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THE SINGULARITIES IN A TWO-DIMENSIONAL UNSTABLE FREE BOUNDARY PROBLEM

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ABSTRACT. We introduce a new method for the analysis of singularities in the unstable problem

$$\Delta u = -\chi_{\{u>0\}},$$

which arises in solid combustion as well as the composite membrane problem. Our study is confined to points of "*supercharacteristic*" growth of the solution, i.e. points at which the solution grows faster than the characteristic/invariant scaling of the equation would suggest. At such points the classical theory is doomed to fail, due to incompatibility of the invariant scaling of the equation and the scaling of the solution.

In the case of a second-order non-degenerate solution in two dimensions our result shows that in a neighborhood of the set at which the second derivatives of u are unbounded, the level set $\{u = 0\}$ consists of two C^1-curves meeting at right angles. Our estimates hold uniformly when considering a class of solutions. It is important that our result is not confined to the minimal solution of the equation but holds for *all* solutions.

1. INTRODUCTION

This paper contains an announcement and heuristics of results to be published elsewhere concerning the unstable obstacle problem

$$(1) \qquad \Delta u = -\chi_{\{u>0\}} \qquad \text{in } \Omega \subset \mathbf{R}^n,$$

related to traveling wave solutions in solid combustion with ignition temperature (see the introduction of [16] for more details), to the composite membrane problem (see [9], [8], [3], [17], [10], [11]) as well as

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the shape of self-gravitating rotating fluids describing stars (see [6, equation (1.26)]). Solutions of equation (1) may exhibit "*supercharacteristic*" growth of order

$$r^2|\log r|$$

not suggested by the invariant/characteristic scaling $u(rx)/r^2$ of the equation.

In this paper we introduce a new method to analyze the fine structure of singular sets close to points of supercharacteristic growth of the solution.

Equation (1) has been investigated by R. Monneau-G.S. Weiss in [16]. They establish partial regularity for *second order non-degenerate* solutions of (1). More precisely they show that the singular set has Hausdorff dimension less than or equal to $n-2$, and that in two dimensions the free boundary of the *minimal solution* consists close to singular points of four Lipschitz graphs meeting at right angles. They also show that energy-minimising solutions are in the two-dimensional case of class $C^{1,1}$ and that their free boundaries are locally analytic.

J. Andersson-G.S. Weiss have constructed a cross-shaped counter-example proving that the solution need not be of class $C^{1,1}$ (see [1]). In [16] it has been shown that the second variation of the energy at that particular solution takes the value $-\infty$. In this sense the cross-solution is completely unstable. Moreover, it cannot be obtained by naive numerical schemes.

In this paper we analyze the behavior of solutions at points at which the second derivatives are unbounded. Difficulties in the analysis are:

(i) At cross-like singular points the solution has the "wrong scaling", i.e. $u(rx)$ scales like $r^2|\log(r)|$ which is different from the characteristic scaling r^2 of the equation. The lack of a suitable local Lyapunov functional/monotonicity formula implies that methods like the Lojasiewicz inequality (see for example [19], [20]) would be hard to apply even at isolated singularities.

(ii) The cross-like singularities are unstable.

(iii) The comparison principle does not hold.

Instead we use knowledge about the Newtonian potential of the right-hand side to derive a quantitative estimate for the projection of the solution onto the homogeneous harmonic polynomials of degree 2. This leads to an estimate of order

$$\int_0^r \frac{\sqrt{|\log|\log s||}}{s|\log s|^{3/2}}\,ds$$

for how much the projection of $u(x+s\cdot)$ and also the approximate tangent space of the singular set can turn as s moves from r to 0 (see

Theorem 5.2). Our main result Theorem 5.2 shows that close to a non-degenerate singular point, the level set $\{u = 0\}$ consists of two C^1-curves meeting at right angles. The result holds uniformly when considering a class of solutions. Both uniformity and the fact that our result is not confined to the minimal solution are important differences to the results in [16].

We also prove a growth estimate at the highest-dimensional part of the non-degenerate singular set which holds in any dimension (Remark 3.4).

2. A Newtonian potential and its projection

Let us recall the definition of the spaces P and $P_{2-\text{dim}}$ of Definition A:

Definition 2.1. *Let us first define in each dimension $n \geq 2$ the space P of 2-homogeneous harmonic polynomials, i.e. harmonic polynomials of degree 2. In dimension 2 we define $P_{2-\text{dim}}$ as the space of homogeneous harmonic polynomials of degree 2. In dimension $n > 2$ we define $P_{2-\text{dim}} := \{p : (p \circ Q)(x_1, \ldots, x_n) = q(x_1, x_2)$ for some $q \in P_{2-\text{dim}}$ and some orthogonal (n, n) matrix $Q\}$.*

Definition 2.2. *(i) Let us define the projection*

$$\Pi : W^{2,2}(B_1) \to P$$

as follows: for $v \in W^{2,2}(B_1)$, let $\Pi(v)$ be the by Lemma 2.3 unique minimizer of

$$p \mapsto \int_{B_1} |D^2 v - D^2 p|^2$$

on P, where $|A| = \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ is the Frobenius norm of the matrix A.
(ii) Let us also define $\tau(v) \geq 0$ by

$$\Pi(v) = \tau(v) p, \; p \in P, \; \sup_{B_1} |p| = 1.$$

Lemma 2.3. *(i) For each $v \in W^{2,2}(B_1)$ the minimizer of Definition 2.2 exists and is unique. Thus $\Pi : W^{2,2}(B_1) \to P$ is well-defined.*
(ii) Π is a linear operator.
(iii) If $h \in W^{2,2}(B_1)$ is harmonic in B_1 then $\Pi(h(x)) = \Pi(h(rx)/r^2)$ for all $r \in (0,1)$.
(iv) For every $v, w \in W^{2,2}(B_1)$

$$\sup_{B_1} |\Pi(v + w)| \leq \sup_{B_1} |\Pi(v)| + \sup_{B_1} |\Pi(w)|.$$

Lemma 2.4. *Define* $v : (0,+\infty) \times [0,+\infty) \to \mathbf{R}$ *by* $v(x_1,x_2) :=$

$$-4x_1x_2\log(x_1^2+x_2^2) + 2(x_1^2 - x_2^2)\left(\frac{\pi}{2} - 2\arctan\left(\frac{x_2}{x_1}\right)\right) - \pi(x_1^2+x_2^2).$$

Moreover let

$$w(x_1,x_2) := \begin{cases} v(x_1,x_2), & x_1x_2 \geq 0, x_1 \neq 0, \\ -v(-x_1,x_2), & x_1 < 0, x_2 \geq 0, \\ -v(x_1,-x_2), & x_1 > 0, x_2 \leq 0, \end{cases}$$

and let

$$z(x_1,x_2) := \frac{w(x_1,x_2) - \pi(x_1^2+x_2^2) + 8x_1x_2}{8\pi}.$$

Then
(i) $\Delta z = -\chi_{\{x_1x_2>0\}}$ *in* $\mathbf{R}^2$.
(ii) $z(0) = |\nabla z(0)| = 0$.
(iii) $\lim_{x\to\infty} \frac{z(x)}{|x|^3} = 0$.
(iv) $\Pi(z) = 0$.
(v) $\Pi(z_{1/2}) = \log(2)x_1x_2/\pi,\ \tau(z_{1/2}) = \log(2)/(2\pi)$.

3. A quantitative result for the projection

Lemma 3.1. *If* $(u_\eta)_{\eta\in I}$ *is a family of solutions of (1) such that* $\eta \in [0,1/4], x^\eta \in B_{1/4}$, u_η *solves in* $B_{2\eta}(x^\eta)$ *and satisfies* $\sup_{B_{2\eta}(x^\eta)} |u_\eta| \leq M, u_\eta(x^\eta) = |\nabla u_\eta(x^\eta)| = 0$ *for every* $\eta \in I$, *then for each* $\alpha \in [1,+\infty)$ *and each* $\beta \in (0,1)$

$$\left\{ d_\eta(\cdot) := \frac{u_\eta(x^\eta + r_\eta\cdot)}{{r_\eta}^2} - \Pi\left(\frac{u_\eta(x^\eta + r_\eta\cdot)}{{r_\eta}^2}\right) \ :\ \eta \in I \right\}$$

is bounded in $W^{2,\alpha}(B_1)$ *and relatively compact in* $C^{1,\beta}(\overline{B_1})$.

Lemma 3.2. *For each* $\epsilon > 0, n \in \mathbf{N}, d > 0, M < +\infty, \alpha \in [1,+\infty)$ *and* $\beta \in (0,1)$ *there exists* $\delta > 0$ *with the following property:*
Suppose that $0 < r \leq \delta, x \in \Omega_d$ *and that* u *is a solution of (1) in* Ω *satisfying* $\sup_{\Omega_d} |u| \leq M$, $u(x) = |\nabla u(x)| = 0$ *and*

$$\mathcal{L}^n((\{u(x+r\cdot) > 0\}\Delta\{x_1x_2 > 0\}) \cap B_1) \leq \delta.$$

Then

$$\left\| \frac{u(x+r\cdot)}{r^2} - \Pi\left(\frac{u(x+r\cdot)}{r^2}\right) - z \right\|_{C^{1,\beta}(\bar{B}_1)} \leq \epsilon.$$

Lemma 3.3. *For each* $\gamma \in (0, \log(2)/(2\pi)), n \in \mathbf{N}, d > 0$ *and* $M < +\infty$ *there is* $\delta > 0$ *with the following property:*

Suppose that $0 < r \le \delta, x \in \Omega_d$ *and that* u *is a solution of (1) in* Ω *satisfying* $\sup_{\Omega_d} |u| \le M$, $u(x) = |\nabla u(x)| = 0$ *and*

$$\text{dist}_{W^{2,2}(B_1)}\left(\frac{u(x+r\cdot)}{\sup_{y\in B_1}|u(x+ry)|}, P_{2-\text{dim}}\right) \le \delta.$$

Then $\tau(4u(x + r\cdot/2)/r^2) \ge \tau(u(x+r\cdot)/r^2) + \gamma$.

Remark 3.4. *Note that if the hypothesis in Lemma 3.3 is satisfied for* $r \in (0,\delta)$, *then Lemma 3.3 implies*

$$\tau(u(x+r\cdot)/r^2) \ge c|\log(r/\delta)|$$

and thereby logarithmic growth of the norm of $u(x+r\cdot)/r^2$ *as* $r \to 0$.

4. Controlling the movement of $\Pi(u(x+r\cdot))$

Lemma 4.1. *For each* $n \in \mathbf{N}, d > 0$ *and* $M < +\infty$ *there is* $\delta > 0$ *with the following property:*
Suppose that $0 < r_0 \le \delta, x \in \Omega_d$ *and that* u *is a solution of (1) in* Ω *satisfying* $\sup_{\Omega_d} |u| \le M$, $u(x) = |\nabla u(x)| = 0$ *and for all* $r \in (\rho, r_0)$

$$\text{dist}_{W^{2,2}(B_1)}\left(\frac{u(x+r\cdot)}{\sup_{y\in B_1}|u(x+ry)|}, P_{2-\text{dim}}\right) \le \delta.$$

Then

$$\mathcal{L}^n(\{u(x+r\cdot) > 0\}\Delta\{\Pi(u(x+r\cdot)) > 0\}) \cap B_1)$$
$$\le C(n)\frac{|\log(|\log(r/r_0)|)|}{|\log(r/r_0)|} \quad \textit{for } r \in (\rho, r_0).$$

The above lemma gives some control on how much the solution can "turn" when passing to a smaller scale. In two dimensions the estimate leads to unique tangent cones:

Proposition 4.2. *In the case* $n = 2$ *there is* $\delta > 0$ *with the following property:*
Suppose that $0 < r_0 \le \delta, x \in \Omega_d$ *and that* u *is a solution of (1) in* Ω *satisfying* $\sup_{\Omega_d} |u| \le M$, $u(x) = |\nabla u(x)| = 0$ *and for all* $r \in (\rho, r_0)$

$$\text{dist}_{W^{2,2}(B_1)}\left(\frac{u(x+r\cdot)}{\sup_{y\in B_1}|u(x+ry)|}, P\right) \le \delta.$$

Then for $r \in (\rho, r_0)$,

$$\sup_{B_1}\left|\frac{\Pi(u(x+r\cdot))}{|\Pi(u(x+r\cdot))|} - \frac{\Pi(u(x+r\cdot/2))}{|\Pi(u(x+r\cdot/2))|}\right| \le C\frac{\sqrt{|\log(|\log(r/r_0)|)|}}{|\log(r/r_0)|\sqrt{|\log(r/r_0)|}}.$$

Theorem 4.3. *Let $n = 2$. Then for each $\delta \in (0, \delta_0)$ there is $\theta_\delta > 0$ with the following property:*
Suppose that $0 < r_0 \leq \delta_0, x \in \Omega_d$ and that u is a solution of (1) in Ω satisfying $\sup_{\Omega_d} |u| \leq M$, $u(x) = |\nabla u(x)| = 0$,

$$r_0^{-n-4} \int_{B_{r_0}(x)} u^2 \geq 1/\theta_\delta$$

$$\text{and } \operatorname{dist}_{W^{2,2}(B_1)}\Big(\frac{u(x + r_0 \cdot)}{\sup_{y \in B_1} |u(x + r_0 y)|}, P\Big) \leq \theta_\delta.$$

Then for all $r \in (0, r_0)$

$$\sup_{B_1} \left| \frac{\Pi(u(x + r\cdot))}{|\Pi(u(x + r\cdot))|} - \frac{\Pi(u(x + r\cdot/2))}{|\Pi(u(x + r\cdot/2))|} \right|$$

$$\leq C(n) \frac{\sqrt{|\log(|\log(r/r_0)|)|}}{|\log(r/r_0)|\sqrt{|\log(r/r_0)|}},$$

$$\sup_{B_1} \left| \frac{\Pi(u(x + r_0\cdot))}{|\Pi(u(x + r_0\cdot))|} - \frac{\Pi(u(x + r\cdot))}{|\Pi(u(x + r\cdot))|} \right| \leq \delta,$$

$$\operatorname{dist}_{W^{2,2}(B_1)}\Big(\frac{u(x + r\cdot)}{\sup_{y \in B_1} |u(x + ry)|}, P\Big) \leq \delta/2 \text{ and}$$

$$\operatorname{dist}_{W^{2,\alpha}(B_1)}\Big(\frac{u(x + r\cdot)}{\sup_{y \in B_1} |u(x + ry)|}, P\Big) \leq C(n, \alpha) \min\Big(\delta/4, \frac{1}{|\log(r/r_0)|}\Big).$$

5. Uniform Estimates close to the Singular Set

Definition 5.1. *Let u be a solution of (1) in $\Omega \subset \mathbf{R}^2$ satisfying $\sup_{\Omega_d} |u| \leq M$. We define for $r_0 \in (0, \delta_0)$ the set*

$$\Sigma^u_{\delta,r_0} := \{x : x \in \Omega_d, u(x) = |\nabla u(x)| = 0, r_0^{-6} \int_{B_{r_0}(x)} u^2 \geq 1/\theta_\delta,$$

$$\operatorname{dist}_{W^{2,2}(B_1)}\Big(\frac{u(x + r_0\cdot)}{\sup_{y \in B_1} |u(x + r_0 y)|}, P\Big) \leq \theta_\delta/2\};$$

in what follows $\delta_0 > 0$ and $\theta_\delta > 0$ will be the constants of Theorem 4.3.

Theorem 5.2. *Let u be a solution of (1) in $\Omega \subset \mathbf{R}^2$ satisfying $\sup_{\Omega_d} |u| \leq M$. Then Σ^u_{δ,r_0} consists of isolated points in Ω. Moreover $\{u = 0\}$ is in $\overline{B_{c(n,r_0)}(\Sigma^u_{\delta,r_0})}$ the union of two C^1-curves intersecting each other at right angles. For a family of solutions as above the family of C^1-curves above is relatively compact in C^1. The estimate*

$$\left\| \frac{u(x + r\cdot)}{\sup_{y \in B_1} |u(x + ry)|} - p^{x,u}(\cdot) \right\|_{C^{1,\beta}(\bar{B}_1)} \leq C(\beta, r_0) \int_0^r \frac{\sqrt{|\log|\log s||}}{s|\log s|^{3/2}}\, ds$$

holds for $x \in \Sigma^u_{\delta,r_0}$, *some* $p^{x,u} \in P$ *and* $r \leq r_0$.

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