(z) + \frac{1}{c+p} z F^{(p+1)}(z)\right\} > 0 \qquad (z \in \mathbb{U})$$

and $\dfrac{1}{c+p} \geqq 1 \quad (-p < c \leqq 1-p)$. Using Lemma 2 for $p(z) = \dfrac{F^{(p)}(z)}{p!}$, we obtain

$$1 + \frac{1}{c+p}\operatorname{Re}\frac{zF^{(p+1)}(z)}{F^{(p)}(z)} > 0 \qquad (z \in \mathbb{U}).$$

Then we have

$$p + \operatorname{Re}\frac{zF^{(p+1)}(z)}{F^{(p)}(z)} > -c \geqq p-1 > 0 \qquad (z \in \mathbb{U}).$$

From Lemma 3, we have

$$k + \operatorname{Re}\frac{zF^{(p+1)}(z)}{F^{(p)}(z)} > 0 \qquad (z \in \mathbb{U})$$

for $k = 0, 1, 2, \cdots, p-1$.

Taking $k = 0$, we have $F(z) \in \mathcal{S}^*(p)$, also letting $k = 1$, we obtain $F(z) \in \mathcal{K}(p)$. □

Applying $c = 1-p$ in Theorem 4, we can prove

Corollary 4 *Let $f(z) \in \mathcal{T}(p)$ be analytic in $\mathbb{U}$ and satisfies $\operatorname{Re} f^{(p)}(z) > 0 \quad (z \in \mathbb{U})$, then the function*

$$g(z) = \frac{1}{z^{1-p}}\int_0^z \frac{f(t)}{t^p}dt$$

belongs to $\mathcal{S}^*(p)$ and $\mathcal{K}(p)$.

Applying Lemma 4, we can prove

Theorem 5 *If $f(z) \in \mathcal{T}(p)$ be analytic in $\mathbb{U}$ with $\operatorname{Re}\dfrac{f^{(p)}(z)}{p!} > \beta$. If the function $F(z)$ given by (3.9), then*

$$\operatorname{Re}\frac{F^{(p)}(z)}{p!} > \beta + (1-\beta)\left\{\int_0^1 \frac{1}{1+\rho^{\frac{1}{c+p}}}d\rho - 1\right\} \qquad (z \in \mathbb{U}), \tag{3.10}$$

where $\beta < 1$.

Proof. By differentiating (3.9), we can show that

$$\frac{F^{(p)}(z)}{p!} + \frac{1}{c+p}\frac{zF^{(p+1)}(z)}{p!} = \frac{f^{(p)}(z)}{p!}.$$

Letting $p(z) = \frac{F^{(p)}(z)}{p!}$ and $n = 1$, $\alpha = \frac{1}{c+p}$ in Lemma 4, we have (3.10). □

Putting $p = 1$ in Theorem 5, we obtain

Corollary 5 *If* $f(z) \in \mathcal{T}(1) = \mathcal{T}$ *and* $\operatorname{Re} f'(z) > 0$, *let the function* $F(z)$ *given by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \qquad (c > -1), \tag{3.11}$$

then we have

$$\operatorname{Re} F'(z) > \beta + (1-\beta)\left\{2\int_0^1 \frac{1}{1+\rho^{\frac{1}{c+1}}}d\rho - 1\right\}.$$

References

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