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HAAGERUP PROPERTY FOR WREATH PRODUCTS

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If H and G are any discrete groups, the *standard wreath product* of H by G is the semidirect product

$$H \wr G = H^{(G)} \rtimes G,$$

where $H^{(G)}$ denotes the direct sum of copies of H indexed by G, and G acts by shifting. If H and G are finitely generated, so is the wreath product $H \wr G$.

A discrete group Γ has the *Haagerup Property* if the constant function 1 can be pointwise approximated by positive definitive functions on Γ. When Γ is countable, Akemann and Walter [AW] proved that this holds if and only if there exists a metrically proper action of Γ on a Hilbert space by affine isometries.

A nice feature about Haagerup groups is that they satisfy the strongest form of the Baum-Connes conjecture, namely the conjecture with coefficients [HK].

On the other hand, in known examples, there was a striking coincidence between the class of groups with the Haagerup Property and the class of groups with the *complete metric approximation property* [CH], and it was conjectured by Cowling that the two properties are actually equivalent.

Then it was proved by Ozawa and Popa that [OP] if H is any non-trivial group and G is any non-amenable group, then $H \wr G$ does not satisfy the complete metric approximation property.

In contrast, we prove, disproving one implication in Cowling's conjecture

Theorem 1 (joint with Y. Stalder and A. Valette). *Let H, G be any groups with the Haagerup Property. Then the wreath product $H \wr G$ has the Haagerup Property as well.*

This applies in the case of the wreath product of a non-trivial finite cyclic group and a non-abelian free group, so that Ozawa-Popa's result shows that it does not satisfy the complete metric approximation property. The Haagerup Property for this example is established in [CSV], and the redaction for the general case is currently in preparation. In both cases, the proof relies on a characterization of the Haagerup Property by the existence of a proper action on a space with walls, or a space with measured walls. It is currently unknown how to translate the proof of the stability of the Haagerup Property by wreath products, in terms of unitary representations.

References

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