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A two-stage approach for Russell measure in DEA

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1 Introduction

DEA models are commonly recast as a linear programming (LP) problem, which is easy to compute the efficiency measurement and to find an optimal projection and and a reference set. Their outputs of the LP problem are useful to measure activity performance level of each Decision making unit (DMU) and to propose feasible improvement targets for an inefficient DMU. Computational practicality of the DEA models plays a key role of the popularity over various applications. The LP formulation appears in radial measurement DEA models (e.g., CCR and BCC) and non-radial ones (e.g., AM, RAM, SBM and ERGM).

Development of the axiomatic foundations of efficiency measurement began with Färe and Lovell [5], who suggested three desirable axioms for efficiency indices: homogeneity, monotonicity, and indication of efficient input/output vectors. To meet the axiomatic foundations, Färe and Lovell [5] introduced an input-oriented non-radial model which was later extended in Färe et al [6] into a jointly aggregate measure of output and input efficiency. The DEA model extended by Färe et al [6] is not reduced to the LP formulation. The efficiency measure of the model is referred to as Russell Measure (RM).

The nonlinearity of RM is an obstacle for DEA users. Therefore, various modified RM are developed in [3, 11, 12] and used in [7, 9, 10, 17]. For example, Pastor et al [11] propose Enhanced Russell Graph Efficiency Measure (ERGM) incorporating the analytical feature of RM into the LP formulation.

The LP formulations can not avoid an occurrence of multiple optimal solutions. The existence of multiple optimal solutions is a potential problem in a multi-stage approach such as the two-phase computational procedure for finding a max-slack solution in the radial DEA models [4, 14] and Returns to scale (RTS) measurement in the non-radial measurement DEA models [1, 13]. Inputs of the last stage of the multi-stage approach are optimal solutions that are obtained in the preceding stages. Hence, the last stage output depends on the choice of optimal solutions in the preceding stages. If some preceding stages have multiple optimal solutions and an alternative one is chosen, the last stage may reach a different conclusion.

Recently, such a shortcoming of the LP formulation is overcome by Sueyoshi and Sekitani, who develop a primal-dual DEA approach [13, 14] for checking occurrence of multiple optimal solutions of RAM or radial DEA models.

Does the RM including nonlinearity also have the shortcoming? The corresponding answer is "yes" since the occurrence of multiple reference sets is independent of the choice of an efficiency measurement [14]. Therefore, the RM DEA model also suffers from the problem of multiple optimal solutions. Moreover, the primal-dual DEA approach [13, 14] can not be directly applied to the RM DEA model because of the difference between the LP and non-LP formulations.

The aim of this paper is to analyze the nonlinearity of RM and to develop a computational method of both solving the RM DEA model and checking the uniqueness of an optimal projection and and a reference set. As will be shown, the nonlinearity of RM is a convex function and the RM DEA model is a convex programming problem that can be exactly solved by a standard optimization technique such as a steepest descent method or the newton method [2]. Therefore, DEA users may deal with the RM DEA model as easily as the LP formulations of DEA models.

This paper is organized as follows: Section 2 introduces a nonlinear formulation of the RM DEA model and a certain type of the uniqueness of its optimal solution. Section 3 shows a necessary and sufficient condition of the occurrence of multiple optimal solutions, that lead to multiple projection points and multiple reference sets. The multiple reference sets are independent of the choice of an efficiency measurement. Section 4 proposes a two-stage approach and shows how to solve the RM DEA model for an illustrative example The final section concludes with future research.

2 Some properties of Russell measure

Suppose that we have n DMUs (decision making units) where each DMU$_j$, $j = 1, \dots, n$, produces the same s outputs in (possibly) different amounts, y_{rj} $(r = 1, \dots, s)$, using the same m inputs, x_{ij} $(i = 1, \dots, m)$, also in (possibly) different amounts. In the sequel, we assume that every DMU$_j$ has $x_{ij} > 0$ for all $i = 1, \dots, m$ and $y_{rj} > 0$ for all $r = 1, \dots, s$. The specific DMU to be currently evaluated is listed by the subscript "k".

Following Färe and Lovell [5] and Cooper et al [3]; the RM for the kth DMU $(k = 1, \dots, n)$ can be formulated as follows:

$$\begin{array}{lll} \min & \dfrac{1}{m+s}\left(\displaystyle\sum_{i=1}^{m}\theta_i + \sum_{r=1}^{s}\frac{1}{\phi_r}\right) & \\ \text{s.t.} & -\displaystyle\sum_{j=1}^{n} x_{ij}\lambda_j + \theta_i x_{ik} \geq 0 & (i = 1, \dots, m) \\ & \displaystyle\sum_{j=1}^{n} y_{rj}\lambda_j - \phi_r y_{rk} \geq 0 & (r = 1, \dots, s) \\ & \theta_i \leq 1 & (i = 1, \dots, m) \\ & \phi_r \geq 1 & (r = 1, \dots, s) \\ & \lambda_j \geq 0 & (j = 1, \dots, n). \end{array} \tag{1}$$

The variables (θ_i and ϕ_r) indicate the level of efficiency/inefficiency related to the ith input and the rth output, respectively. The variables (λ_j for $j = 1, \dots, n$) are used for a structural connection among DMUs in the input-output space.

The RM model (1) attains the equalities of the first and second inequality constraints as follows:

Proposition 1. *Let $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$ be an optimal solution of (1), then we have*

$$-\sum_{j=1}^{n} x_{ij}\lambda_j^* + \theta_i^* x_{ik} = 0 \text{ for all } i = 1, \dots, m \tag{2}$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j^* - \phi_r^* y_{rk} = 0 \text{ for all } r = 1, \dots, s. \tag{3}$$

For a given optimal solution of the RM (1) model, we define a projection point of DMU$_k$ as follows:

Definition 1. *For an optimal solution $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$ of (1), we define a pair of*

$$\boldsymbol{x}^* = (\theta_1^* x_{1k}, \cdots, \theta_m^* x_{mk}), \ \boldsymbol{y}^* = (\phi_1^* y_{1k}, \cdots, \phi_s^* y_{sk}) \tag{4}$$

as a projection of DMU$_k$. A projection of DMU$_k$ is denoted by $(\boldsymbol{x}^*, \boldsymbol{y}^*)$.

Note that a projection point depends on the choice of an optimal solution of the RM (1) model. Hence, the projection point is not unique if multiple optimal solutions occurs on the RM (1) model. This paper defines a production possibility P set and its efficiency frontier EF as follows:

Definition 2. *The production possibility set is defined as*

$$P = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \;\middle|\; \begin{array}{ll} x_i \geq \sum_{j=1}^{n} x_{ij}\lambda_j & i = 1, \ldots, m \\ y_r \leq \sum_{j=1}^{n} y_{rj}\lambda_j & r = 1, \ldots, s \\ \lambda_j \geq 0 & j = 1, \ldots, n \end{array} \right\}. \tag{5}$$

The efficiency frontier is defined as

$$EF = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in P \;\middle|\; \begin{array}{l} \text{there is no } (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in P \\ \text{such that } (-\boldsymbol{x}, \boldsymbol{y}) \leq (-\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \\ \text{and } (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \neq (\boldsymbol{x}, \boldsymbol{y}) \end{array} \right\}. \tag{6}$$

The next proposition guarantees that any projection point of the RM model (1) always exists on the efficiency frontier EF:

Proposition 2. *Any projection of DMU_k belongs to the efficiency frontier. That is $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in EF$.*

The RM model (1) has an advantage of the existence of any projection point on EF that is not guaranteed on any radial measurement such as CCR or BCC.

As defined in (4), the projection$(\boldsymbol{x}^*, \boldsymbol{y}^*)$ depends on $\boldsymbol{\lambda}^*$ of an optimal solution $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$, however, $\boldsymbol{\lambda}^*$ is not in one to one correspondence with the projection pint $(\boldsymbol{x}^*, \boldsymbol{y}^*)$. This phenomenon is caused by multiple reference sets [13]. Note that multiple reference sets may occurs on any DEA model. The remaining part of the optimal solution $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$, that is $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)$, is in one to one correspondence with $(\boldsymbol{x}^*, \boldsymbol{y}^*)$.

Proposition 3. *Let $(\bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\phi}}^*, \bar{\boldsymbol{\lambda}}^*)$ and $(\hat{\boldsymbol{\theta}}^*, \hat{\boldsymbol{\phi}}^*, \hat{\boldsymbol{\lambda}}^*)$ be distinct optimal solutions of (1). Let $(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{y}}^*)$ be a projection of DMU_k generated by $(\bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\phi}}^*)$ and $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$ by $(\hat{\boldsymbol{\theta}}^*, \hat{\boldsymbol{\phi}}^*)$, then $(\bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\phi}}^*) \neq (\hat{\boldsymbol{\theta}}^*, \hat{\boldsymbol{\phi}}^*)$ if and only if $(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{y}}^*) \neq (\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$.*

The objective function of the RM model (1) is a convex function and the output-part of the objective function is a strictly convex function.

Proposition 4. *Let*

$$(\boldsymbol{\theta}, \boldsymbol{\phi}) = (\theta_1, \cdots, \theta_m, \phi_1, \cdots, \phi_s) \tag{7}$$

and $\Omega = \{(\boldsymbol{\theta}, \boldsymbol{\phi}) \mid (\boldsymbol{\theta}, \boldsymbol{\phi}) > 0\}$. Let

$$f(\boldsymbol{\theta}, \boldsymbol{\phi}) = \frac{1}{m+s} \left(\sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} \frac{1}{\phi_r} \right) \tag{8}$$

and suppose that $(\boldsymbol{\theta}^1, \boldsymbol{\phi}^1)$, $(\boldsymbol{\theta}^2, \boldsymbol{\phi}^2) \in \Omega$, then

(i) $f\left((1-\alpha)(\boldsymbol{\theta}^1, \boldsymbol{\phi}^1) + \alpha(\boldsymbol{\theta}^2, \boldsymbol{\phi}^2)\right) \leq (1-\alpha) f\left(\boldsymbol{\theta}^1, \boldsymbol{\phi}^1\right) + \alpha f\left(\boldsymbol{\theta}^2, \boldsymbol{\phi}^2\right)$ *for all* $\alpha \in (0, 1)$.

(ii) $f\left((1-\alpha)(\boldsymbol{\theta}^1, \boldsymbol{\phi}^1) + \alpha(\boldsymbol{\theta}^2, \boldsymbol{\phi}^2)\right) = (1-\alpha) f(\boldsymbol{\theta}^1, \boldsymbol{\phi}^1) + \alpha f(\boldsymbol{\theta}^2, \boldsymbol{\phi}^2) =$ *for all* $\alpha \in (0,1) \iff \phi_r^1 = \phi_r^2$ *for all* $r = 1, \cdots, s$.

(iii) $f\left((1-\alpha)(\boldsymbol{\theta}^1,\boldsymbol{\phi}^1)+\alpha(\boldsymbol{\theta}^2,\boldsymbol{\phi}^2)\right) < (1-\alpha)f\left(\boldsymbol{\theta}^1,\boldsymbol{\phi}^1\right)+\alpha f\left(\boldsymbol{\theta}^2,\boldsymbol{\phi}^2\right)$ *for all* $\alpha \in (0,1)$, $\Longleftrightarrow$ *there exists an index* $\bar{r} \in \{1,\cdots,s\}$ *such that* $\phi^1_{\bar{r}} \neq \phi^2_{\bar{r}}$.

Proposition 4 means that the RM model (1) has a unique optimal output-part $\boldsymbol{\phi}^*$ of all optimal solutions $(\boldsymbol{\theta}^*,\boldsymbol{\phi}^*,\boldsymbol{\lambda}^*)$. Furthermore, it follows from Proposition 3 that the projection point $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ has the uniqueness of an output-part $\boldsymbol{y}^*$.

The RM model (1) has two advantages of the existence of all projection points on the efficiency frontier EF and the uniqueness of output-part of all projection points. The uniqueness is caused by the non-linear term of the output-part, $\sum_{r=1}^{s} 1/\phi_r$, in the objective function of (1). Since the output-part $\sum_{r=1}^{s} 1/\phi_r$ is strictly convex, the RM model (1) can be solved exactly. In fact, Sueyoshi and Sekitani [15] show the solvability of the RM model (1) by reducing (1) into a Second-Order Cone Programming (SOCP) problem.

3 A characterization of multiple solutions

This section discusses examination of multiple projections and finding of multiple reference sets. Since the RM model (1) is a convex programming problem, it is easy to find an optimal solution of (1) by using the steepest descent algorithm [2] which is a typical nonlinear programming method, instead of SOCP.

Let $(\boldsymbol{\theta}^*,\boldsymbol{\phi}^*,\boldsymbol{\lambda}^*)$ be an optimal solution of (1), then the problem (1) replacing $\boldsymbol{\phi}$ with $\boldsymbol{\phi}^*$ is equivalent to

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m} \theta_i & \\
\text{s.t.} & -\sum_{j=1}^{n} x_{ij}\lambda_j + \theta_i x_{ik} \geq 0 & (i=1,\dots,m) \\
& \sum_{j=1}^{n} y_{rj}\lambda_j - y^*_r \geq 0 & (r=1,\dots,s) \\
& 0 \leq \theta_i \leq 1 & (i=1,\dots,m) \\
& \lambda_j \geq 0 & (j=1,\dots,n).
\end{array}
\tag{9}
$$

where $\boldsymbol{y}^* = (\phi^*_1 y_{1k},\cdots,\phi^*_s y_{sk})$.

Any optimal solution of the RM model (1) satisfies the following properties:

Proposition 5. *Let* $(\boldsymbol{\theta}^*,\boldsymbol{\phi}^*,\boldsymbol{\lambda}^*)$ *be an optimal solutions of (1) and let* γ^* *be the optimal objective function value of (9), then* $\sum_{i=1}^{m}\theta^*_i = \gamma^*$ *and* $\theta^*_i > 0$ *for all* $i = 1,\dots,m$.

From propositions 1 and 5, a projection point set Ω_k of DMU$_k$ is equivalently given as follows:

$$
\left\{ (\theta_1 x_{1k},\cdots,\theta_m x_{mk},\boldsymbol{y}^*) \;\middle|\; \begin{array}{l}
\sum_{i=1}^{m}\theta_i = \sum_{i=1}^{m}\theta^*_i \\
\sum_{j=1}^{n} x_{ij}\lambda_j = \theta_i x_{ik} \\
\quad i=1,\dots,m \\
\sum_{j=1}^{n} y_{rj}\lambda_j = y^*_r \\
\quad r=1,\dots,s, \\
0 \leq \theta_i \leq 1 \\
\quad i=1,\dots,m, \\
\lambda_j \geq 0 \\
\quad j=1,\dots,n
\end{array} \right\}. \tag{10}
$$

The linear programming problem (9) is a dual form of

$$
\begin{array}{lll}
\max. & \sum_{r=1}^{s} w_r y_r^* - \sum_{i=1}^{m} u_i & \\
\text{s.t.} & -\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} w_r y_{rj} \leq 0 & j = 1, \dots, n \\
& v_i x_{ik} - u_i \leq 1 & i = 1, \dots, m \\
& v_i \geq 0 & i = 1, \dots, m \\
& w_r \geq 0 & r = 1, \dots, s \\
& u_i \geq 0 & i = 1, \dots, m.
\end{array}
\tag{11}
$$

Furthermore, it follows from Proposition 5 that the dual problem (11) can be simplified by eliminating dual variables $\boldsymbol{u}$ and it is reduced into (12) as follows:

Proposition 6. *Let $(\boldsymbol{v}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)$ be an optimal solution of (11) and let $(\boldsymbol{v}^\#, \boldsymbol{w}^\#)$ be an optimal solution of*

$$
\begin{array}{lll}
\max. & \sum_{r=1}^{s} w_r y_r^* - \sum_{i=1}^{m} v_i x_{ik} + m & \\
s.t. & -\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} w_r y_{rj} \leq 0 & j = 1, \dots, n \\
& v_i x_{ki} \geq 1 & i = 1, \dots, m \\
& w_r \geq 0 & r = 1, \dots, s
\end{array}
\tag{12}
$$

then $(\boldsymbol{v}^*, \boldsymbol{w}^*)$ is an optimal solution of (12) and $(\boldsymbol{v}^\#, \boldsymbol{w}^\#, \boldsymbol{u}^\#)$ is an optimal solution of (11), where $u_i^\# = v_i^\# x_{ik} - 1$ for all $i = 1, \dots, m$.

The pair problem between (11) and (9) satisfies the Strong Complementary Slackness Conditions (SCSC):
There exists a pair of an optimal solution $(\bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\lambda}})$ of (9) and an optimal solution $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{u}})$ of (11) such that

$$\bar{\lambda}_j + \sum_{i=1}^{m} \bar{v}_i x_{ij} - \sum_{r=1}^{s} \bar{w}_r y_{rj} > 0 \quad j = 1, \dots, n \tag{13}$$

$$\bar{v}_i - \sum_{j=1}^{n} x_{ij} \bar{\lambda}_j + \bar{\theta}_i x_{ik} > 0 \quad i = 1, \dots, m \tag{14}$$

$$\bar{w}_r + \sum_{j=1}^{n} y_{rj} \bar{\lambda}_j - y_r^* > 0 \quad r = 1, \dots, s \tag{15}$$

$$(\bar{v}_i x_{ik} - 1) + (1 - \bar{\theta}_i) = \bar{v}_i x_{ik} - \bar{\theta}_i > 0 \quad i = 1, \dots, m \tag{16}$$

From Proposition 6, the above SCSC also are valid between (12) and (9). It follows from Proposition 3, (14) and (15) that $\bar{\boldsymbol{v}} > 0$ and $\bar{\boldsymbol{w}} > 0$. Two conditions (13) and (16) play the key role of characterizing multiple optimal solutions as follows:

Proposition 7. *Let $(\bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\lambda}}^*)$ and $(\bar{\boldsymbol{v}}^*, \bar{\boldsymbol{w}}^*)$ be an optimal solution pair of (9) and (12) satisfying (13) and (16). Choose arbitrarily an optimal solution $(\boldsymbol{\theta}^*, \boldsymbol{\lambda}^*)$ of (9) and an optimal solution*

$(\boldsymbol{v}^*, \boldsymbol{w}^*)$ *of (11), then*

$$\begin{aligned}\left\{j \,\middle|\, \lambda_j^* > 0\right\} \subseteq\ & \left\{j \,\middle|\, \bar{\lambda}_j^* > 0\right\} \\ &= \{j \mid \textstyle\sum_{i=1}^m \bar{v}_i^* x_{ij} = \sum_{r=1}^s \bar{w}_r^* y_{rj}\} \\ &\subseteq \{j \mid \textstyle\sum_{i=1}^m v_i^* x_{ij} = \sum_{r=1}^s w_r^* y_{rj}\}\end{aligned} \tag{17}$$

and

$$\left\{i \,\middle|\, \bar{\theta}_i^* = 1\right\} \subseteq \left\{i \,\middle|\, \theta_i^* = 1\right\}. \tag{18}$$

Proposition 7 means that any optimal solution $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$ of the RM model (1) satisfies

$$\left\{j \,\middle|\, \lambda_j^* > 0\right\} \subseteq \left\{j \,\middle|\, \bar{\lambda}_j^* > 0\right\} \tag{19}$$

and

$$\left\{i \,\middle|\, \bar{\theta}_i^* = 1\right\} \subseteq \left\{i \,\middle|\, \theta_i^* = 1\right\} \tag{20}$$

for a optimal solution $(\bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\lambda}}^*)$ specified in Proposition 7. The set inclusion of (19) implies that $\left\{j \,\middle|\, \bar{\lambda}_j^* > 0\right\}$ is a unique maximal of any reference set $\left\{j \,\middle|\, \lambda_j^* > 0\right\}$. Similarly, the set inclusion of (20) implies that $\{i \mid \bar{\theta}_i^* = 1\}$ is a unique minimal of $\left\{j \,\middle|\, \bar{\lambda}_j^* > 0\right\}$. By using the maximal reference set $\left\{j \,\middle|\, \bar{\lambda}_j^* > 0\right\}$ and the minimum set $\{i \mid \bar{\theta}_i^* = 1\}$, we develop two discriminant equations for multiple projection and multiple reference sets as follows:

Proposition 8. *Suppose that $(\bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\lambda}}^*)$ is an optimal solution of (9) satisfying (13) and (16) for some optimal solution of (12). Let*

$$\begin{aligned} J^+ &= \{j \mid \bar{\lambda}_j^* > 0\} \qquad \textit{and} & (21)\\ I^+ &= \{i \mid \bar{\theta}_i^* = 1\}. & (22)\end{aligned}$$

Let $X^+ = [\boldsymbol{x}_j \mid j \in J^+]$ and $Y^+ = [\boldsymbol{y}_j \mid j \in J^+]$. Let $\mathbf{e}_i$ be a unit vector whose ith element is 1 *and let $\mathbf{e} = \sum_{i=1}^m \mathbf{e}_i^\top$. Let $E^+ = [\mathbf{e}_i \mid i \in I^+]^\top$ and let* $\operatorname{diag}(\boldsymbol{x}_k) = \begin{bmatrix} x_{1k} & & 0 \\ & \ddots & \\ 0 & & x_{mk}\end{bmatrix}$. *Let $|I^+|$ be the number of elements of I^+ and let* $\mathbf{1}$ *be an $|I^+|$ vector whose element is all 1, then consider a linear equation system*

$$\begin{bmatrix} -X^+ & \operatorname{diag}(\boldsymbol{x}_k) \\ Y^+ & \mathbf{0} \\ \mathbf{0} & E^+ \\ \mathbf{0} & \boldsymbol{e}\end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}^+ \\ \boldsymbol{\theta}\end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}^* \\ \mathbf{1} \\ \sum_{i\in I} \bar{\theta}_i^*\end{bmatrix}, \tag{23}$$

where $\boldsymbol{\lambda}^+ = (\lambda_j \mid j \in J^+)^\top$. Let $\operatorname{rank}(k)$ *be the rank of the coefficient matrix* $\begin{bmatrix} -X^+ & \operatorname{diag}(\boldsymbol{x}_k) \\ Y^+ & \mathbf{0} \\ \mathbf{0} & E^+ \\ \mathbf{0} & \boldsymbol{e}\end{bmatrix}$ *of (23) and let $(\boldsymbol{\theta}^*, \boldsymbol{\lambda}^*)$ be an optimal solution of (9), then*

1. $(\boldsymbol{\theta}^*, \boldsymbol{\lambda}^*)$ *is unique if and only if*

$$\text{rank}(k) = |J^+| + m. \tag{24}$$

That is, DMU_k has a unique projection and a unique reference set.

2. *Let $\boldsymbol{x}_i^+$ be the ith row vector of X^+ and let X^{++} be an $|I^+| \times |J^+|$ matrix whose row vectors are $\{\boldsymbol{x}_i^+ \mid i \in I^+\}$, then*

$$\text{rank}\begin{bmatrix} X^+ \\ Y^+ \end{bmatrix} = \text{rank}\begin{bmatrix} X^{++} \\ Y^+ \\ \sum_{i \in I \setminus I^+} \boldsymbol{x}_i^+ / x_{ki} \end{bmatrix} \tag{25}$$

if and only if $\boldsymbol{\theta}^*$ is unique. That is, DMU_k has.a unique projection.

4 Finding an optimal solution with two conditions of SCSC

Consider a linear programming problem combined with (9) and (12) as follows:

$$\begin{array}{lll}
\max. & \eta & \\
\text{s.t.} & \text{all constraints of (9).} & \\
 & \text{all constraints of (12).} & \\
 & \displaystyle\sum_{i=1}^{m} \theta_i = \sum_{r=1}^{s} w_r y_r^* - \sum_{i=1} v_i x_{ik} + m & \\
 & \displaystyle\lambda_j + \sum_{i=1} v_i x_{ij} - \sum_{r=1}^{s} w_r y_{rj} \geq \eta & j = 1, \dots, n \\
 & v_i x_{ik} - \theta_i \geq \eta & i = 1, \dots, m.
\end{array} \tag{26}$$

Any optimal solution $(\bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\lambda}})$ of the primal problem (9) and any optimal solution $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{w}})$ of the dual problem (12) provides a feasible solution $(\bar{\eta}, \bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{w}})$ of the model (26) for some $\bar{\eta} \geq 0$ and, vice versa. Hence, the model (26) is a problem to find an optimal solution pair of (9) and (12) satisfying two conditions (13) and (16) of SCSC. The existence of an optimal solution pair of (9) and (12) satisfying SCSC guarantees the existence of a feasible solution $(\bar{\eta}, \bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{w}})$ of (26) with $\bar{\eta} > 0$. Since a set $\{\bar{\theta} | (\bar{\theta}, \bar{\lambda})$ is an optimal solution of (9) $\}$ is compact, the model (26) has an optimal solution. These properties of (26) are summarized into the following proposition:

Proposition 9. *Let $(\bar{\eta}^*, \bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\lambda}}^*, \bar{\boldsymbol{v}}^*, \bar{\boldsymbol{w}}^*)$ be an optimal solution of (26), then $\bar{\eta}^* > 0$ and, $(\bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\lambda}}^*)$ and $(\bar{\boldsymbol{v}}^*, \bar{\boldsymbol{w}}^*)$ are an optimal solution of (9) and (12), respectively. Hence, any pair of $(\bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\lambda}}^*)$ and $(\bar{\boldsymbol{v}}^*, \bar{\boldsymbol{w}}^*)$ satisfies (13) and (16).*

We propose a two-stage approach of the RM model (1):

Stage 1: Solve a convex programming problem (1) by the steepest descent method and let the optimal solution be $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$. Set $\boldsymbol{y}^* = (\phi_1^* y_{k1}, \cdots, \phi_s^* y_{js})^\top$.

Stage 2: Solve a linear programming problem (26) and let the optimal solution be $(\bar{\eta}^*, \bar{\boldsymbol{\theta}}^*, \bar{\boldsymbol{\lambda}}^* . \bar{\boldsymbol{v}}^*, \bar{\boldsymbol{w}}^*)$. Set $\boldsymbol{x}^* = (\bar{\theta}_1^* x_{1k}, \cdots, \bar{\theta}_m^* x_{km})^\top$. If (24) is valid, then the optimal solution of (1) is unique. If (24) is invalid but (25) is valid, then the projection point $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is unique and multiple reference sets occurs. If (24) and (25) are invalid, multiple reference sets and multiple projection occurs.

In order to illustrate visually the existence of multiple optimal solutions and how to check it numerically, we consider an example of 5 DMUs with three inputs and two outputs, whose data is given in Table 1:

Table 1: Input-Output data

DMU	A	B	C	D	E
x_1	2	4	5	1	3
x_2	2	4	1	5	3
x_3	1	1	1	1	1
y_1	4	8	6	6	6
y_2	1	1	1	1	1
Eff. Score	1	1	1	1	1

The last row of Table 1 indicates efficiency scores by the RM model (1) as follows:

$$\begin{aligned}
\min \quad & \frac{1}{3+2}\left(\sum_{i=1}^{2}\theta_i + 1 + \frac{1}{\phi_1} + \frac{1}{1}\right) \\
\text{s.t.} \quad & -\sum_{j=A}^{E} x_{ij}\lambda_j + \theta_i x_{ik} \geq 0 \qquad\qquad i = 1,2 \\
& \sum_{j=A}^{E} y_{1j}\lambda_j - \phi_1 y_{1k} \geq 0 \\
& \sum_{j=A}^{E} \lambda_j = 1, \qquad \lambda_j \geq 0 \; j = A,\ldots,E \\
& \theta_i \leq 1 \; i = 1,2 \qquad \phi_1 \geq 1.
\end{aligned} \tag{27}$$

Since $x_{3j} = y_{2j} = 1$ for all DMUs, $\theta_3 \leq 1$ and $\phi_2 \geq 1$, $\sum_{j=A}^{E} x_{3j}\lambda_j = \sum_{j=A}^{E}\lambda_j \leq \theta_3 x_{3k} = \theta_3 \leq 1$ and $\sum_{j=A}^{E} y_{2j}\lambda_j = \sum_{j=A}^{E}\lambda_j \geq \phi_2 y_{2k} = \phi_2 \geq 1$. Hence, $\sum_{j=A}^{E}\lambda_j = 1$ and $\theta_3 = \phi_2 = 1$. The equation $\sum_{j=A}^{E}\lambda_j = 1$ corresponds to an assumption that the production possibility set P is under variable returns to scale. Therefore, problem (27) is reduced to a DEA model with two inputs(x_1 and x_2) and one output (y_1) whose the production possibility set P is visually illustrated in Figure 1.

As shown in Figure 1, all five DMUs are on the common face of of the production possibility set, which implies that all DMUs are full efficient. All four DMUs except DMU$_E$ are a vertex on the common face, however, DMU$_E$ is not a vertex. DMU$_E$ is convex combination of remained four DMUs. Since the dimension of the common face is 2, DMU$_E$ has multiple reference sets, e.g., $\{E\}$, $\{A,B\}$, $\{A,B,E\}$,$\{C,D\}$, $\{C,D,E\}$, $\{A,B,C,D\}$ and $\{A,B,C,D,E\}$.

Consider the example according to our two-stage approach. Since the RM model (1) is a convex problem, the steepest descent method provides an optimal solution $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$ of (1) independently of the choice of an initial point for the method. Table 2 documents the stage 1 whose outputs are all optimal solutions $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$ and output-part projections $\boldsymbol{y}^* = (\phi_1^* y_{k1}, \phi_2^* y_{k2})^\top$. Proposition 4 implies that $\boldsymbol{y}^*$ is unique, however, the uniqueness of $\boldsymbol{x}^*$ and $\boldsymbol{\lambda}^*$ is not guaranteed. The stage 2 checks the uniqueness of all optimal $\boldsymbol{\theta}^*$ and $\boldsymbol{\lambda}^*$ by the stage 1. Each optimal solution of (26) with the unique output projection $\boldsymbol{y}^*$ is shown in Table 3. By using the optimal solution of (26), the stage 2 determines validity of (24) and that of (25), respectively, and the maximum reference set, that are shown in Table 4. As shown in Table 4, all projection points are unique

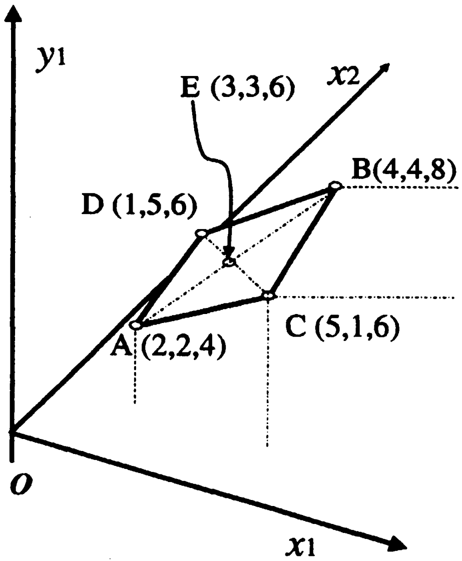

Figure 1: Location of DMUs in Production possibility set

Table 2: Stage 1 of the illustrative example

DMU	$(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \boldsymbol{\lambda}^*)$	Eff. Score	$\boldsymbol{y}^*$
A	$\theta_1^* = \theta_2^* = \theta_3^* = 1$ $\phi_1^* = \phi_2^* = 1,$ $\lambda_A^* = 1$	1	$(4, 1)$
B	$\theta_1^* = \theta_2^* = \theta_3^* = 1$ $\phi_1^* = \phi_2^* = 1$ $\lambda_B^* = 1$	1	$(8, 1)$
C	$\theta_1^* = \theta_2^* = \theta_3^* = 1$ $\phi_1^* = \phi_2^* = 1$ $\lambda_C^* = 1$	1	$(6, 1)$
D	$\theta_1^* = \theta_2^* = \theta_3^* = 1$ $\phi_1^* = \phi_2^* = 1$ $\lambda_D^* = 1$	1	$(6, 1)$
E	$\theta_1^* = \theta_2^* = \theta_3^* = 1$ $\phi_1^* = \phi_2^* = 1$ $\lambda_E^* = 1$	1	$(6, 1)$

Table 3: Opt. Sol. of (26) of the illustrative example

DMU	$(\eta^*, \boldsymbol{\theta}^*, \boldsymbol{\lambda}^*)$ $(\boldsymbol{v}^* \boldsymbol{w}^*)$	$\boldsymbol{x}^*$
A	$\eta^* = 1, \theta_1^* = \theta_2^* = \theta_3^* = 1,\ \lambda_A^* = 1$ $v_1^* = v_2^* = 1, v_3^* = 2,\ w_1^* = 1/2, w_2^* = 4$	$(2,2,1)$
B	$\eta^* = 1, \theta_1^* = \theta_2^* = \theta_3^* = 1,\ \lambda_B^* = 1$ $v_1^* = v_2^* = 1/2, v_3^* = 4,\ w_1^* = 3/2$	$(4,4,1)$
C	$\eta^* = 1, \theta_1^* = \theta_2^* = \theta_3^* = 1,\ \lambda_C^* = 1$ $v_1^* = 1, v_2^* = v_3^* = 2,\ w_1^* = 3/2$	$(5,1,1)$
D	$\eta^* = 1, \theta_1^* = \theta_2^* = \theta_3^* = 1,\ \lambda_D^* = 1$ $v_1^* = v_3^* = 2, v_2^* = 2,\ w_1^* = 1$	$(1,5,1)$
E	$\eta^* = 1/5, \theta_1^* = \theta_2^* = \theta_3^* = 1$ $\lambda_A^* = \lambda_B^* = \lambda_C^* = \lambda_D^* = \lambda_E^* = 1/5$ $v_1^* = v_3^* = 1, v_2^* = w_1^* = 1, w_2^* = 6/5$	$(3,3,1)$

Table 4: Uniqueness of opt. sol. and max. reference set

DMU	max. reference set	(24)	(25)	Status: reference set	Status: projection
A	A	○	○	unique	unique
B	B	○	○	unique	unique
C	C	○	○	unique	unique
D	D	○	○	unique	unique
E	A,B,C,D,E	○	X	multiple	unique

○:the corresponding discriminant equation is valid.
X:the corresponding discriminant equation is invalid.

and all DMUs except DMU_E have unique reference sets. However, DMU_E has multiple reference sets where maximal one is $\{A, B, C, D, E\}$.

5 Conclusion

This study shows that the RM model (1) is a convex programming problem and its optimal output-part projection is unique. Hence, the RM model (1) is easy to solve in practice. Furthermore, our proposal two-stage approach not only identifies the uniqueness of an optimal input-part projection and a reference set but also provides the maximum reference set.

A multiple-stage approach generally has a link that connects between output of its preceding stages and input of current stage. If some preceding stages have multiple outputs and an alternative one is chosen from them, the analysis on the current stage can reach a different conclusion. However, our two-stage approach is free from the bias link since the first stage of our approach gives a unique optimal output-part projection to the second one.

For DEA with a single input or a single output , all variations of RM, e.g., ERGM, input-oriented RM and output-oriented RM, are reduced to SBM, which is generally not equivalent to the RM model (1). In the presentation, we report numerical results by application of the RM model (1) and SBM to performance data of soccer players in Japanese professional league [8].

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