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Convexity of Information Theoretical Quantities in Tripartite Quantum Communication Systems

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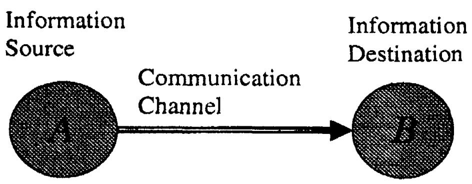
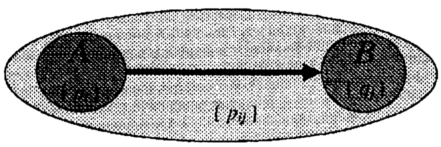
1. Introduction

• Communication System



• Classical Communication System



$\{p_{ij}\}$: Joint Probability
$\{p_i\}$: Source Probability $p_i = \sum_j p_{ij}$
$\{q_j\}$: Destination Probability
$q_j = \sum_i p_{ij}$
Classical Entropies

$$H(A) = S(\{p_i\}),\ H(B) = S(\{q_j\}),$$
$$H(A, B) = S(\{p_{ij}\})$$
$$H(A, B) \geq \max\{H(A), H(B)\}$$

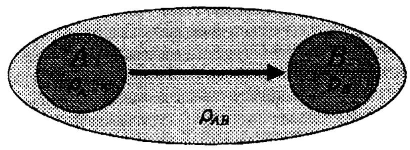
Classical Mutual Entropy

$$I(A, B) \equiv H(A) + H(B) - H(A, B) \leq \min\{H(A), H(B)\}$$

Classical Conditional Entropy

$$H(B|A) \equiv H(A, B) - H(A) \geq 0$$

• Quantum Communication System (QCS)



ρ_{AB} : "Joint" State
ρ_A : Source State $\rho_A = \mathrm{tr}_B \rho_{AB}$
ρ_B : Destination State $\rho_B = \mathrm{tr}_A \rho_{AB}$
Quantum Entropies

$$H(A) = S(\rho_A), \quad H(B) = S(\rho_B),$$
$$H(A, B) \equiv S(\rho_{AB})$$
$$H(A, B) \not\geq \max\{H(A), H(B)\}$$

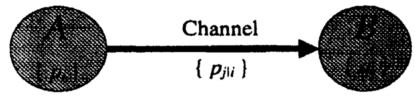
Quantum Mutual Entropy

$$I(A, B) \equiv H(A) + H(B) - H(A, B) \not\leq \min\{H(A), H(B)\}$$

Quantum Conditional Entropy

$$H(B|A) \equiv H(A, B) - H(A) \not\geq 0$$

• Alternative Description (Classical Case)



Channel

$\{p_{j|i}\}$: Conditional Probability

Conditional Entropy

$H(B|A) \equiv \sum_i p_i S(\{p_{j|i}\})$

Mutual Entropy

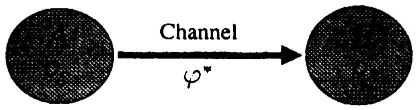
$I(A,B) \equiv H(B) - H(B|A)$

Joint Entropy

$H(A,B) \equiv H(A) + H(B|A)$

equivalent to the previous description

• **Alternative Description (Quantum Case)**



Channel

φ^* : Trace-Preserving CP-map

Conditional Entropy $H(B|A) \equiv$?

Mutual Entropy

$I(A,B) \equiv H(B) - H(B|A)$

Joint Entropy

$H(A,B) \equiv H(A) + H(B|A)$

• **Criteria for the Definition of Informational Quantities in QCS**

(1) Inclusion of classical information theory

(2) Relations among the informational quantities which hold in classical information theory

(3) Information theoretical *naturality* of the values

($\leftrightarrow$) · Mutual Entropy not more than source and destination entropies

· Conditional Entropy not less than 0

Additional Condition

(4) Convexity properties w.r.t. the source state and the channel which are possessed by the classical theory

• **Framework**

$A = (\mathcal{H}_A, \mathcal{A} = B(\mathcal{H}_A))$,

$B = (\mathcal{H}_B, \mathcal{B} = B(\mathcal{H}_B))$

: subsystems in QCS

$\mathcal{H}_A, \mathcal{H}_B$: Hilbert spaces

$\mathfrak{S}(\mathcal{H}_A), \mathfrak{S}(\mathcal{H}_B)$:

state space on $\mathcal{H}_A, \mathcal{H}_B$

= set of density operators on $\mathcal{H}_A, \mathcal{H}_B$

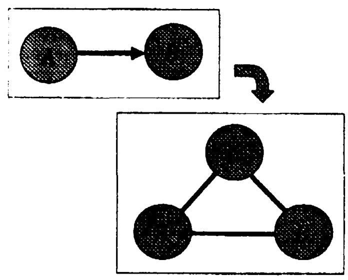
φ^* : channel from A to B

= Trace-Preserving CP-map from $\mathfrak{S}(\mathcal{H}_A)$ to $\mathfrak{S}(\mathcal{H}_B)$

= dual of *operation* (unital CP-map) φ from $B(\mathcal{H}_B)$ to $B(\mathcal{H}_A)$

Ch(A,B) : Set of channels from A to B

2. Tripartite Structures of QCS



• **Subsystem** $C = (\mathcal{H}_C, \mathcal{C} = B(\mathcal{H}_C))$

Quantum communication channel

$\varphi^* : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$

Kraus form

$\varphi^*(\rho) = \sum_k V_k \rho V_k^*, \quad \rho \in \mathfrak{S}(H_A)$

$V_k : \mathcal{H}_A \to \mathcal{H}_B, \ \sum_k V_k^* V_k = I_A$

New Hilbert space $\mathcal{H}_C$ with CONS $\{g_k\}$

Isometry $U_\varphi \ : \ \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$

$U_\varphi \equiv \sum_k V_k \otimes |g_k\rangle$ or

$U_\varphi e_i \equiv \sum_{jk} (V_k)_{ji} f_j \otimes g_k$

$\{e_i\}, \{f_j\}$: CONS's of $\mathcal{H}_A$ and $\mathcal{H}_B$

Theorem 1.

U_φ can be defined independently of the choice of the Kraus operators $\{V_k\}$ of φ^*. Namely, another set of operators $\{V_l'\}$ of φ^* corresponds to the choice of another orthonormal basis $\{g_l'\}$ of $\mathcal{H}_C$.

Proof. Let $\varphi^*(\cdot) = \sum_l V_l' \cdot V_l'^*$ be another Kraus form. Then there exist a unitary matrix $\{u_{kl}\}$ such that $V_k = \sum_l u_{kl} V_l'$, where zero operators should be added to the shorter list of $\{V_k\}$ or $\{V_l'\}$. Then

$$U_\varphi = \sum_k V_k \otimes |g_k\rangle = \sum_{kl} u_{kl} V_l' \otimes |g_k\rangle$$
$$= \sum_l V_l' \otimes |\sum_k u_{kl} g_k\rangle$$

where additional basis vectors should be included to $\{g_k\}$ if $\{V_k\}$ is shorter than $\{V_l'\}$ extending $\mathcal{H}_C$. This extension is superficial because the extended directions are out of the range of U_φ.

Selecting $\{g_l' \equiv \sum_k u_{kl} g_k\}$ as the basis of $\mathcal{H}_C$ corresponding to the Kraus oerators $\{V_l'\}$, we obtain the result. Q.E.D.

Example.1

$$V_1 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi^*(\cdot) = V_1 \cdot V_1^* + V_2 \cdot V_2^* \quad \to \{g_1, g_2\}$$
$$= P_1 \cdot P_1^* + P_2 \cdot P_2^* \quad \to \{g_1', g_2'\}$$

$$V_1 = \tfrac{1}{\sqrt{2}} P_1 + \tfrac{1}{\sqrt{2}} P_2,$$

$$V_2 = \tfrac{1}{\sqrt{2}} P_1 - \tfrac{1}{\sqrt{2}} P_2,$$

$$g_1' = \tfrac{1}{\sqrt{2}} g_1 + \tfrac{1}{\sqrt{2}} g_2,$$

$$g_2' = \tfrac{1}{\sqrt{2}} g_1 - \tfrac{1}{\sqrt{2}} g_2,$$

$$U_\varphi = V_1 \otimes |g_1\rangle + V_2 \otimes |g_2\rangle$$
$$= P_1 \otimes |g_1'\rangle + P_2 \otimes |g_2'\rangle$$

Example.2

$$\varphi^*(\cdot) = U \cdot U^* \ (+\ 0 \cdot 0) \ \to \{g_1(, g_2)\}$$
$$= \lambda U \cdot U^* + (1-\lambda) U \cdot U^* \to \{g_1', g_2'\}$$

$$U = \sqrt{\lambda}(\sqrt{\lambda} U) + \sqrt{1-\lambda}(\sqrt{1-\lambda} U)$$

$$0 = -\sqrt{1-\lambda}(\sqrt{\lambda} U) + \sqrt{\lambda}(\sqrt{1-\lambda} U)$$

$$g_1' = \sqrt{\lambda} g_1 - \sqrt{1-\lambda} g_2,$$

$$g_2' = \sqrt{1-\lambda} g_1 + \sqrt{\lambda} g_2 U_\varphi = U \otimes |g_1\rangle$$
$$= \sqrt{\lambda} U \otimes |g_1'\rangle + \sqrt{1-\lambda} U \otimes |g_2'\rangle$$

Using the operator U_φ, φ^* is written as
$\varphi^*(\rho) = \mathrm{tr}_C U_\varphi \rho U_\varphi^*$ for $\rho \in \mathfrak{S}(\mathcal{H}_A)$.

• Complementary map

$\tilde{\varphi}^* \ : \ \mathfrak{S}(\mathcal{H}_A) \ \to \ \mathfrak{S}(\mathcal{H}_C)$ is defined as
$\tilde{\varphi}^*(\rho) \equiv \mathrm{tr}_B U_\varphi \rho U_\varphi^*\,, \quad \rho \in \mathfrak{S}(\mathcal{H}_A),$
$= \sum_j \tilde{V}_j \rho \tilde{V}_j^*, \quad (\tilde{V}_j)_{ki} = (V_k)_{ji}$
or $\tilde{V}_j = \langle f_j | U_\varphi = \langle f_j | \sum_k V_k \otimes |g_k\rangle,$

satifying $\sum_j \tilde{V}_j^* \tilde{V}_j = I_A$.

While, $\tilde{\varphi}^*(\rho) = \mathrm{tr}_B U_\varphi \rho U_\varphi^*$
$= \sum_{kk'} \mathrm{tr}_B (V_k \otimes |g_k\rangle) \rho (V_{k'}^* \otimes \langle g_{k'}|)$
$= \sum_{kk'} \mathrm{Tr}(V_k \rho V_{k'}^*) |g_k\rangle\langle g_{k'}| \equiv M_\rho^{\mathrm{L}}(\varphi),$
is the (operator version of) Lindblad matrix of φ w.r.t. ρ

Note: symmetry between subsystems B and C:

$\tilde{\varphi}^*(\rho) = M_\rho^{\mathrm{L}}(\varphi), \quad M_\rho^{\mathrm{L}}(\tilde{\varphi}) = \varphi^*(\rho)$

Definition 1.

Let $\rho = \sum_n \lambda_n |e_n'\rangle\langle e_n'|$ be the spectral decomposition with $\lambda_n > 0, \sum_n \lambda_n = 1$ and an appropriate ONS $\{e_n'\}$ of $\mathcal{H}_A$.

a) The (*symmetric*) *purification* $\hat{\rho}$ of ρ
$\hat{\rho} \equiv |\xi_\rho\rangle\langle\xi_\rho| \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_A)$ with unit vector
$\xi_\rho = \sum_n \lambda_n^{1/2} e_n' \otimes e_n' \in \mathcal{H}_A \otimes \mathcal{H}_A$

Partial trace of $\hat{\rho}$ w.r.t. each $\mathcal{H}_A$ coincides with ρ.

b) The *purification operator* (or *canonical square root*) $r_\rho \in B(\mathcal{H}_A)$ of ρ is

$$r_\rho \equiv \sum_n \lambda_n^{1/2} |e_n'\rangle\langle \bar{e}_n'|,$$

where the orthonormal vectors $\bar{e}_n' \in \mathcal{H}_A$ are defined by the equations
$\langle e_i | \bar{e}_n' \rangle = \langle e_n' | e_i \rangle$ for all e_i.

It has the properties
$r_\rho^{\mathrm{T}} = r_\rho, \quad r_\rho r_\rho^* = \rho$ and $r_\rho^* r_\rho = \bar{\rho},$
where r_ρ^{T} is the transposed operator of r_ρ w.r.t. the basis $\{e_i\}$.

c) Lindblad state
$\rho_\varphi^{\mathrm{L}} \equiv (I \otimes \varphi^*)(\hat{\rho}) \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$
$\mathrm{tr}_B \rho_\varphi^{\mathrm{L}} = \rho, \quad \mathrm{tr}_A \rho_\varphi^{\mathrm{L}} = \varphi^*(\rho)$

Note: We will not consider $H(A,B) = S(\rho_\varphi^{\mathrm{L}})$!

• Generating State ω

Let $\omega = |\zeta\rangle\langle\zeta| \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ be a pure state defined by the unit vector
$\zeta = (I_A \otimes U_\varphi)\xi_\rho = \sum_{ik} r_\rho e_i \otimes V_k e_i \otimes g_k$
$= \sum_{ijk} (V_k r_\rho)_{ji} e_i \otimes f_j \otimes g_k.$

This state has a form of a composition of the Lindblad state $(I \otimes \varphi^*)(\hat{\rho})$ and the expression of the channel by an isometry $\varphi^*(\rho) = \mathrm{tr}_C U_\varphi \rho U_\varphi^*$.

ω is also regarded as a (non-symmetric) purification of the Lindblad state.

It completely determines the communication system and various quantities can be deduced from this state by taking partial traces or appropriate operations. Hereafter, we call

ω : generating state,

ζ : generating vector

of QCS.

• **Symmetric Notations**

(X, Y, Z) : a permutation of (A, B, C)

$\rho_{XY} \equiv \mathrm{tr}_Z\omega$, $\rho_X \equiv \mathrm{tr}_{YZ}\omega$.

Then,

$\rho_A = \rho$, $\rho_B = \varphi^*(\rho)$, $\rho_C = M^{\mathrm{L}}_\rho(\varphi)$

$\rho_{BC} = U_\varphi \rho_A U_\varphi^*$

$\rho_{AB} = (I \otimes \varphi^*)(\hat{\rho})$: Lindblad state

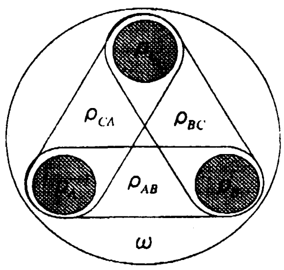
ρ_{AB} has the same non-zero eigenvalues with the same multiplicities as the Lindblad matrix ρ_C, since $\rho_{AB} = \mathrm{tr}_C\omega$, $\rho_C = \mathrm{tr}_{AB}\omega$ and ω is a pure state. Especially, we have

$$S(\rho_{AB}) = S(\rho_C) \text{ and } S(\rho_{BC}) = S(\rho_A)$$

where $S(\cdot)$ is the von-Neumann entropy.

We note here again that ρ_{XY}'s are not regarded as the joint states which give the joint entropies of composite systems (X, Y).

Fig. of Tripartite Structure:



3. Symmetrization

To make the theory completely symmetric w.r.t. the subsystems A, B and C, we define, generalizing $\zeta = \sum_{ijk}(V_k r_\rho)_{ji} e_i \otimes f_j \otimes g_k$, a unit vector

$$\zeta = \sum_{ijk} d_{ijk} e_i \otimes f_j \otimes g_k$$

with $d_{ijk} \in \mathbb{C}$ and $\sum_{ijk}|d_{ijk}|^2 = 1$ and start from the pure state $\omega = |\zeta\rangle\langle\zeta|$.

We can obtain φ^* and ρ from ω conversely. The definition $\rho_X \equiv \mathrm{tr}_{YZ}\omega$ yields

$$(\rho_A)_{ii'} = \sum_{jk} d_{ijk}\bar{d}_{i'jk},$$

$$(\rho_B)_{jj'} = \sum_{ki} d_{ijk}\bar{d}_{ij'k},$$

$$(\rho_C)_{kk'} = \sum_{ij} d_{ijk}\bar{d}_{ijk'}.$$

For simplicity, we consider the case where ρ_X is faithful on $\mathcal{H}_X$ or, equivalently, restrict $\mathcal{H}_X$ to the support subspace of ρ_X for $X{=}A$, B, C.

Notations:

$\hat{\rho}_X$: purification of ρ_X

r_X the purification operator of ρ_X

r_X^{-1} : inverse of r_X, i.e.,

$$r_A^{-1}(= r_\rho^{-1}) \equiv \sum_n \lambda_n^{-1/2}|\bar{e}_n{}'\rangle\langle e_n'|$$

and analogously for r_B and r_C.

$r_X^{-1}r_X = r_X r_X^{-1} = I_X$ identity operator on $\mathcal{H}_X$.

Isometries $U_X : \mathcal{H}_X \to \mathcal{H}_Y \otimes \mathcal{H}_Z$ are generalization of U_φ defined by

$$U_A e_i \equiv \sum_{i'jk} d_{i'jk}(r_A^{-1})_{i'i} f_j \otimes g_k,$$

$$U_B f_j \equiv \sum_{ij'k} d_{ij'k}(r_B^{-1})_{j'j} e_i \otimes g_k,$$

$$U_C g_k \equiv \sum_{ijk'} d_{ijk'}(r_C^{-1})_{k'k} e_i \otimes f_j.$$

We can define the channel $\varphi^*_{YX} \in \mathrm{Ch}(X, Y)$ for any pair of subsystems (X, Y) by

$$\varphi^*_{YX}(\cdot) \equiv \mathrm{tr}_Z U_X \cdot U_X^*.$$

and the operation $\varphi_{YX} : B(\mathcal{H}_Y) \to B(\mathcal{H}_X)$ which has φ^*_{YX} as its dual. If $d_{ijk} = (V_k r_\rho)_{ji}$ of Section 2, $\varphi^*_{BA} = \varphi^*$.

Theorem 2.

$\rho_{YZ} \equiv \mathrm{tr}_X\omega = U_X \rho_X U_X^*$,

$\varphi^*_{YX}(\rho_X) = \rho_Y$,

$$(I_X \otimes \varphi^*_{YX})(\hat{\rho}_X) = \rho_{XY}$$

φ^*_{YX} has the Kraus form of

$\varphi^*_{YX}(\cdot) = \sum_n V_n^{YX} \cdot V_n^{YX*}$

with $(V_n^{YX})_{ml} = \sum_{l'} d^{XYZ}_{l'mn}(r_X^{-1})_{l'l}$

and $\sum_n V_n^{YX*} V_n^{YX} = I_X$

where $d^{XYZ}_{l_X l_Y l_Z} \equiv d_{l_A l_B l_C}$

(e.g. $d^{CAB}_{lmn} = d_{mnl}$).

• Another way to symmetric structure

Start from a state $\tau \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ (or more generally $\tau \in \mathfrak{S}(\mathcal{H}_X \otimes \mathcal{H}_Y)$) defined as

$$\tau = \sum_k |\eta_k\rangle\langle\eta_k| \text{ with } \eta_k = \sum_{ij} d_{ijk} e_i \otimes f_j,$$

d_{ijk} is simply defined here as the coefficient of the vector η_k w.r.t. the basis $\{e_i \otimes f_j\}$

Quite the same way as Section 2, we introduce a Hilbert space $\mathcal{H}_C$ with CONS $\{g_k\}$ and define a unit vector ζ using d_{ijk}. ζ is independent of the expression $\tau = \sum_k |\eta_k\rangle\langle\eta_k|$ by the same arguments as the proof of Theorem 1.

Then, according to the discussions of this section we can define the channel φ^*_{BA} with

$\varphi^*_{BA}(\mathrm{tr}_B\tau) = \mathrm{tr}_A\tau,$

$\tau = \rho_{AB} = (I_A \otimes \varphi^*_{AB})\rho_A.$

· This is an inverse procedure to obtain the channel φ^* from a Lindblad state τ.

· This procedure is also regarded as a general purification process and most simply, if this is applied to a state on $\mathcal{H}_A$, the purification of the state defined in Definition 1 will be generated.

• Relation to CP-map defined from a State

From a state $\tau (= \rho_{XY}) \in \mathfrak{S}(\mathcal{H}_X \otimes \mathcal{H}_Y)$, we can define a CP-map $\psi : B(\mathcal{H}_Y) \to T(\mathcal{H}_X)$ by

$$\tau(a \otimes b) = \mathrm{Tr}\psi(b^{\mathrm{T}})a,$$

$$a \in B(\mathcal{H}_X),\ b \in B(\mathcal{H}_Y)$$

where b^{T} is the transposed operator of b w.r.t. the basis of $\mathcal{H}_Y$. Let us examine the relation between ψ and φ_{YX} of the last discussion.

Since $\psi(I) = \rho_X$, we can define an operation $\phi : B(\mathcal{H}_Y) \to B(\mathcal{H}_X)$ by

$$\phi(\cdot) \equiv \rho_X^{-1/2}\psi(\cdot)\rho_X^{-1/2},$$

which satisfies $\phi(I_Y) = I_X$, but $\phi^*(\rho_X) = \rho_Y^{\mathrm{T}}$. To obtain the channel φ^*_{YX} which maps ρ_X to ρ_Y, we define an opeartion with the help of the purification operator r_X of ρ_X as

$$\varphi_{YX}(\cdot) = (r_X^*)^{-1}\psi^{\mathrm{T}}(\cdot)r_X^{-1}$$

where $\psi^{\mathrm{T}}(b) = \psi(b^{\mathrm{T}})^{\mathrm{T}}$, whose dual map turns out to be φ^*_{YX}.

If $d_{ijk} = (V_k r_\rho)_{ji}$, we have $\rho_A = \rho$, $\varphi^*_{BA} = \varphi^*$ and $V_k^{BA} = V_k$ etc. Among φ^*_{XY}'s, only φ^*_{BA} $(= \varphi^*)$ and φ^*_{CA} $(= \tilde{\varphi}^*)$ are independent of ρ $(= \rho_A)$, while others depend on both ρ and φ.

4. Informational Quantities

Start from the definition of $H(B|A)$

$\to I(A, B) \equiv H(B) - H(B|A),$

$H(A, B) \equiv H(A) + H(B|A).$

The classical theory is given as

$H(B|A) \equiv \sum_i p_i S(\{p_{\cdot|i}\})$

$\boldsymbol{p} \equiv (p_1\ p_2 \cdots p_n)^{\mathrm{T}},$

$S(\boldsymbol{p}) = -\sum_i p_i \log p_i$

$\boldsymbol{q} \equiv (q_1\ q_2 \cdots q_m)^{\mathrm{T}},\ q_j = \sum_i p_{j|i}p_i$

$\to\ \boldsymbol{q} = \Phi^*\boldsymbol{p},\quad \Phi^*$: Classical Channel

i.e. Conditional Probability Matrix :

$$\Phi^* = \begin{pmatrix} p_{1|1} & p_{1|2} & \cdots & p_{1|n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m|1} & p_{m|2} & \cdots & p_{m|n} \end{pmatrix}$$

Since $\boldsymbol{p} = \sum_i p_i \boldsymbol{e}_i$ with

$\boldsymbol{e}_i = (0 \cdots 0\ 1\ 0 \cdots 0)^{\mathrm{T}},$

i-th element

$H(B|A) = \sum_i p_i S(\Phi^* \boldsymbol{e}_i)$

$= \inf\{\sum_i \lambda_i S(\Phi^* \boldsymbol{p}_i)\ ;\ \boldsymbol{p} = \sum_i \lambda_i \boldsymbol{p}_i\}.$

Extension to Quantum System may be given by the substitution:

$\boldsymbol{p} \to \rho,\ \Phi^* \to \varphi^*$

Definition 2.

For $\rho \in \mathfrak{S}(\mathcal{H}_A)$, $\varphi^* \in \mathrm{Ch}(A, B)$,

· Conditional Entropy (Dissemination)

$H(B|A) = \inf\{\sum_i \lambda_i S(\varphi^*(\rho_i))\ ;$
$\rho = \sum_i \lambda_i \rho_i,\ \lambda_i > 0,\ \sum_i \lambda_i = 1,$
$\rho_i \in \mathfrak{S}(\mathcal{H}_A)\}$

· Mutual Entropy

$I(A,B) \equiv H(B) - H(B|A)$

This definition coincides with Holevo mutual entropy and Ohya's pseudo mutual entropy.

· Joint Entropy

$H(A,B) \equiv H(A) + H(B|A)$

Lemma 1.

$0 \le I(A,B) \le \min\{H(A), H(B)\}$.

Proof. $I(A,B) \le H(B)$ is clear, since $H(B|A) \ge 0$. Rewriting $I(A,B)$ as

$$I(A,B) = \sup\{\textstyle\sum_i \lambda_i S(\varphi^*(\rho_i)|\varphi^*(\rho));\ \rho = \sum_i \lambda_i \rho_i\}$$

where $S(\cdot|\cdot)$ is the Umegaki relative entropy, we have the positivity of the mutual entropy, and the property of relative entropy that it is non-increasing under the action of a channel CP-map implies

$I(A,B) = \sup \sum_i \lambda_i S(\varphi^*(\rho_i)|\varphi^*(\rho))$
$\le \sup \sum_i \lambda_i S(\rho_i|\rho) = S(\rho) = H(A).$■

Corollary.

$H(A,B) \ge \max\{H(A), H(B)\}$,

For,

$I(A,B) \equiv H(B) - H(B|A) \le H(A)$

implies

$H(B) \le H(A) + H(B|A) \equiv H(A,B)$.

Rewriting the definition of $H(B|A)$ as

$$H(B|A) \equiv \inf\{\textstyle\sum_i \lambda_i S(\varphi^*_{BA}(\rho_i));\ \rho_A = \sum_i \lambda_i \rho_i\}$$

and extending this definition to φ^*_{YX} for any X, Y out of A, B, C as

$$H(Y|X) \equiv \inf\{\textstyle\sum_i \lambda_i S(\varphi^*_{YX}(\rho_i));\ \rho_X = \sum_i \lambda_i \rho_i\},$$

we can define

$I(X,Y) \equiv H(Y) - H(Y|X)$,
$H(X,Y) \equiv H(X) + H(Y|X)$.

Then, differently from the classical case, $I(X,Y)$ and $H(X,Y)$ are not symmetric under the exchange of their arguments i.e.

$I(X,Y) \ne I(Y,X)$,
$H(X,Y) \ne H(Y,X)$.

Instead of regarding this situation as a problem, we consider it as a phenomena characteristic to the quantum systems i.e. an expression of quantum non-commutativity.

Also we note that neither $H(X,Y)$ nor $H(Y,X)$ is equal to $S(\rho_{XY}) = H(Z)$ in general.

To define the equivocation of $\varphi^* = \varphi^*_{BA}$, we use the notation:

$H(A/B) \equiv H(A,B) - H(B)$,

because it is different from the dissemination $H(A|B)$ of reversed channel $\varphi^*_{AB} \in \mathrm{Ch}(B,A)$:

$H(A|B) = H(B,A) - H(B)$.

• Relation to the Entanglement

The entanglement of formation(EoF) of a state σ on the tensored algebra $B(\mathcal{H}_Y) \otimes B(\mathcal{H}_Z)$ is defined by

$$E_{YZ}(\sigma) \equiv \inf\{\textstyle\sum_i \lambda_i S(\mathrm{tr}_Z \sigma_i)\ ;\ \sum_i \lambda_i \sigma_i = \sigma\}$$

Then defining $E(Y,Z) \equiv E_{YZ}(\rho_{YZ})$, we have the following theorem:

Theorem 3.

$H(B|A) = E(B,C)$,

more generally,

$H(Y|X) = E(Y,Z)$.

Proof. Since $\varphi^*(\rho_i) = \mathrm{tr}_C U_\varphi \rho_i U_\varphi^*$, we have

$$\begin{aligned} H(B|A) &= \inf\{\textstyle\sum_i \lambda_i S(\varphi^*(\rho_i))\ ;\ \rho = \sum_i \lambda_i \rho_i\} \\ &= \inf\{\textstyle\sum_i \lambda_i S(\mathrm{tr}_C U_\varphi \rho_i U_\varphi^*)\ ;\ \rho = \sum_i \lambda_i \rho_i\} \\ &= \inf\{\textstyle\sum_i \lambda_i S(\mathrm{tr}_C \sigma_i)\ ;\ U_\varphi \rho U_\varphi^* = \sum_i \lambda_i \sigma_i\} \\ &= E_{B,C}(U_\varphi \rho U_\varphi^*) = E_{B,C}(\rho_{BC}) \\ &= E(B,C) \end{aligned}$$

The third equality holds by the equivalence of $\rho = \sum_i \lambda_i \rho_i$ and $U_\varphi \rho U_\varphi^* = \sum_i \lambda_i \sigma_i$, or equivalently

$\sigma_i = U_\varphi \rho_i U_\varphi^*$ for appropriate ρ_i, which holds because the inequality $\lambda_i \sigma_i \leq U_\varphi \rho U_\varphi^*$ leads to $\sigma_i \in U_\varphi \mathfrak{S}(\mathcal{H}_A) U_\varphi^*$. The generalization of the proof to $H(Y|X)$ is clear. QED.

By the symmetry of EoF w.r.t. its arguments,

$$H(X,Y) = H(X) + E(Y,Z)$$
$$= H(X) + E(Z,Y) = H(X,Z),$$

and using $S(YZ) \equiv S(\rho_{YZ})$ $(= H(X))$

$$H(X,Y) = H(X,Z)$$
$$= S(YZ) + E(Y,Z).$$

As a corollary of Theorem 3 and Lemma 1,

$$H(X,Y) \geq H(Z) = S(XY)$$

because $H(X,Y) = H(X,Z) \geq H(Z)$.

Hence, summarizing the above results, we have

Theorem 4.

$$H(X,Y) \geq \max\{H(A), H(B), H(C)\},$$
$$H(X,Y) \leq H(X) + H(Y),$$
$$I(X,Y) \leq \min\{H(X), H(Y)\}.$$

Note: Fundamental Quantities of Tripartite systems are $H(A)$, $H(B)$, $H(C)$, $E(B,C)$, $E(C,A)$, $E(A,B)$.

The other quantities are calculated by

$$H(Y|X) = E(Y,Z),$$
$$I(X,Y) = H(Y) - E(Y,Z),$$
$$H(X,Y) = H(X) + E(Y,Z),$$
$$H(X/Y) = H(X,Y) - H(Y).$$

5. Convex Theory

For QCS constructed from $\rho \in \mathfrak{S}(\mathcal{H}_A)$ and $\varphi^* \in \mathrm{Ch}(A,B)$, we shall investigate the convexities of information theoretical quantities w.r.t. ρ and φ^* for finite dimensional $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$. We will discuss about

• *Naturalness* of a quantum information theory (as an extension of classical information theory)

$\leftrightarrow$ Preservation of convexity properties of informational quantities which hold in the classical theory

· **Convexity in Classical Theory**

$\rho = \{p_i\}$: source probability,

$\varphi^* = \Phi^*$: conditional probability matrix

	$H(A)$	$H(B)$	$H(B\|A)$
ρ	$\frown$	$\frown$	affine
φ^*	independent	$\frown$	$\frown$

	$I(A,B)$	$H(A,B)$	$H(A/B)$
ρ	$\frown$	$\frown$	$\frown$
φ^*	$\smile$	$\frown$	$\frown$

$\frown$: concave, $\smile$: convex

· **Convexity in Quantum Theory**

$H(B|A){=}E_{BC}(\rho_{BC})$, ρ_{BC}: ρ-affine

$\to H(B|A)$: ρ-convex (property of EoF)

$\to$ breaking of ρ-concavities of $H(A,B)$, $H(B/A)$

	$H(A)$	$H(B)$	$H(B\|A)$
ρ	$\frown$	$\frown$	$\smile$
φ^*	independent	$\frown$	$\frown$

	$I(A,B)$	$H(A,B)$	$H(A/B)$
ρ	$\frown$	$\times$	$\times$
φ^*	$\smile$	$\frown$	$\frown$

· **Quantities Including Subsystem C**

Symmetry between subsystems B and C gives ρ-convexity(concavity) for B replaced by C:

	$H(C)$	$I(A,C)$	$H(A/C)$
ρ	$\frown$	$\frown$	$\times$

$$H(C|A) = E(C,B) = E(B,C)$$
$$= H(B|A),\ \ H(A,C) = H(A,B).$$

φ^*-convexity(concavity) is not in the same situation except $H(C|A)$ and $H(A,C)$, because complementary map $\tilde{\varphi}^*$ is not φ^*-affine.

$\rho_{AB}{=}(I \otimes \varphi^*)\hat{\rho}$: φ^*-affine implies

$H(C){=}S(\rho_{AB})$: φ^*-concave,

and $E(A,B)$ $=H(A|C)$ $=H(B|C)$

: φ^*-convex implies

$I(C,A){=}H(A) - H(A|C)$

: φ^*-concave,

$I(C,B){=}H(B) - H(B|C)$

: φ^*-concave.

The ρ-convexity properties of these quantities (except $H(C)$) are difficult to discuss because the purification $\hat{\rho}$ is always pure and out of the discussions about convexity.

6. Examples

In order to understand the role of subsystem C in the following examples, let us observe the flow of the entropy. From

$$H(B) = I(A,B) + E(C,B),$$

we see that the entropy of B is supplied by the information from A and the entanglement with C, as a noise.

Concerning the outgoing information from A, we use the equation

$$H(A) = I(A,B) + H(A/B).$$

The equivocation $H(A/B)$ is rewritten as

$$H(A/B) = H(A,B) - H(B)$$
$$= H(A,C) - S(AC) \equiv F(A,C)$$

meaning the difference of the joint entropy $H(A,C)$ from the classical one, $S(AC)$. Then the equality

$$H(A) = I(A,B) + F(A,C),$$

means that $I(A,B)$ in the entropy of A is sent to B as the information and the remaining part of $H(A)$ is absorbed to fertilize the joint entropy of $(A,\ C)$ from $S(AC)$ to $H(A,C)$.

So, the subsysytem C works as the absorber of the information from A and the generator of the noise to B.

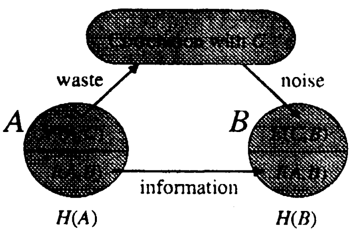
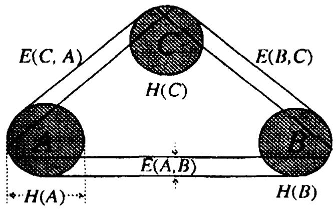


Fig. Flow of Entropy

Remind that the fundamental quantities of a tripartite system are $H(A)$, $H(B)$, $H(C)$, $E(B,C)$, $E(C,A)$, $E(A,B)$



Legend of figures for Examples

Ex. 1 **Classical Channel**

Discrete classical communication system in the quantum formalism is described as follows:

ρ : diagonal matrix on $\mathcal{H}_A$ with diagonal elements p_i

φ^* : diagonal matrix $\mapsto$ diagonal matrix

$$\varphi^*(\cdot) = \textstyle\sum_{ij} p_{j|i} E_{ji} \cdot E_{ij}$$

E_{ij} : matrix unit,

$p_{j|i}$: conditional probabitity

Kraus operators : $\sqrt{p_{j|i}}E_{ji}$

(i,j) in E_{ji} corresponds to basis of $\mathcal{H}_A \otimes \mathcal{H}_B$.

$\to \{g_{ij}\}$ corresponding to $\{e_i \otimes f_j\}$

$\mathcal{H}_C$ has the structure of $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$U_\varphi e_i = \textstyle\sum_j \sqrt{p_{j|i}} f_j \otimes g_{ij}$$
$$\zeta = \textstyle\sum_{ij} \sqrt{p_{ij}}\, e_i \otimes f_j \otimes g_{ij}$$

p_{ij} : Joint probability of (A,B)

$\rho_C = \mathrm{tr}_{AB}|\zeta\rangle\langle\zeta| = \sum_{ij} p_{ij}|g_{ij}\rangle\langle g_{ij}|$, $\to C$: equivalent to joint probability system $(A,\ B)$

(Peculiar feature of the classical theory)

$\to H(C) = H_{\mathrm{cl}}(A,B) \equiv S(\{p_{ij}\})$

H_{cl} denotes the value of the classical theory. Clearly we have

$H(A) = H_{\mathrm{cl}}(A)$, $H(B) = H_{\mathrm{cl}}(B)$,
$E(B,C) = H(B|A) = H_{\mathrm{cl}}(B|A)$,
$E(C,A) = H(A|B) = H_{\mathrm{cl}}(A|B)$.

Since $\rho_{AB} = \sum_{ij} p_{ij}|e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|$ is a separable state, $E(A,B) = 0$.

Consequently,

$$I(A,B) = I(B,A) = I_{\mathrm{cl}}(A,B),$$
$$I(A,C) = I(C,A) = H_{\mathrm{cl}}(A),$$
$$I(B,C) = I(C,B) = H_{\mathrm{cl}}(B),$$
$$H(X,Y) = H_{\mathrm{cl}}(A,B) \text{ for any } X,Y.$$

Equivocations $H(Y/X) = H(Y|X)$ for any X,Y because $H(X,Y) = H(Y,X)$.

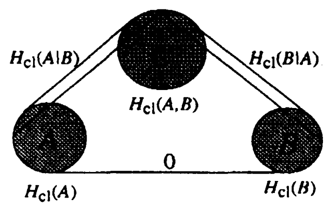


Fig. Classical Channel

Ex.2 **Unitary Channel**

When φ^* is a unitary map, since $\varphi^*(\cdot) = U \cdot U^*$ is a Kraus form, $\mathcal{H}_C$ is one-dimensional and

$$\zeta = \sum_i r_\rho e_i \otimes U e_i \otimes g_1.$$

$H(A) = H(B)$ and $H(C) = 0$, ρ_{AB}: pure $\rightarrow E(A,B) = H(A)$, $E(B,C) = E(C,A) = 0$. $I(A,B) = I(B,A) = H(A)$ and $I(X,Y) = 0$ for others. $H(X,Y) = H(A)$ for all X and Y.

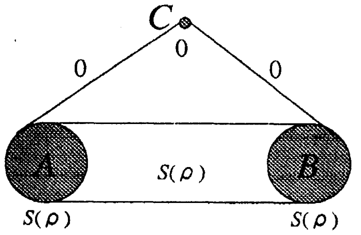


Fig. Unitary Channel

Ex.3 **Trivial Channel**

This channel maps all the state of A to an identical state ρ_B.

If ρ_B is pure $(=|\xi\rangle\langle\xi|)$, Kraus operators are given by $V_k = |\xi\rangle\langle e_k|$.

Since $(\tilde{V})_{ki} = (V_k)_{1i} = (\sum_l |g_l\rangle\langle e_l|)_{ki}$ regarding $\xi = f_1$, this is the complementary case of the above example where B and C are exchanged.

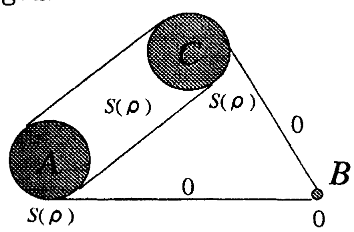


Fig. Trivial Channel (ρ_B: pure)

When ρ_B is not pure, Kraus operators may be $V_{kl} = r_B|f_l\rangle\langle e_k|$ to yield $\sum_{kl} V_{kl}\rho V_{kl}^* = \rho_B$.

Since V_{kl} has the indices corresponding to both $\mathcal{H}_A$ and $\mathcal{H}_B$, $\mathcal{H}_C = \mathcal{H}_{C1} \otimes \mathcal{H}_{C2}$ unitarily equivalent to $\mathcal{H}_A \otimes \mathcal{H}_B$ and

$$\begin{aligned}\zeta &= \sum_{ij} r_\rho e_i \otimes r_B f_j \otimes g_{ij} \\ &= (\sum_i r_\rho e_i \otimes g_i) \otimes (\sum_j r_B f_j \otimes g_j')\end{aligned}$$

rewriting $g_{kl} = g_k \otimes g_l'$, where g_k and g_l' are the basis vectors of $\mathcal{H}_{C1}$ and $\mathcal{H}_{C2}$, respectively. Hence, $\omega = |\zeta\rangle\langle\zeta|$ is the tensor-product of two pure states $\rho_1 \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_{C1})$ and $\rho_2 \in \mathfrak{S}(\mathcal{H}_B \otimes \mathcal{H}_{C2})$, which leads to $H(C) = H(A) + H(B)$. Since all the components of the pure decomposition of ρ_{BC} include ρ_2, we have $E(B,C) = H(B)$ and $I(A,B) = 0$. Analogously, $E(A,C) = H(A)$ holds and $\rho_{AB} = \rho_A \otimes \rho_B$ implies $E(A,B) = 0$.

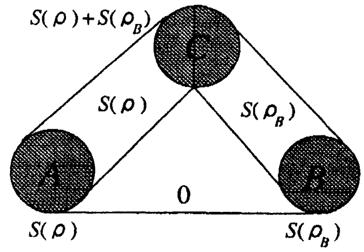


Fig. Trivial Channel (ρ_B: not pure)

The classical version of this case corresponds to Ex.1 where A and B are independent each other.

The case where ρ is pure is interesting as a sub-complementary type of Unitary Channel.

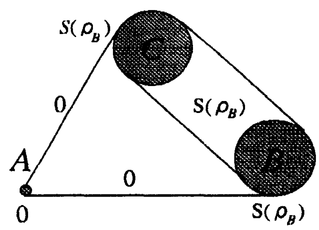


Fig. Trivial Channel (ρ : pure)

Ex.4 **Partial Trace Channel**

When $\mathcal{H}_A = \mathcal{H}_1 \otimes \mathcal{H}_2$, partial trace of $\rho \in \mathfrak{S}(\mathcal{H}_A)$ w.r.t. $\mathcal{H}_2$ is regarded as a channel by identifying $\mathcal{H}_B$ with $\mathcal{H}_1$. Let $\{e_{jk} = f_j \otimes e_k'\}$ be CONS of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\{f_j\}$, $\{e_k'\}$ the CONS's of $\mathcal{H}_B$ and $\mathcal{H}_2$, respectively. The channel is given by

$$\begin{aligned}\varphi^*(\rho) &= \mathrm{tr}_{\mathcal{H}_2}\rho \\ &\equiv \sum_k (I_1 \otimes \langle e_k'|)\rho(I_1 \otimes |e_k'\rangle) \\ &= \sum_k V_k \rho V_k^*\end{aligned}$$

with $V_k = I_1 \otimes \langle e_k'|$, I_1= identity on $\mathcal{H}_1$. Then we have

$$U_\varphi = \sum_k V_k \otimes |g_k\rangle = I_1 \otimes \sum_k |g_k\rangle\langle e_k'|.$$

Identifying $\mathcal{H}_C = \mathcal{H}_2$ and $\{g_k\} = \{e'_k\}$, U_φ becomes the identity on $\mathcal{H}_A$ and complementary channel is given by

$$\tilde{\varphi}^*(\rho) = \mathrm{tr}_B U_\varphi \rho U_\varphi^* = \mathrm{tr}_{\mathcal{H}_1}\rho.$$

If $\rho = \sum_{jk} \lambda_{jk}|e_{jk}\rangle\langle e_{jk}|$ (i.e. diagonal w.r.t. the basis $\{e_{jk}\}$), $\zeta = \sqrt{\lambda_{jk}} e_{jk} \otimes f_j \otimes g_k$, the situation is the same as Ex.1 by exchanging the role of A and C.

So, the value of the informational quantities are obtained from Ex.1 by exchanging A and C, $E(B,C) = 0$, $H(A,B) = H(B,A) = H(A)$, $I(A,B) = H(B)$, for example.

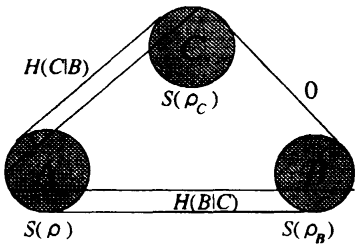


Fig. Partial Trace Channel (ρ : diagonal)

For general ρ, $E(B,C)$ is the EoF $E(\rho)$ of ρ itself between $\mathcal{H}_1$ and $\mathcal{H}_2$, which gives

$$I(A,B) = S(\mathrm{tr}_{\mathcal{H}_2}\rho) - E(\rho),$$

$$H(A,B) = S(\rho) + E(\rho).$$

Other quantities are not so simple. Because, a general channel may be regarded as a partial trace channel from equality $\varphi^*(\rho) = \mathrm{tr}_C U_\varphi \rho U_\varphi^*$ by considering U_φ as embedding.

A different point of view :

This system can be considered as the one generated from $\rho_{BC} = \rho$ like τ in Section 3 by regarding $\mathcal{H}_1$ and $\mathcal{H}_2$ as $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively. Then applying the process analogous to the one there, we can obtain the system given by ω as the purification of ρ_{BC}.

***Ex.5* Sharp Measurement Channel**

This channel φ^* is expressed as the sum of the orthogonal 1-dimensional CP-map from A to B, i.e.

$$\varphi^*(\cdot) = \textstyle\sum_k E_k \cdot E_k^*$$

with $E_k = |f_k\rangle\langle e_k|$ being Kraus operators. Then

$$U_\varphi = \textstyle\sum_i |f_i \otimes g_i\rangle\langle e_i|,$$

which shows the symmetry between B and C.

The complementary channel is $\tilde{\varphi}^*(\cdot) = \sum_j \tilde{E}_j \cdot \tilde{E}_j^*$ with $\tilde{E}_j = |g_j\rangle\langle e_j|$.

If $\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|$, the situation is very simple and we have

$$\zeta = \sum_i \sqrt{\lambda_i} e_i \otimes f_i \otimes g_i.$$

In this case, the structure of the system is totally symmetric w.r.t. A, B and C.

Consequently, all ρ_X's and ρ_{XY}'s are respectively identical for all X and Y disregarding the differences of the bases of the Hilbert spaces and so is for all φ^*_{XY}.

Since the dissemination $H(X|Y)$ vanishes for all X and Y, $H(X,Y) = H(X) = S(\rho)$ and $I(X,Y) = S(\rho)$ hold.

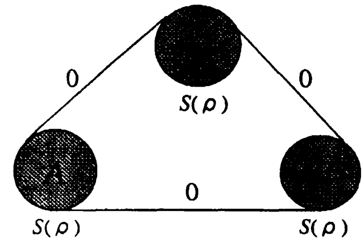


Fig. Sharp Measurement Channel (ρ: diagonal)

When e_i's are not eigenvectors of ρ,

$$\zeta = \textstyle\sum_i r_\rho e_i \otimes f_i \otimes g_i,$$

which violates the total symmetry keeping the symmetry between B and C. Though, both $H(B)$ and $H(C)$ are equal to the entropy calculated by the diagonal value of ρ. The calculation of $E(X,Y)$ is not so simple, but Ex.6 will give the one for 2-dimensional case.

***Ex.6* 2-dimensional Measurement Channel**

As the last example, we will investigate the channel of Ex.5 of 2-dimensions,

$$\varphi^*(\cdot) = P_1 \cdot P_1 + P_2 \cdot P_2$$

with ρ non-diagonal w.r.t. this basis. Here, we identify the Hilbert spaces of A and B ($\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}$) for simplicity and express E_i of Ex.5 by the orthogonal projections P_1 and P_2. This map converts the 2×2-matrices to diagonal ones keeping the diagonal elements as they are. We introduce the parametrization for a two-dimensional density matrix such as

$$\rho_{\alpha,\beta} = \frac{1}{2}\begin{pmatrix} 1+\alpha & \bar{\beta} \\ \beta & 1-\alpha \end{pmatrix},$$

$$\alpha^2 + |\beta|^2 \leq 1,$$

with entropy $S(\rho_{\alpha,\beta}) = h(\sqrt{\alpha^2 + |\beta|^2})$.
Here

$$h(x) \equiv -\tfrac{1+x}{2}\log\tfrac{1+x}{2} - \tfrac{1-x}{2}\log\tfrac{1-x}{2}$$

is a concave even function of $x \in [-1,1]$ with minimum $h(-1)=h(1)=0$ and maximum $h(0)=\log 2$ (we set $0\cdot\log 0 = 0$). We note the affineness of ρ w.r.t. α and β, i.e.,

$$\lambda\rho_{\alpha,\beta} + (1-\lambda)\rho_{\alpha',\beta'} = \rho_{\lambda\alpha+(1-\lambda)\alpha',\lambda\beta+(1-\lambda)\beta'}.$$

Let us consider the case $\rho = \rho_{0,\beta}$ for real β. Then $H(A) = h(\beta)$ and

$$\begin{aligned} H(B|A) &= \inf\{\textstyle\sum_i \lambda_i S(\varphi^*(\rho_i))\ ;\ \rho = \sum_i \lambda_i \rho_i\} \\ &= \tfrac{1}{2}\{S(\varphi^*(\rho_{-\sqrt{1-\beta^2},\beta})) + S(\varphi^*(\rho_{\sqrt{1-\beta^2},\beta}))\} \\ &= \tfrac{1}{2}\{S(\rho_{\sqrt{1-\beta^2},0}) + S(\rho_{-\sqrt{1-\beta^2},0})\} \\ &= h(\sqrt{1-\beta^2}), \end{aligned}$$

since infimum is realized by the decomposition

$$\rho_{0,\beta} = \tfrac{1}{2}\rho_{\sqrt{1-\beta^2},\beta} + \tfrac{1}{2}\rho_{-\sqrt{1-\beta^2},\beta}.$$

Hence,

$$\begin{aligned} I(A,B) &= H(B) - H(B|A) \\ &= \log 2 - h(\sqrt{1-\beta^2}) \\ H(A,B) &= H(A) + H(B|A) \\ &= h(\beta) + h(\sqrt{1-\beta^2}). \end{aligned}$$

As a function of $\beta \in [-1,1]$, $H(A,B)$ is a double-peaked continuous function which attains its minimum value $H(B)=\log 2$ at $\beta=-1,0,1$, hence neither convex nor concave w.r.t. ρ.

The equivocation

$$\begin{aligned} H(A/B) &= H(A,B) - H(B) \\ &= H(A,B) - \log 2 \end{aligned}$$

has the same behavior.
$I(A,B)$ is a convex function w.r.t. ρ as assured by the general discussion.
In this case, the generating vector is

$$\begin{aligned} \zeta = &\ d_+ e_1 \otimes f_1 \otimes g_1 + d_- e_1 \otimes f_2 \otimes g_2 \\ &+ d_- e_2 \otimes f_1 \otimes g_1 + d_+ e_2 \otimes f_2 \otimes g_2 \end{aligned}$$

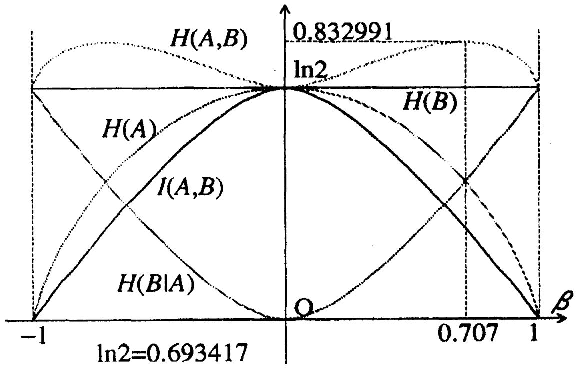
with $d_\pm = \frac{1}{2\sqrt{2}}(\sqrt{1+\beta} \pm \sqrt{1-\beta})$.
Because of the symmetry w.r.t. B and C as stated in Ex.5, $H(C) = H(B)$ and $I(A,C) = I(A,B)$ in addition to $H(A,C) = H(A,B)$.
Since a calculation derives

$$E(C,A) = E(A,B) = 0,$$

we see

$$\begin{aligned} H(C,A) &= H(C,B) = H(B,A) \\ &= H(B,C) = \log 2, \\ I(C,A) &= I(B,A) = H(A) = h(\beta), \\ I(C,B) &= I(B,C) = H(B) \\ &= H(C) = \log 2. \end{aligned}$$

Graphs of the values are shown below.



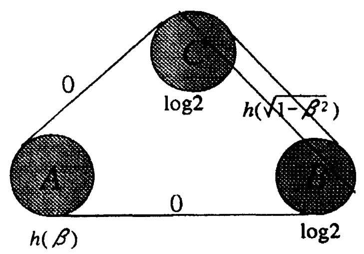


Fig. 2-dimensional Measurement Channel ($\rho = \rho_{0,\beta}$)

Remark of this section:

As above, new aspects of well-known channels and similarities of different channels can be seen in the tripartite structure.

It may be interesting to consider the structure of other channels or, conversely, to lead channels from specific structures (e.g. from the ones of the above examples exchanging the positions of A, B and C).

7. Discussions

Another definition of mutual entropy of natural value has been given by M. Ohya as

$$I_{\mathrm{Ohya}}(\varphi^*;\rho) \equiv S(\varphi^*(\rho)) - \inf\{\textstyle\sum_i \lambda_i S(\varphi^*(\rho_i)) \; ; \rho = \sum_i \lambda_i \rho_i,\ \rho_i \perp \rho_j\ i \neq j\}$$

where the infimum is taken for "orthogonal" pure decompositions differently from ours. The merit of this definition is that it is derived from the joint state (named "compound state")

$$\rho_{\mathrm{Ohya}} = \textstyle\sum_i \lambda_i \rho_i \otimes \varphi^*(\rho_i)$$

for the orthogonal pure decomposition $\rho = \sum_i \lambda_i \rho_i$ of ρ which realizes the infimum in the definition of $I_{\mathrm{Ohya}}(\varphi^*;\rho)$ above and defining $H(A,B) = S(\rho_{\mathrm{Ohya}})$. It breaks, however, the continuity of the mutual entropy w.r.t. ρ together with the concavity.

In fact, for QCS given by Ex.6 of the last section we have for $\beta \neq 0$,

$$\begin{aligned} I_{\mathrm{Ohya}}(\varphi_0^*;\rho_{0,\beta}) &= S(\varphi_0^*(\rho_{0,\beta})) \\ &\quad - \tfrac{1+\beta}{2} S(\varphi_0^*(\rho_{0,1})) - \tfrac{1-\beta}{2} S(\varphi_0^*(\rho_{0,-1})) \\ &= S(\rho_{0,0}) - \tfrac{1+\beta}{2} S(\rho_{0,0}) - \tfrac{1-\beta}{2} S(\rho_{0,0}) \\ &= 0, \end{aligned}$$

since the orthogonal decomposition of $\rho_{0,\beta}$ to pure states is unique when $\beta \neq 0$. On the other hand, for $\rho_{0,0}$ the orthogonal decomposition is not unique and the one which derives I_{Ohya} is given by $\rho_{0,0} = \frac{1}{2}\rho_{1,0} + \frac{1}{2}\rho_{-1,0}$ yielding

$$\begin{aligned} I_{\mathrm{Ohya}}(\varphi_0^*;\rho_{0,\beta=0}) &= S(\varphi_0^*(\rho_{0,0})) \\ &\quad - \tfrac{1}{2} S(\varphi_0^*(\rho_{1,0})) - \tfrac{1}{2} S(\varphi_0^*(\rho_{-1,0})) \\ &= S(\rho_{0,0}) - \tfrac{1}{2} S(\rho_{1,0}) - \tfrac{1}{2} S(\rho_{-1,0}) \\ &= \log 2 - 0 - 0 = \log 2. \end{aligned}$$

This shows the discontinuity of $I_{\mathrm{Ohya}}(\varphi_0^*;\rho_{0,\beta})$ at $\beta = 0$ as well as that of $H(B|A)$ and $H(A,B)$ and consequently violation of convexity or concavity of these quantities.

Coming back to our present theory, although the preservation of the concavity of $I(A,B)$ and the continuities of $H(B|A)$ and $H(A,B)$ in the quantum theory seems preferable, the violation of concavity of the latter two quantities might include some insufficiency in our definition of the mutual entropy.

For example, the double-peaked structure of $H(A,B)$ of Ex.6 might suggest some deficiency or excessive convexity arround β=0 of the dissemination $H(B|A)$ which makes $H(A,B)$ non-concave.

A subject in the future may be to look for a better definition of mutual entropy which makes the other quantities preserve the convexity properties better than ours.

Though our consideration is quite theoretical, we conjecture that the properties of classical theory will carry a practical meaning in constructing a physical communication system.