数理解析研究所講究録
第 1628 巻 2009 年 101-108

Existence of traveling waves for a nonlocal monostable equation

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July 24, 2008

Abstract We consider the nonlocal analogue of the Fisher-KPP equation

$$u_t = \mu * u - u + f(u),$$

where μ is a Borel-measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and f satisfies $f(0) = f(1) = 0$ and $f > 0$ in $(0, 1)$. We do not assume that μ is absolutely continuous. The equation may have a standing wave solution (a traveling wave solution with speed 0) whose profile is a monotone but discontinuous function. We show that there is a constant c_* such that it has a traveling wave solution with monotone profile and speed c when $c \geq c_*$ while no periodic traveling wave solution with average speed c when $c < c_*$. In order to prove it, we modify a recursive method for abstract monotone discrete dynamical systems by Weinberger. We note that the monotone semiflow generated by the equation does not have compactness with respect to the compact-open topology.

Keywords: discontinuous profile, convolution model, integro-differential equation, discrete monostable equation, nonlocal evolution equation, Fisher-Kolmogorov equation.

AMS Subject Classification: 35K57, 35K65, 35K90, 45J05.

1 Introduction

We consider the following nonlocal analogue of the Fisher-KPP equation:

$$u_t = \mu * u - u + f(u).$$

Here, μ is a Borel-measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and the convolution is defined by

$$(\mu * u)(x) = \int_{y\in\mathbb{R}} u(x-y)d\mu(y)$$

for a bounded and Borel-measurable function u on $\mathbb{R}$. The nonlinearity f is a Lipschitz continuous function with $f(0) = f(1) = 0$ and $f > 0$ in $(0,1)$. Then, we would show that there is a constant c_* such that the nonlocal monostable equation has a traveling wave solution with monotone profile and speed c when $c \geq c_*$ while it has no periodic traveling wave solution with average speed c when $c < c_*$, if there is a positive constant λ satisfying

$$\int_{y\in\mathbb{R}} e^{\lambda|y|}d\mu(y) < +\infty.$$

Here, we say that the solution $u(t,x)$ is *a periodic traveling wave solution with average speed* c, if $u(t+\tau,\cdot) \equiv u(t,\cdot+c\tau)$ holds for some positive constant τ with $0 \leq u(t,\cdot) \leq 1$, $u(t,+\infty) = 1$ and $u(t,\cdot) \not\equiv 1$ for all $t \in \mathbb{R}$. In order to prove this result, we employ the recursive method for monotone dynamical systems introduced by Weinberger [22] and Li, Weinberger and Lewis [14]. We note that the semiflow generated by the nonlocal monostable equation does not have compactness with respect to the compact-open topology. In fact, there is a smooth and monostable nonlinearity f such that the equation has a standing wave solution (i.e., a traveling wave solution with speed 0) whose profile is a monotone but discontinuous function, if μ satisfies the extra condition $\int_{y\in\mathbb{R}} y d\mu(y) > 0$. In our results, we do not assume that μ is absolutely continuous with respect to the Lebesgue measure. For example, not only the integro-differential equation

$$\frac{\partial u}{\partial t}(t,x) = \int_0^1 u(t,x-y)dy - u(t,x) + f(u(t,x))$$

but also the discrete equation

$$\frac{\partial u}{\partial t}(t,x) = u(t,x-1) - u(t,x) + f(u(t,x))$$

satisfies all the assumptions for the measure μ.

For the nonlocal monostable equation, Schumacher [18, 19] proved that there is the minimal speed c_* and the equation has a traveling wave solution with speed c when $c \geq c_*$, if the nonlinearity f satisfies the extra condition

$$f(u) \leq f'(0)u.$$

Recently, Coville, Dávila and Martínez [5] showed that if the monostable nonlinearity $f \in C^1(\mathbb{R})$ satisfies $f'(1) < 0$ and the Borel-measure μ has a density function $J \in C(\mathbb{R})$ with

$$\int_{y\in\mathbb{R}} (|y| + e^{-\lambda y}) J(y) dy < +\infty$$

for some positive constant λ, then there is a constant c_* such that the nonlocal monostable equation has a traveling wave solution with monotone profile and speed c when $c \geq c_*$ while it has no such solution when $c < c_*$. The approach employed in [5] is not of dynamical systems, but they directly solved the stationary problem

$$J * u - u - cu_x + f(u) = 0, \quad u(-\infty) = 0, \quad u(+\infty) = 1.$$

When " 1. Introduction and main results " in [5] was read, it might be misunderstood that Schumacher [18] and Weinberger [22] assumed the isotropy of dynamical systems. The nonlocal equation is isotropic if and only if μ is symmetric with respect to the origin. Here, to make sure, we note that the isotropy is not assumed in the results by [18] and [22]. Further, the result by [22] is not limited at a linear determinate. If $f(u) \leq f'(0)u$ holds, then it is a linear determinate. See, e.g., [2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 20, 21, 23, 24] on traveling waves and long-time behavior in various monostable evolution systems, [1, 3] nonlocal bistable equations and [17] Euler equation.

The proof of our results is given in [25] or [26], and it is self-contained. We would believe that it might be rather simple than in [5].

2 Abstract theorems for monotone semiflows

In the abstract, we would treat a monostable evolution system. Put a set of functions on $\mathbb{R}$;

$$\mathcal{M} := \{u \,|\, u \text{ is a monotone nondecreasing}$$

$$\text{and left continuous function on } \mathbb{R} \text{ with } 0 \leq u \leq 1\}.$$

The followings are our basic conditions for discrete dynamical systems:

Hypotheses 1 *Let Q_0 be a map from $\mathcal{M}$ into $\mathcal{M}$.*

(i) *Q_0 is continuous in the following sense: If a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ converges to $u \in \mathcal{M}$ uniformly on every bounded interval, then the sequence $\{Q_0[u_k]\}_{k\in\mathbb{N}}$ converges to $Q_0[u]$ almost everywhere.*

(ii) *Q_0 is order preserving; i.e.,*

$$u_1 \le u_2 \Longrightarrow Q_0[u_1] \le Q_0[u_2]$$

for all u_1 and $u_2 \in \mathcal{M}$. Here, $u \le v$ means that $u(x) \le v(x)$ holds for all $x \in \mathbb{R}$.

(iii) *Q_0 is translation invariant; i.e.,*

$$T_{x_0}Q_0 = Q_0T_{x_0}$$

for all $x_0 \in \mathbb{R}$. Here, T_{x_0} is the translation operator defined by $(T_{x_0}[u])(\cdot) := u(\cdot - x_0)$.

(iv) *Q_0 is monostable; i.e.,*

$$0 < \alpha < 1 \Longrightarrow \alpha < Q_0[\alpha]$$

for all constant functions α.

The following states that existence of suitable *super*-solutions of the form $\{v_n(x+cn)\}_{n=0}^{\infty}$ implies existence of traveling wave solutions with speed c in the discrete dynamical systems on $\mathcal{M}$:

Proposition 2 *Let a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfy* Hypotheses 1, *and $c \in \mathbb{R}$. Suppose there exists a sequence $\{v_n\}_{n=0}^{\infty} \subset \mathcal{M}$ with $(Q_0[v_n])(x-c) \le v_{n+1}(x)$, $\inf_{n=0,1,2,\cdots} v_n(x) \not\equiv 0$ and $\liminf_{n\to\infty} v_n(x) \not\equiv 1$. Then, there exists $\psi \in \mathcal{M}$ with $(Q_0[\psi])(x-c) \equiv \psi(x)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.*

In the discrete dynamical system on $\mathcal{M}$ generated by a map Q_0 satisfying Hypotheses 1, if there is a *periodic* traveling wave *super*-solution with *average* speed c, then there is a traveling wave solution with speed c:

Theorem 3 *Let a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfy* Hypotheses 1, *and $c \in \mathbb{R}$. Suppose there exist $\tau \in \mathbb{N}$ and $\phi \in \mathcal{M}$ with $({Q_0}^{\tau}[\phi])(x-c\tau) \le \phi(x)$, $\phi \not\equiv 0$ and $\phi \not\equiv 1$. Then, there exists $\psi \in \mathcal{M}$ with $(Q_0[\psi])(x-c) \equiv \psi(x)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.*

The infimum c_* of the speeds of traveling wave solutions is not $-\infty$, and there is a traveling wave solution with speed c when $c \ge c_*$:

Theorem 4 *Suppose a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfies* Hypotheses 1. *Then, there exists $c_* \in (-\infty, +\infty]$ such that the following holds :*

Let $c \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $(Q_0[\psi])(x - c\tau) \equiv \psi(x)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ if and only if $c \geq c_*$.

We add the following conditions to Hypotheses 1 for continuous dynamical systems on $\mathcal{M}$:

Hypotheses 5 *Let Q^t be a map from $\mathcal{M}$ to $\mathcal{M}$ for $t \in [0, +\infty)$.*

(i) *Q is a semigroup; i.e., $Q^t \circ Q^s = Q^{t+s}$ for all t and $s \in [0, +\infty)$.*

(ii) *Q is continuous in the following sense: Suppose a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0, +\infty)$ converges to 0, and $u \in \mathcal{M}$. Then, the sequence $\{Q^{t_k}[u]\}_{k\in\mathbb{N}}$ converges to u almost everywhere.*

As we would have Theorems 3 and 4 for the discrete dynamical systems, we would have the following two for the continuous dynamical systems:

Theorem 6 *Let Q^t be a map from $\mathcal{M}$ to $\mathcal{M}$ for $t \in [0, +\infty)$. Suppose Q^t satisfies* Hypotheses 1 *for all $t \in (0, +\infty)$, and Q* Hypotheses 5. *Then, the following holds :*

Let $c \in \mathbb{R}$. Suppose there exist $\tau \in (0, +\infty)$ and $\phi \in \mathcal{M}$ with $(Q^\tau[\phi])(x - c\tau) \leq \phi(x)$, $\phi \not\equiv 0$ and $\phi \not\equiv 1$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $(Q^t[\psi])(x - ct) \equiv \psi(x)$ holds for all $t \in [0, +\infty)$.

Theorem 7 *Let Q^t be a map from $\mathcal{M}$ to $\mathcal{M}$ for $t \in [0, +\infty)$. Suppose Q^t satisfies* Hypotheses 1 *for all $t \in (0, +\infty)$, and Q* Hypotheses 5. *Then, there exists $c_* \in (-\infty, +\infty]$ such that the following holds :*

Let $c \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $(Q^t[\psi])(x - ct) \equiv \psi(x)$ holds for all $t \in [0, +\infty)$ if and only if $c \geq c_*$.

3 A key lemma to prove the abstract theorems

To prove the theorems stated in Section 2, we would modify the recursive method introduced by Weinberger [22] and Li, Weinberger and Lewis [14]. At that time, the following lemma becomes a key. It states that Hypotheses 1 imply more strong continuity than Hypothesis 1 (i):

Lemma 8 *Let a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfy* Hypotheses 1 (i), (ii) *and* (iii). *Suppose a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ converges to $u \in \mathcal{M}$ almost everywhere. Then,* $\lim_{k\to\infty}(Q_0[u_k])(x) = (Q_0[u])(x)$ *holds for all continuous points $x \in \mathbb{R}$ of $Q_0[u]$.*

4 The main results for the nonlocal monostable equation

Let a Lipschitz continuous function f on $\mathbb{R}$ be a monostable nonlinearity; $f(0) = f(1) = 0$ and $f(u) > 0$ in $(0,1)$. Let a Borel-measure μ on $\mathbb{R}$ satisfy $\mu(\mathbb{R}) = 1$. (We do not assume that μ is absolutely continuous with respect to the Lebesgue measure.) Then, we consider the following nonlocal monostable equation:

$$u_t = \mu * u - u + f(u), \tag{4.1}$$

where $(\mu * u)(x) := \int_{y\in\mathbb{R}} u(x-y)d\mu(y)$ for a bounded and Borel-measurable function u on $\mathbb{R}$. Then, $G(u) := \mu * u - u + f(u)$ is a map from the Banach space $L^\infty(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and it is Lipschitz continuous. (We note that $u(x-y)$ is a Borel-measurable function on $\mathbb{R}^2$, and $\|u\|_{L^\infty(\mathbb{R})} = 0$ implies $\|\mu * u\|_{L^1(\mathbb{R})} \leq \int_{y\in\mathbb{R}}(\int_{x\in\mathbb{R}} |u(x-y)|dx)d\mu(y)=0$.) So, because the standard theory of ordinary differential equations works, we have well-posedness of (4.1) and the equation generates a flow in $L^\infty(\mathbb{R})$. Here, we recall that $\mathcal{M}$ has been defined at the beginning of Section 2.

If the semiflow generated by (4.1) has a *periodic* traveling wave solution with *average* speed c (even if the profile is not a monotone function), then it has a traveling wave solution with *monotone* profile and speed c:

Theorem 9 *Let a Borel-measure μ have $\lambda \in (0,+\infty)$ satisfying*

$$\int_{y\in\mathbb{R}} e^{\lambda|y|}d\mu(y) < +\infty, \tag{4.2}$$

and $c \in \mathbb{R}$. Suppose there exist $\tau \in (0,+\infty)$ and a solution $\{u(t,x)\}_{t\in\mathbb{R}} \subset L^\infty(\mathbb{R})$ to (4.1) with $0 \leq u(t,x) \leq 1$, $\lim_{x\to+\infty} u(t,x) = 1$ and $\|u(t,x) - 1\|_{L^\infty(\mathbb{R})} \neq 0$ such that

$$u(t+\tau, x) = u(t, x+c\tau)$$

holds for all t and $x \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x+ct)\}_{t\in\mathbb{R}}$ is a solution to (4.1).

The infimum c_* of the speeds of traveling wave solutions is not $\pm\infty$, and there is a traveling wave solution with speed c when $c \geq c_*$:

Theorem 10 *Let a Borel-measure μ have $\lambda \in (0, +\infty)$ satisfying (4.2). Then, there exists $c_* \in \mathbb{R}$ such that the following holds :*

Let $c \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x + ct)\}_{t\in\mathbb{R}}$ is a solution to (4.1) if and only if $c \geq c_*$.

Acknowledgments. I thank Prof. Hiroshi Matano, Dr. Xiaotao Lin and Dr. Masahiko Shimojo for their discussion.

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