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GEOMETRIC PROPERTIES IN PARABOLIC FLOWS AND ITS APPLICATIONS

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ABSTRACT. In this paper, we are going to introduce the recent development in the study of the geometric properties of parabolic flows. It provides a parabolic approach on geometric properties of the solutions of the nonlinear eigen value problems.

1. INTRODUCTION

Let us present the problems and concepts to motivate our issue in the geometric properties . Let the function $\varphi(x)$ satisfy the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta\varphi & = \lambda\varphi^p \quad \text{in } \Omega, \\ \varphi & > 0 \quad \text{in } \Omega, \\ \varphi & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The main question we address is the following: assuming that Ω is a strictly convex domain in $\mathbb{R}^N$, are the level sets of the positive first eigen-function convex? A stronger version of this question is the following: is there a monotone real function f such that $f(\varphi(x))$ is convex or concave? Since φ and $f(\varphi)$ share the same level sets, the convexity or concavity of $f(\varphi)$ will imply an affirmative answer to the main question; and strict convexity or concavity will imply the existence of a unique peak of φ (i.e., the point of maximum, also called hot spot).

If Ω is a ball, then there is a unique rotationally symmetric solution by the Alexandrov reflection argument, and this function is decreasing as $|x|$ increases. Then each level set of φ is a ball as φ has a unique peak. Somehow, we are asking whether similar geometric properties are preserved under a large convex perturbation of the domain. The case $p = 1$ corresponds to the linear eigenvalue problem for the Laplace equation. H.J. Brascamp and E.H. Lieb [BL] have shown that $\log(\varphi)$ is concave by a probability method, and the proof has been simplified by N. Korevaar's new approach which will be discussed below, [Ko]. B. Kawohl

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[Ka] has extended Korevaar's idea to the case $0 < p < 1$ by considering φ^q for some $q > 0$ instead of $\log(\varphi)$.

For $0 < p < p_s$ where p_s is the Sobolev exponent ($p_s = \frac{n+2}{n-2}$ for $n \geq 3$, infinity for $n = 1, 2$), C.S. Lin [Li] shows the uniqueness of the energy minimizer of (1.1) and the convexity of the level sets of the energy minimizer in two dimensions. F. Gladis and M. Grossi [GG] show that there is a small $\varepsilon_o > 0$ such that the energy minimizing sequence u_ε such that

$$\lim_{\varepsilon \to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2 \, dx}{\left(\int_\Omega u_\varepsilon^{2^*} \, dx\right)^{\frac{2}{2^*}}} = S$$

(where S is the best Sobolev constant and $2^* = \frac{2n}{n-2}$) has strictly convex level sets. L. Caffarelli and J. Spruck [CS] use Korevaar's idea to show such geometric property for the solution of the following elliptic free boundary problems:

$$-\triangle u = \lambda u_+ \quad \text{with } \lambda \int_\Omega u_+ \, dx = \text{constant}.$$

X. Cabré and S. Chanillo [CCh] show, in two dimensions, that the semi-stable solution for general $p \leq 1$ has a unique critical point, which is a nondegenerate maximum: this means that, in a neighborhood of the peak, the level sets will be convex. And we recall that for $p > 1$ all positive solutions are unstable.

1.1. **A simple computation.** Let us introduce the main difficulties and ideas through a simple computation. For example, if we try to show the log-concavity of φ in (1.1), we can put $v = \log(\varphi)$ and replace φ by e^v in the equation. We get

$$\triangle v + |\nabla v|^2 = -e^{(p-1)v}. \tag{1.2}$$

The concavity of v is equivalent to the non-positivity of the quantity: $Z = \sup_x \sup_\beta v_{\beta\beta}$. Let us assume that the supremum is achieved at a point x_o in the direction α, i. e.,

$$\sup_x \sup_\beta v_{\beta\beta}(x) = v_{\alpha\alpha}(x_o) = \delta.$$

Notice that x_o may be located in the interior or on the boundary of the domain Ω. We want to eliminate the possibility $\delta > 0$.

CASE 1. The non-degeneracy of $|D\varphi|$ (i. e., $|D\varphi| > 0$) is enough to rule out the possible maximum point on the boundary. Let ν be the outward normal direction to $\partial\Omega$ at 0, set $\tau = (\tau_1, \cdots, \tau_{n-1})$ to be orthogonal tangential coordinates, and let $x_\nu = \gamma(\tau)$ be the representation of the boundary near 0. Then, we have $D_{\tau\tau} v(\tau, \gamma(\tau)) = 0$ and $\gamma_\tau(0) = 0$. From the convexity of the boundary $\partial\Omega$, the tangential second derivative in the direction τ, $D_{\tau\tau} v = -v_\nu D_{\tau\tau}\gamma \leq 0$. Besides, $-\triangle_\tau \gamma$ is the mean curvature, $H(\partial\Omega)$, of $\partial\Omega$ at 0 (for example, for a rotationally symmetric function, $v(x) = v(|x|)$, $\triangle v = v_{\nu\nu} + \sum_i v_{\tau_i\tau_i} = v_{\nu\nu} + \frac{N-1}{r} v_\nu$ where $\nu = e_r$,

$1/r$ is the curvature in the direction τ_i, and $(n-1)/r$ is the mean curvature of the boundary).

Now, only the normal second derivative may be positive. But $|D\varphi| = -\varphi_\nu > 0$ tells us that

$$v_{\nu\nu} = \frac{\varphi_{\nu\nu}}{\varphi} - \frac{\varphi_\nu^2}{\varphi^2}.$$

We conclude that the maximum of Z can only be achieved at an interior point.

CASE 2. When x_o is an interior point, we note that $v_{,\alpha\alpha}$ satisfies the following equation:

$$\Delta v_{,\alpha\alpha} + 2\nabla v \cdot \nabla v_{,\alpha\alpha} + \sum_\beta v_{,\alpha\beta}^2 = -(p-1)e^{(p-1)v} v_{,\alpha\alpha} - (p-1)^2 e^{(p-1)v} v_{,\alpha}^2.$$

Since the supremum of the pure second derivative has been achieved in the direction e_α, e_α will be an eigen-direction of $D^2 v$ at x_o, which means $v_{,\alpha\beta}(x_o) = 0$ for $\beta \neq \alpha$. Therefore, we have at this point

$$\Delta v_{,\alpha\alpha} + 2\nabla v \cdot \nabla v_{,\alpha\alpha} = -v_{,\alpha\alpha}^2 - (p-1)e^{(p-1)v} v_{,\alpha\alpha} - (p-1)^2 e^{(p-1)v} v_{,\alpha}^2.$$

We also have $\Delta v_{,\alpha\alpha}(x_o) \leq 0$ and $\nabla v_{,\alpha\alpha} = 0$. To have a contradiction we expect a nonnegative term at the right hand side of the equation above. Since $v_{\alpha\alpha}(x_o) = \delta > 0$, we impose $p - 1 \leq 0$; to treat the last term we also need $-(p-1)^2 = 0$ i. e., $p = 1$, which is the reason that log-concavity of φ holds only for $p = 1$. For a general p, φ^q can be considered and q will be selected in order to kill the third term in right-hand side. But we still need to impose $p - 1 \leq 0$ so that the second term is nonnegative. Korevaar's idea is brought to treat the first term $-v_{,\alpha\alpha}^2 = -\delta^2$, and will be presented in next subsection.

1.2. **Korevaar's idea.** Equation (1.2) can be written in a more general form:

$$Lu := a_{ij}(Du)D_{ij}u - b(x, u, Du) = 0, \tag{1.3}$$

with the restrictions equivalent to the condition on p above:

$$\frac{\partial b}{\partial u} \geq 0, \qquad b \text{ is jointly concave in } (x, u), \tag{1.4}$$

see [Ko, Theorem 1.3]. The second difference of u,

$$C(x, y) = \frac{1}{2}(u(x) + u(y)) - u(\frac{x+y}{2}),$$

is then considered. The point is that the concavity of u is equivalent to the non-positivity of $C(x, y)$. The paper shows that there is a contradiction if $C(x, y)$ has a positive maximum. In this introduction, we are going to show only how to deal with the gradient term $|Du|$ in (1.3) at an interior maximum point, since this is important for the sequel (the other details can be found in [Ko]). Let us assume $C(x, y)$ has a positive maximum at an interior point (x_o, y_o). Then for any unit vector e, $C(x_o+te, y_o)$ and $C(x_o, y_o+te)$, for $t \in \mathbb{R}$, will have a maximum at $t=0$. This implies that $D_e u(x_o) = D_e(\frac{x_o+y_o}{2}) = D_e u(y_o)$. Set $Du(x_o) = U$ and

$$M_{ij} = D_{e_i} D_{e_j} C(x_o, y_o) = \frac{1}{2}(D_{ij}u(x_o) + D_{ij}u(y_o)) - D_{ij}u(\frac{x_o+y_o}{2}) \leq 0.$$

From (1.3),(1.4), we have

$$\begin{aligned} a_{ij}(U)M_{ij} &= \frac{1}{2}(b(x_o, u(x_o), U) + b(y_o, u(y_o), U)) - b(\frac{x_o+y_o}{2}, u(\frac{x_o+y_o}{2}), U) \\ &\geq b(\frac{x_o+y_o}{2}, \frac{u(x_o)+u(y_o)}{2}, U) - b(\frac{x_o+y_o}{2}, u(\frac{x_o+y_o}{2}), U) \geq 0, \end{aligned}$$

which is a contradiction to $(M_{ij}) \leq 0$ after a simple modification.

Note that the condition $\frac{\partial b}{\partial u} \geq 0$ in (1.4) imposes $p \leq 1$ in (1.1) through (1.2). Therefore we may need to create different approach for the nonlinear eigen value problems. In the next chapter, we will overview the recent development on the geometric properties in nonlinear parabolic flows.

2. Geometric Properties and Regularities in Degenerate Diffusion Equations

One of the important class of nonlinear equations is the degenerate diffusion equations. The *porous medium equation*

$$(PME) \qquad u_t = \Delta u^m$$

describes the isentropic gas through a porous medium. u and $v = \frac{m}{m-1}u^{m-1}$ represent the density of mass and its corresponding pressure respectively. And the *pressure* v satisfies

$$(PME_p) \qquad v_t = (m-1)v\Delta v + |\nabla v|^2$$

We see that the diffusion coefficient is mu^{m-1} which vanishes for $m > 1$ wherever u is zero. In the other words, (PME) is *degenerate parabolic equation.* For $m=1$, we recover the *heat equation,* $u_t = \Delta u$ which is not degenerate. For $0 < m < 1$, the diffusion coefficient $\frac{m}{u^{1-m}} \to \infty$ as $u \to 0$ and then we call it *fast diffusion equation.*

The existence of weak solution and strong solution can be found in [V3]. And the concept of viscosity solution and its existence can be found in [HV]. The degeneracy in (PME) for $m > 1$ results in the interesting phenomenon of the *finite speed of propagation*: if the initial data u^0 is compactly supported in $\mathbb{R}^n$, the solution $u(x,t)$ remains supported for all time t. Therefore the boundary of the support of u, $\Gamma = \partial \overline{\text{supp} u}$ may have finite speed. If the initial configuration of the support of $u(x,0) = u_o(x)$ and mass distribution is complicated, the advancing free boundary may collide with each other and create some singularity. And a small empty hall can be filled out by advancing mass which also create a singularity. The global C^{α}-regularity has been proved by L. Caffarelli and A. Friedman, [CF]. L. Caffarelli, J.L. Vazquez and N.I. Wolanski show that the solution will be Lipschitz for $t > T$ after the support of $u(x,t)$ overflows a ball containing the support of initial data, $u_0(x)$, for $t > T$, [CVW]. L. Caffarelli and N.I. Wolanski show that the solution is C^1 and that the free boundary $\partial\{u > 0\}$ is $C^{1,\alpha}$ for $t > T$, [CW]. H. Koch show that u and $\partial\{u > 0\}$ are C^{∞} for $t > T$. The short time existence of the smooth solution is proved by P. Daskalopolous and R. Hamilton, [DH], under the condition that the the initial speed of the free boundary is nondegenerate. As we observed, it is important to prohibit the collision of free boundaries in order to have the long time existence of smooth solution and the smoothness of the free boundaries.

Let us briefly summarize the recent development of geometric properties in parabolic flows. We start by some results on minimal curvature flows. Gage, Hamilton, and Grayson show that any convex curve or surface will stay convex (the property is called *all-time convexity*) and, in the 2-dimension minimal curvature flow, even any simply connected curve will become convex in finite time (*eventual convexity*) in [GH][G]. And they show that the convex curve converges to a circle after a normalization.

These issues have been pursued by the author on nonlinear diffusion equations. All-time square-root concavity of the pressure in the porous medium equation has been shown at [DHL] and, through a simpler computation, it has been extended to degenerate parabolic nonlinear equation with various homogeneity, for example parabolic p-Laplace equation where all-time $\frac{p-2}{p}$–concavity of the density is proved, [Le]. And all time log-concavity of the solution has been shown in one-phase free boundary problems of flame type, [DL1], and of Stephan type, [DL2]. Recently Su Jung Kim found the similar geometric properties for the Fully nonlinear Parabolic flows , [KsL], with the author. In addition, Sung Hoon Kim and the author showed geometric properties of the ground state eigen functions for non-local equation, [KL1], conjectured by Bauelos, R., Kulczycki, T., and Mndez-Hernndez, P. J. , [BKM].

The geometric properties of parabolic flows prevent the collision of the advancing free boundaries considered in the Porous Medium Equations, [DHL], Flame propagation, [DL1], and Stephan Problems, [DL2] and then let us prove the existence of smooth solutions in those flows under a natural conditions. The initial conditions consist of two parts: the first is the smoothness of the initial data and the second is on the finite and nondegenerate initial speed of the free boundary, without which there may be waiting time and no improvement of the regularity of the solutions.

3. Long Time Behavior of Parabolic Flows

In [LV1], J.L. Vazquez and the author considered the long time behavior of Porous medium equations (PME), Fast Diffusion equations (FDE), and a heat equation (HE). It is well-known that any solution of (PME) with finite L^1-data converges uniformly to rotationally symmetric self-similar solution called a Barenblatt Solution in L^1-norm in $\mathbb{R}^n$ and L^∞-norm in an expanding domain. And similar convergence to the heat kernel is true in (HE). Barenblatt solution is concave for (PME) and convex for (FDE). And the heat kernel is log-concave in (HE). The key idea is to scale the solution in the time $[2^k, 2^{k+1}]$ to a scaled solution in the time $[1,2]$ following the scale invariance in the space, the time, and the value satisfied by the Barenblatt solution. Now the scaled solution will represent the original solution in the different time intervals. The key estimate is the uniform estimate of the derivatives of the scaled solutions so that the second derivate of parabolic flows converge to those of the self-similar solutions, which will imply the eventual geometric properties of parabolic flows.

In [LPV], A. Petrosyan, J.L. Vázquez, and the author showed the similar long time behavior of the solutions in the parabolic p-Laplace equations.

4. Applitcations to Nonlinear Eigen Value Problems

4.1. **Heat equation and linear eigen value problem.** Another important application of the geometric properties of parabolic flows is characterizing the long-time behavior of the parabolic flows and finding the geometric properties of the ground state eigen functions of nonlinear eigen value problems proposed in the introduction, [LV2]. Let us summarize the key steps in the proof for the Heat equations and the linear eigen value problems. Similar method can be applicable for the non linear eigen value problems.

We consider the solutions $u(x,t)$ of the problem

$$(4.5)\qquad \begin{cases} u_t(x,t) = \triangle u(x,t) & \text{in } Q = \Omega \times (0,T), \\ u(x,0) = u_o(x) \in W_o^{1,2}(\Omega), & \\ u(x,t) = 0 & \text{for } x \in \partial\Omega \times (0,T), \end{cases}$$

where Ω is a bounded sub-domain of $\mathbb{R}^N$ with smooth boundary. Our geometrical results will be derived under the extra assumption that Ω is strictly convex.

It is well-known, cf. Theorem 8.37 in [GT], that (even without the last assumption) the Laplace operator has a countable discrete set of eigenvalues $\Sigma = \{\lambda_i \mid \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots\}$, whose eigen-functions $\{\phi_n\}$ span $W_o^{1,2}(\Omega)$, where ϕ_n is a normalized eigen-function corresponding to λ_n. For $u_o \in L^2(\Omega)$ there are coefficients $\{a_n\}$ such that $u_o = \sum_{n=1}^{\infty} a_n \phi_n$. Hence,

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n = a_1 e^{-\lambda_1 t} \varphi + e^{-\lambda_2 t} \eta(x,t) \tag{4.6}$$

where $||\eta(x,t)||_{L^2_x(\Omega)} < C < \infty$. Then $\varphi(x)$ will be the unique solution of

$$\begin{cases} \Delta\varphi(x) = -\lambda_1 \varphi(x) & \text{in } \Omega \\ \varphi(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{\textbf{EV}}$$

In this section, $\varphi(x)$ will be the solution of (**EV**). We have the following well-known result.

Lemma 4.1 (Approximation lemma). *For every $u_0 \in L^2(\Omega)$ we have*

$$|e^{\lambda_1 t} u(x,t) - a_1 \varphi(x)| \leq C e^{-(\lambda_2 - \lambda_1)t} \tag{4.7}$$

and

$$||e^{\lambda_1 t} u(x,t) - a_1 \varphi(x)||_{C^k_x(\Omega)} \leq CK\, e^{-(\lambda_2 - \lambda_1)t} \tag{4.8}$$

for $k = 1, 2, \cdots$.

Next, coming to our subject, we have the following result about *preservation of log concavity*, which is easy but allows to present the basic technique.

Lemma 4.2. *Let Ω be a convex bounded domain and let $u_0 \geq 0$ be a continuous and bounded initial function that vanishes on the boundary. If $\log(u_0)$ is concave, then the solution of the heat equation, $u(x,t)$, is log-concave in the space variable for all $t > 0$, i. e., $D^2 \log(u(x,t)) \leq 0$.*

PROOF. (i) Let us also assume that u_0 is smooth in $\overline{\Omega}$, that $D^2 \log u_0(x) \leq -c\mathbf{I} < 0$ in Ω, and $u_0 = 0$ on $\partial\Omega$, and that this border is C^2 smooth. There is a smooth solution of (4.5) with initial data u_0. Let us put $g(x,t) = \log u(x,t)$, which is finite and smooth for $x \in \Omega$ and takes the value $g = -\infty$ on the lateral boundary $S = \partial\Omega \times (0,\infty)$. It also satisfies the equation

$$\partial_t g = \Delta g + |\nabla g|^2. \tag{4.9}$$

To estimate the maximum of the second derivatives, we look at the quantity

$$Z(t) = \sup_{y\in\Omega, s\in[0,t]} \sup_{\beta} g_{,\beta\beta}(y,s)$$

$(1 \leq \beta \leq N)$, which is taken along a direction α in which the maximum of the second directional derivative is achieved, $Z(t) = g_{,\alpha\alpha}(x_o,t)$. Therefore α is an eigen-direction of the symmetric matrix $D^2 g(x_o,t)$, which means that, using orthonormal coordinates in which α is taken as one of the coordinate axes, we have $g_{,\alpha\beta} = 0$ at (x_o, t) for $\beta \neq \alpha$. Then, we notice that

$$g_{,\alpha\alpha} = \frac{u\, u_{,\alpha\alpha} - u_{,\alpha}^2}{u^2} \longrightarrow -\infty$$

as $x \in \Omega \to \partial\Omega$, since $\partial\Omega$ is smooth and $|\nabla u| > 0$ on $\partial\Omega$ by Hopf's principle. We conclude that the maximum of Z can only be achieved at an interior point (x_o, t_o).

Next, we see that the evolution of $g_{,\alpha\alpha}(x,t)$ is given by the equation

$$g_{,\alpha\alpha t} = \triangle g_{,\alpha\alpha} + 2\nabla g \cdot \nabla g_{,\alpha\alpha} + 2\nabla g_{,\alpha} \cdot \nabla g_{,\alpha}\,. \tag{4.10}$$

At the point of maximum we have $\nabla g_{,\alpha\alpha} = 0$, $\Delta g_{,\alpha\alpha} \leq 0$, as well as $g_{,\alpha\beta} = 0$ for $\beta \neq \alpha$, hence at this point

$$g_{,\alpha\alpha t} \leq 2g_{,\alpha\alpha}^2. \tag{4.11}$$

Then, we have $Z'(t) \leq 2Z^2$ and $Z(0) < 0$ which implies $Z(t) \leq 0$ for all $t \geq 0$. The proof is finished when the initial data and domain are as regular as assumed.

(ii) The proof in the general case uses a density argument which is more or less standard. Briefly, if u_0 is not smooth and strictly log-concave, we first perform a mollification to obtain an approximating sequence u_{0n} of smooth and log-concave functions; we then modify u_{0n} to make it strictly log-concave. We may put for instance,

$$\tilde{u}_{on}(x) = u_{on}(x) \exp\left(-c_n x^2/2\right)$$

for some $c_n > 0$, $c_n \to 0$ as $n \to \infty$. Then,

$$\tilde{g}_{on}(x) = \log(\tilde{u}_{on}(x)) = g_{on}(x) - c_n x^2,$$

so that $D^2\tilde{g}_{on} \leq -2c\mathbf{I}$. We get the conclusion for $\tilde{u}_n$, the solution of the problem with data $\tilde{u}_{on}$ and pass to the limit $n \to \infty$ to get the result for u.

(iii) When the domain is not smooth, we make the approximation of the domain with smooth convex domains and use the uniform convergence due to the Hölder-estimate in the Lipschitz domain to show the sign of second difference preserved. □

The approximation lemma and the preservation of log-concavity will give us the following lemma for the first eigen function.

Corollary 4.3 (Log-concavity). *If Ω is convex the stationary profile $\varphi(x)$ is log-concave, i. e., $D^2\log(\varphi(x)) \leq 0$.*

Now the any pure second derivative of $\log(\varphi(x))$ is non-positive. To show the strict log-concavity of $\varphi(x)$, we need to show the strict negativity of any pure second derivative, which requires a kind of strong maximum principle for the pure second derivatives. The detail of the proof can be found in [LV2].

Lemma 4.4 (Strict log-concavity). *If Ω is smooth and strictly convex, φ is strictly log-concave: there exists a constant $c_1 > 0$ such that*

$$D^2 \log(\varphi(x)) \le -c_1 \mathbf{I}. \tag{4.12}$$

The constant c_1 depends only on the shape of Ω.

From the Approximation Lemma, the second derivative of $u(x,t)$ converges uniformly to that of φ.

Theorem 4.5 (Eventual log-concavity). *Let u_0 be a nonnegative and integrable initial function. Then, the solution $u(x,t)$ of Problem* (4.5) *is strictly log-concave in the space variable for all large $t > 0$. More precisely, for every $\varepsilon > 0$ there is $t_0 = t_0(u, \varepsilon)$ such that*

$$D^2 \log(u(x,t)) \le -(c_1 - \varepsilon)\mathbf{I} \qquad \text{for all } t \ge t_0, \tag{4.13}$$

where $c_1 = c(\varphi) > 0$ is the constant of Lemma 4.4.

4.2. **Porous Medium Equations and Nonlinear Eigen value problem ($0 < p < 1$).** We address now the long-time geometrical properties of solutions of the initial-value problem for the Porous Medium Equation

$$u_t = \Delta u^m, \qquad m > 1, \tag{4.14}$$

posed in a bounded domain Ω with homogeneous Dirichlet conditions

$$u = 0 \quad \text{on } \partial\Omega, \tag{4.15}$$

and initial data

$$u(x,0) = u_0(x) \text{ nonnegative and integrable.} \tag{4.16}$$

By known regularity theory, cf. [Ar, V1, V2], we may also assume without loss of generality that u_0 is continuous and bounded. We assume for convenience that $\partial\Omega$ is $C^{2,\alpha}$ smooth.

The large-time stabilization for the solutions of the above problem has been studied by Aronson and Peletier [AP], who prove that as $t \to \infty$, they tend in the L^∞ norm to the similarity solution $U(x,t) = f(x)/(1+t)^{1/(m-1)}$ with an error of order $O(1/(1+t)^{m/(m-1)})$. Here, f is the unique solution of the elliptic equation (profile equation)

$$-\Delta f^m = \frac{1}{m-1} f \tag{4.17}$$

satisfying the conditions $f > 0$ in Ω and $f = 0$ on $\partial\Omega$.

Lemma 4.6 (Approximation Lemma). *Let $u(x,t)$ be a nonnegative weak solution of* (4.14) *satisfying the conditions* (4.15)–(4.16) *in a smooth domain Ω. Set $U(x,t) = f(x)/(1+t)^{1/(m-1)}$ where f is defined above. Then, we have the following properties.*

(i) There is a time $t_o(u_o, \Omega) > 0$ such that $u(x,t) > 0$ for $t > t_o$.
(ii) We have the estimate

$$\lim_{t\to\infty} t^{1/(m-1)}|u(x,t) - U(x,t)| \to 0 \tag{4.18}$$

uniformly in $x \in \Omega$.
(iii) There is a $t_o^*(u_o, \Omega) \geq t_o(u_o, \Omega) > 0$ such that u^m is C^1 up to the boundary and $0 < c_o < t^{m/(m-1)}|\nabla u^m(x,t)| < C_o$ for some uniform constants c_o and C_o.

After a simple transformation $\varphi = \left(\frac{1}{m-1}\right)^{\frac{m}{m+1}} f^m$, we recover from (4.17) the equation (1.1) with $0 < p = \frac{1}{m} < 1$. In this section, $\varphi(x)$ will be the solution of the following equation:

$$\text{(NLEV)} \qquad \begin{cases} \Delta\varphi(x) = -\varphi(x)^p & \text{in } \Omega, \\ \varphi(x) = 0 & \text{on } \partial\Omega, \\ 0 < p < 1. \end{cases}$$

Let us start our investigation by recalling the equation satisfied by the pressure and its square root. Here we introduce the pressure variable in the form $v = u^{m-1}$. We then have

$$v_t = v\Delta v + r|\nabla v|^2, \quad r = \frac{1}{m-1}. \tag{4.19}$$

Apart from its physical significance in the model of flow of gases in porous media, this variable plays a very important mathematical role in the study of the geometric properties of the solutions: the property of finite speed of propagation, as well as the interface behavior and regularity, cf. [V3], Chapters 14, 15.

Lemma 4.7. *Let Ω be a convex bounded domain and let $v_0 \geq 0$ be a continuous and bounded initial function that v_0 vanishes on the boundary. If $\sqrt{v_0}$ is concave, then the solution of the porous medium equation, $v(x,t)$, is square root-concave in the space variable for all $t > 0$, i. e., $D^2\sqrt{v(x,t)} \leq 0$.*

Set $V(x,t) = U^{1/(m-1)} = h(x)/(1+t)$. By applying the convergence (4.18) on the second difference quotient, we have the following.

Corollary 4.8 (Square-root Concavity). *If Ω is convex the stationary pressure profile $h(x)$ is square root-concave, i. e., $D^2\sqrt{h(x)} \leq 0$.*

We also have the following similar Lemmas as those in the Heat equation, [LV2]

Lemma 4.9 (Strict Square-root Concavity). *If Ω is smooth and strictly convex, $h(x)$ is strictly square root-concave: there exists a constant $c_1 > 0$ such that*

$$D^2\sqrt{h(x)} \leq -c_1 \mathbf{I}. \tag{4.20}$$

The constant c_1 depends only on the shape of Ω.

Theorem 4.10 (Eventual square-root concavity). *Let v_0 be a nonnegative and integrable initial function. Then the pressure $v(x,t)$ is strictly square root-concave in the space variable for all large $t > 0$. More precisely, for every $\varepsilon > 0$ there is $t_0 = t_0(v_o, \varepsilon)$ such that*

$$D^2\sqrt{tv(x,t)} \le -(c_1 - \varepsilon)\,\mathbf{I} \tag{4.21}$$

for all $t \ge t_0$ and $x \in \Omega_{(-\varepsilon)} = \{x \in \Omega |\ d(x, \partial\Omega) > \varepsilon\}$, where $c_1 = c(\varphi) > 0$ is the constant of Lemma (4.9).

4.3. **The Fast Diffusion Equation and Nonlinear Eigen Value Problems** $(1 < p < 2^* - 1)$. We now examine the same geometrical questions for the initial-value problem for the Fast Diffusion Equation

$$u_t = \Delta u^m, \qquad 0 < m < 1, \tag{4.22}$$

posed in a bounded smooth domain Ω with homogeneous Dirichlet conditions

$$u = 0 \quad \text{on } \partial\Omega, \tag{4.23}$$

and initial data

$$u(x,0) = u_0(x) \text{ nonnegative and bounded.} \tag{4.24}$$

By known regularity theory, we may assume without loss of generality that u_0 is continuous and bounded. If we let $m = \frac{1}{q-1}$ for $q > 2$ and $g = u^m$, we have

$$\begin{cases} (g^{1/m})_t = \Delta g & , \\ g = 0 & \text{on } \partial\Omega \\ g(x,0) = g_o(x) = u_o(x)^m & \text{in } \Omega \end{cases} \tag{4.25}$$

Preliminaries. The main difference with the previous analysis is the finite time convergence of the solutions to the zero solution, which replaces the infinite time stabilization that holds for $m \ge 1$. This phenomenon is called *extinction in finite time* and reads as follows.

Proposition 4.11. *For every initial data u_o as above there exists a classical solution $u(x,t)$ of equation* (4.22) *defined in a strip $Q_T = \Omega \times (0, T^*)$ for some $T^* > 0$, and taking the initial data u_0 in the sense of initial trace in $L^1(\Omega)$. Moreover, as $t \to T^*$, $t < T^*$, we have*

$$\lim_{t \to T^*} \|u(\cdot,t)\|_\infty = 0. \tag{4.26}$$

The solution can be continued past the extinction time T^* in a weak sense as $u \equiv 0$.

The study of extinction was initiated in a famous paper by Berryman and Holland [BH]. Further information is found in [DK, Kw1, DKV]. The following is known.

Proposition 4.12. *Let $g(x,t)$ be the unique weak solution of the problem* (4.25) *where $g_o \in L^\infty(\Omega)$, $g_o \neq 0$, and $g_o \geq 0$. Then $g(x,t)$ is a positive classical solution of the equation in Q_{T^*} where T^*. And we have*

(1) $g \in C^{2,1} \cap L^\infty(Q_{T^*})$ *and* $g > 0$ *in* Q_{T^*}.
(2) $(g^{1/m})_t - \Delta g = 0$
(3) $c_1(T^* - t)^{m/(1-m)} \leq |g(x,t)|_{L^q(\Omega)} \leq C_2(T^* - t)^{m/(1-m)}$
(4) $c_1(T^* - t)^{m/(1-m)} d(x, \partial\Omega) \leq |\nabla g(x,t)| \leq C_2$.

SPECIAL SOLUTIONS AND STABILIZATION. The form of extinction is studied in [BH, Kw2, Kw3] and [BV]. The asymptotic description is based on the existence of appropriate solutions that serve as model for the behavior near extinction: there is a self-similar solution of the form

$$U(x,t;T) = (T-t)^{1/(1-m)} f(x)\,, \tag{4.27}$$

for a certain profile $f > 0$, where $\varphi = f^m$ is the solution of the super-linear elliptic equation

$$-\Delta\varphi(x) = \frac{1}{1-m}\varphi(x)^p, \quad p = \frac{1}{m}. \tag{4.28}$$

such that $\varphi > 0$ in Ω with zero boundary data. Hence, similarity means in this case the separate-variables form. The existence and regularity of this solution depends on the exponent p, indeed it exists for $p < (N+2)/(N-2)$, the Sobolev exponent. Since $p = 1/m$, this means that smooth separate-variables solutions exist for

$$(N-2)/(N+2) < m < 1, \tag{4.29}$$

an assumption that will be kept in the sequel. Note that the family of solutions (4.27) has a free parameter $T > 0$.

STABILIZATION. The above family of solutions allows to describe the behavior of general solutions near their extinction time. Indeed, it is proved that as $t \to T$, the solution stabilizes in the L^∞ norm, after the natural rescaling, to the separate variables profile. We have

Proposition 4.13. *Under the above assumptions on u_o and m we have the following property near the extinction time of a solution $u(x,t)$: for any sequence $\{u(x,t_n)\}$, we have a subsequence $t_{n_k} \to T$ and a $\varphi(x)$ such that*

$$\lim_{k\to\infty} (T - t_{n_k})^{-1/(1-m)} |u(x,t_{n_k}) - U(x,t_{n_k};T)| \to 0 \tag{4.30}$$

uniformly in $x \in \Omega$ for $U(x,t) = (T-t)^{1/(m-1)}\varphi^{1/m}(x)$.

Remarks. (1) The result can also be written as

$$\lim_{k\to\infty} |(T - t_{n_k})^{-1/(1-m)} u(x,t_{n_k}) - \varphi(x)| \to 0$$

(2) A very important observation is that solutions of (4.28) need not be unique. That property depends on the geometry of Ω and on p. Now, when the solution of (4.28) is unique (for instance in a ball), then the limit of $(T-t)^{-1/(1-m)}u(x,t)$ is also unique.

(3) Uniform convergence does not describe accurately the similarity between u and U near the boundary, where both are zero. It is proved in [BV] that the convergence is indeed uniform in relative error in the sense that (considering the uniqueness case for simplicity)

$$\lim_{t\to\infty} \left|\frac{u(x,t)}{U(x,t;T)} - 1\right| \to 0 \tag{4.31}$$

uniformly in Ω.

Square root of pressure. When $0 < m < 1$, the constant r in (4.19) becomes negative and the pressure v goes to infinity as x approaches $\partial\Omega$. Hence, in fast diffusion we will consider square-root convexity of the pressure v.

Set $w = v^{1/2} = g^{\frac{m-1}{2m}}$, where $g = v^{\frac{m}{m-1}} = u^m$. Since $g = u^m$ has a linear behavior near the boundary of Ω, we will have

Lemma 4.14. *When $0 < m < 1$, for every $t > 0$ and as $x \to \partial\Omega(v)$*

$$w_{,\alpha\alpha}(x,t) = \frac{m-1}{2mg^{2-\frac{m-1}{2m}}}\left(gg_{,\alpha\alpha} - \left[1 - \frac{m-1}{2m}\right]g_{,\alpha}^2\right) \geq \frac{\delta_1}{\varepsilon^2}.$$

Lemma 4.15. *Let Ω be a convex bounded domain and let $v_0 \geq 0$ be a continuous and bounded initial function that v_0 vanishes on the boundary. If $\sqrt{v_0}$ is convex, then the solution of the fast diffusion equation, $v(x,t)$, is square root-convex in the space variable for all $t > 0$, i. e., $D^2\sqrt{v(x,t)} \geq 0$.*

Similarly, we have

Corollary 4.16 (Square-root Convexity). *If Ω is convex, there is a stationary pressure profile $h(x) = \varphi(x)^{\frac{1-m}{m}} = \varphi^{1-p}, 1 < p < \frac{n+2}{n-2}$ which is square-root convex. i. e., $D^2\sqrt{h(x)} \geq 0$.*
As a consequence, the level sets of φ are convex.

A small modification of Lemma 4.9 with Lemma 4.15 and Corollary 4.16, allow us to derive the strict square-root convexity of $\varphi^{\frac{m-1}{m}}$.

Lemma 4.17 (Strict Square-root Convexity). *If Ω is smooth and strictly convex, $h(x) = \varphi(x)^{\frac{m-1}{m}} = \varphi^{1-p}, 1 < p < \frac{n+2}{n-2}$ is strictly square root-concave: there exists a constant $c_1 > 0$ such that*

$$D^2\sqrt{h(x)} \geq c_1\,\mathbf{I}. \tag{4.32}$$

The constant c_1 depends only on the shape of Ω. As a consequence, the level sets of φ are strictly convex.

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