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Stochastic Differential Equation for Set-Valued Processes

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Abstract

In an M-type 2 Banach space $\mathfrak{X}$, we study the set-valued stochastic differential equation represented as follows

$$X_t = cl\Big\{X_0 + \int_0^t a(s, X_s)ds + \int_0^t c(s, X_s)ds + \int_0^t b(s, X_s)dB_s\Big\},\ t \in [0, T],$$

where cl stands for the closure in $\mathfrak{X}$, the given initial value X_0 and the coefficient $a(\cdot, \cdot)$ are set-valued, coefficients $c(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are single valued. Under suitable conditions, by using the successive approximation method, the existence and uniqueness of strong solutions are obtained. The unique strong solution is measurable, adapted and Hausdorff-continuous in t.

Keywords and phrases: M-type 2 Banach space, integrals of set-valued stochastic processes, set-valued stochastic differential equation.

1 Introduction

Theory of stochastic differential inclusions, as natural generalization of that of stochastic differential equations, has been received much attention with widespread applications to mathematical economics, stochastic control theory etc. In this area, we would like to refer to the nice survey [1, 10, 11]. In the n-dimensional Euclidean space $\mathbb{R}^n$, much work has been done on stochastic differential or integral inclusions (see e.g. [2, 9, 16]).

However there are only a few literatures related to considering the set-valued stochastic differential equation or integral equation because of the complexity of derivative of set-valued functions and the difficulties for defining set-valued stochastic integrals.

In a separable Banach space, Michta ([15]) studied compact convex set-valued random differential equation without the diffusion term:

$$\begin{aligned} D_H X_t &= F(t, X_t) \qquad P.1,\ t \in [0, T] - a.e. \\ X_0 &= U \qquad\qquad\ \ P.1, \end{aligned}$$

where F and U are given set-valued random variables with values in the space of all nonempty, compact and convex subsets of $\mathfrak{X}$. $D_H X_t$ is the Hukuhara derivative of X_t.

In a separable M-type 2 Banach space, Zhang et al. ([20]) studied the following set-valued stochastic differential equation

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s, \ t \in [0, T],$$

where both X_s and $a(s, X_s)$ are set-valued, $b(s, X_s)$ is single valued, and $\{B_t\}$ is a real valued Brownian motion. The sum of a set X and an single point y is defined as $X+y = \{x+y : x \in X\}$.

In this paper, based on the work [20], we will study the strong solution of the set-valued stochastic differential equation presented as follows:

$$X_t = cl\Big\{X_0 + \int_0^t a(s, X_s)ds + \int_0^t c(s, X_s)ds + \int_0^t b(s, X_s)dB_s\Big\}, \ t \in [0, T], \tag{1.1}$$

where cl stands for the closure in $\mathfrak{X}$, X_s and $a(s, X_s)$ are set-valued, $b(s, X_s)$ and $c(s, X_s)$ are single valued, and $\{B_t\}$ is a real valued Brownian motion. When the coefficients satisfy suitable conditions, for any given L^2-integrably bounded initial value X_0, there exists a unique Hausdorff-continuous strong solution to the equation (1.1).

This paper is organized as follows. Section 2 is on definition and preliminary results. Section 3 is devoted to the main results.

2 Definitions and preliminary results

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration satisfying the usual conditions such that $\mathcal{F}_0$ includes all P-null sets in $\mathcal{F}$, the filtration is non-decreasing and right continuous, $\mathcal{B}(E)$ the Borel field of a topological space E, $(\mathfrak{X}, \|\cdot\|)$ a separable Banach space $\mathfrak{X}$ equipped with the norm $\|\cdot\|$, $\mathfrak{X}^*$ the topological dual space of $\mathfrak{X}$ and $\mathbf{K}(\mathfrak{X})$ (resp. $\mathbf{K}_b(\mathfrak{X})$), the family of all nonempty closed (resp. closed bounded) subsets of $\mathfrak{X}$. Let p be $1 \leq p < +\infty$ and $L^p(\Omega, \mathcal{F}, P; \mathfrak{X})$ denoted briefly by $L^p(\Omega; \mathfrak{X})$ the Banach space of equivalence classes of $\mathfrak{X}$-valued $\mathcal{F}$-measurable functions $f : \Omega \to \mathfrak{X}$ such that the norm

$$\|f\|_p = \Big\{ \int_\Omega \|f(\omega)\|^p dP \Big\}^{\frac{1}{p}}$$

is finite. f is called *L^p-integrable* if $f \in L^p(\Omega; \mathfrak{X})$.

A set-valued function $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is said to be *measurable* if for any open set $O \subset \mathfrak{X}$, the inverse $F^{-1}(O) := \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{F}$. Such a function F is called a *set-valued random variable.* Let $\mathcal{M}(\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X}))$ be the family of all set-valued random variables, briefly denoted by $\mathcal{M}(\Omega; \mathbf{K}(\mathfrak{X}))$.

A mapping g from a measurable space $(E_1, \mathcal{A}_1)$ into another measurable space $(E_2, \mathcal{A}_2)$ is called *$\mathcal{A}_1/\mathcal{A}_2$-measurable* if $g^{-1}(B) = \{x \in E; g(x) \in B\} \in \mathcal{A}_1$ for all $B \in \mathcal{A}_2$.

For any open subset $O \subset \mathfrak{X}$, set

$$Z_O := \{E \in \mathbf{K}(\mathfrak{X}) : E \cap O \neq \emptyset\},$$

$$\mathcal{C} := \{Z_O : O \subset \mathfrak{X},\ O \text{ is open}\},$$

and let $\sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$.

Proposition 2.1. *A set-valued function $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is measurable if and only if F is $\mathcal{F}/\sigma(\mathcal{C})$-measurable.*

For $A, B \in 2^{\mathfrak{X}}$ (the power set of $\mathfrak{X}$), $H(A,B) \geq 0$ is defined by

$$H(A,B) := \max\{\sup_{x\in A}\inf_{y\in B} ||x-y||, \sup_{y\in B}\inf_{x\in A} ||x-y||\}.$$

If $A, B \in \mathbf{K}_b(\mathfrak{X})$, then $H(A,B)$ is called the *Hausdorff distance* of A and B. It is well-known that $\mathbf{K}_b(\mathfrak{X})$ equipped with the H-metric denoted by $(\mathbf{K}_b(\mathfrak{X}), H)$ is a complete metric space.

The following results are also well-known. (see for example [6], [13]).

Proposition 2.2. *(i) For $A, B, C, D \in \mathbf{K}(\mathfrak{X})$, we have*

$$H(A+B, C+D) \leq H(A,C) + H(B,D),$$

$$H(A \oplus B, C \oplus D) = H(A+B, C+D),$$

where $A \oplus B := cl\{a+b; a \in A, b \in B\}$.
(ii) For $A, B \in \mathbf{K}(\mathfrak{X})$, $\mu \in \mathbb{R}$, we have

$$H(\mu A, \mu B) = |\mu| H(A,B).$$

For $F \in \mathcal{M}(\Omega, \mathbf{K}(\mathfrak{X}))$, the family of all L^p-integrable selections is defined by

$$S_F^p(\mathcal{F}) := \{f \in L^p(\Omega, \mathcal{F}, P; \mathfrak{X}) : f(\omega) \in F(\omega)\ a.s.\}.$$

In the following, $S_F^p(\mathcal{F})$ is denoted briefly by S_F^p. If S_F^p is nonempty, F is said to be *L^p-integrable.* F is called *L^p-integrably bounded* if there exits a function $h \in L^p(\Omega, \mathcal{F}, P; \mathbb{R})$ such that $x \in F(\omega)$, $\|x\| \leq h(\omega)$ for any x and ω with $x \in F(\omega)$. It is equivalent to that $\|F\|_{\mathbf{K}} \in L^p(\Omega; \mathbb{R})$, where $\|F(\omega)\|_{\mathbf{K}} := \sup_{a\in F(\omega)} \|a\|$. The family of all measurable $\mathbf{K}(\mathfrak{X})$-valued L^p-integrably bounded functions is denoted by $L^p(\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X}))$. Write it for brevity as $L^p(\Omega; \mathbf{K}(\mathfrak{X}))$.

Let Γ be a set of measurable functions $f : \Omega \to \mathfrak{X}$. Γ is called *decomposable* with respect to the σ-algebra $\mathcal{F}$ if for any finite $\mathcal{F}$-measurable partition $A_1, .., A_n$ and for any $f_1, ..., f_n \in \Gamma$ it follows that $\chi_{A_1} f_1 + ... + \chi_{A_n} f_n \in \Gamma$, where χ_A is the indicator function of set A.

Proposition 2.3. *(Hiai-Umegaki [6]) Let Γ be a nonempty closed subset of $L^p(\Omega, \mathcal{F}, P; \mathfrak{X})$. Then there exists an $F \in \mathcal{M}(\Omega; \mathbf{K}(\mathfrak{X}))$ such that $\Gamma = S_F^p$ if and only if Γ is decomposable with respect to $\mathcal{F}$.*

Proposition 2.4. *(Hiai-Umegaki [6]) Let $F_1, F_2 \in \mathcal{M}(\Omega; \mathfrak{X})$ and $F(\omega) = cl(F_1(\omega) + F_2(\omega))$ for all $\omega \in \Omega$. Then $F \in \mathcal{M}(\Omega; \mathfrak{X})$. Moreover if $S_{F_1}^p$ and $S_{F_2}^p$ are nonempty where $1 \leq p < \infty$, then $S_F^p = cl(S_{F_1}^p + S_{F_2}^p)$, the closure in $L^p(\Omega; \mathfrak{X})$.*

Lemma 2.1. *Let $F \in \mathcal{M}(\Omega; \mathbf{K}(\mathfrak{X}))$. Then F is L^p-integrably bounded if and only if S_F^p is nonempty and bounded in $L^p(\Omega; \mathfrak{X})$.*

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers and $\mathcal{B}_+ := \mathcal{B}(\mathbb{R}_+)$. An $\mathfrak{X}$-valued stochastic process $f = \{f_t : t \geq 0\}$ (or denoted by $f = \{f(t) : t \geq 0\}$)is defined as a function $f : \mathbb{R}_+ \times \Omega \longrightarrow \mathfrak{X}$ with $\mathcal{F}$-measurable section f_t, for $t \geq 0$. We say f is *measurable* if f is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable. The process $f = \{f_t : t \geq 0\}$ is called *$\mathcal{F}_t$-adapted* if f_t is $\mathcal{F}_t$-measurable for every $t \geq 0$.

In a fashion similar to the $\mathfrak{X}$-valued stochastic process, a *set-valued stochastic process* $F = \{F_t : t \geq 0\}$ is defined as a set-valued function $F : \mathbb{R}_+ \times \Omega \longrightarrow \mathbf{K}(\mathfrak{X})$ with $\mathcal{F}$-measurable section F_t for $t \geq 0$. It is called *measurable* if it is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable, and *$\mathcal{F}_t$-adapted* if for any fixed t, $F_t(\cdot)$ is $\mathcal{F}_t$-measurable.

Let $T \in \mathbb{R}_+$, for $0 \leq s \leq t \leq T$, $\lambda([s,t])$ be the Lebesgue measure in the interval $[s,t]$. In the following, the Lebesgue integral $\int_{[s,t]} f d\lambda$ will be denoted by $\int_s^t f_s ds$, where f is a Lebesgue integrable functional. Let $L^p\Big(([0,T] \times \Omega), \mathcal{B}([0,T]) \otimes \mathcal{F}, \lambda \times P; \mathfrak{X}\Big)$ denoted briefly by $L^p([0,T] \times \Omega; \mathfrak{X})$ be the Banach space of equivalence classes of $\mathfrak{X}$-valued, $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable functions $f : [0,T] \times \Omega \to \mathfrak{X}$ such that

$$\int_{[0,T]\times\Omega} \|f(t,\omega)\|^p d\lambda dP < +\infty.$$

Let $\mathcal{L}^p(\mathfrak{X})$ be the family of all $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable, $\mathcal{F}_t$-adapted, $\mathfrak{X}$-valued stochastic processes $f = \{f_t, \mathcal{F}_t : t \in [0,T]\}$ such that $E\Big[\int_0^T \|f_s\|^p ds\Big] := \int_{[0,T]\times\Omega} \|f(t,\omega)\|^p d\lambda dP < +\infty$, and $\mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$ the family of all $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable, $\mathcal{F}_t$-adapted, set-valued stochastic processes $F = \{F_t, \mathcal{F}_t : t \in [0,T]\}$ such that $\{\|F_t\|_{\mathbf{K}}\}_{t\in[0,T]} \in \mathcal{L}^p(\mathbb{R})$.

For a $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\}$, a $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable selection $f = \{f_t, \mathcal{F}_t : t \in [0,T]\}$ is called *$\mathcal{L}^p$-selection* if $f = \{f_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathfrak{X})$. The family of all $\mathcal{L}^p$-selections is denoted by $S^p(F(\cdot))$. That is to say

$$S^p(F(\cdot)) = \{f \in \mathcal{L}^p(\mathfrak{X}); f(t,\omega) \in F(t,\omega) \ for \ a.e. \ (t,\omega) \in [0,T] \times \Omega\}.$$

By the Kuratowski-Ryll-Nardzewski Selection Theorem (see e.g. [5]), $S^p(F(\cdot))$ is nonempty for $F \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$.

2.1 Set-valued integrals with respect to Lebesgue measure in time interval $[s,t]$

We briefly state the definitions and properties of the set-valued integral with respect to the Lebesgue measure in time interval $[s,t]$ for $s,t \in [0,T]$, which were studied in detail in [20].

For a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, and for $0 \leq s \leq t \leq T$, define

$$\Lambda_{s,t} := \Big\{ \int_s^t f_u du : (f_u)_{u\in[0,T]} \in S^p(F(\cdot))\Big\}, \tag{2.1}$$

where $\int_s^t f_u(\omega) du$ is the Bochner integral with respect to the Lebesgue measure λ in the interval $[s,t]$. By Theorem 9.41 in [5], f is the Bochner integrable in the interval $[s,t]$ if and only if its norm function $\|f\|$ is the Lebesgue integrable. That is $\int_s^t \|f_u\| du < +\infty$. For each $f \in S^p(F(\cdot))$, we know $\int_0^T \|f_u\|^p du < \infty$ a.s., which means there is a P-null set N_f, such that for all $\omega \in \Omega \backslash N_f$ and for $0 \le s < t \le T$, $\int_s^t \|f(u)\|^p du < \infty$. For $\omega \in N_f$, we define $\int_s^t f_u du = 0$. Then for each $f \in S^p(F(\cdot))$, $\int_s^t f_u du$ is well defined path-by-path. Moreover, the process $\{\int_0^t f_u du : t \in [0,T]\}$ is continuous, measurable and $\mathcal{F}_t$-adapted. So that $\Lambda_{s,t} \subset L^p(\Omega, \mathcal{F}_t, P; \mathfrak{X}) \subset L^p(\Omega; \mathfrak{X})$.

We define the *decomposable closed hull* of $\Lambda_{s,t}$ with respect to $\mathcal{F}_t$ by

$$\begin{aligned}\overline{de}\Lambda_{s,t} := \Big\{ & g \in L^p(\Omega, \mathcal{F}_t, P; \mathfrak{X});\ \textit{for any } \varepsilon > 0,\ \textit{there exist a finite } \mathcal{F}_t-\textit{measurable} \\ & \textit{partition } \{A_1, ..., A_n\} \textit{ of } \Omega \textit{ and } f^1, ..., f^n \in S^p(F(\cdot)),\ \textit{such that} \\ & \|g - \sum_{i=1}^n \chi_{A_i} \int_s^t f_u^i du\|_{L^p(\Omega,\mathcal{F}_t,P;\mathfrak{X})} < \varepsilon \Big\}\end{aligned}$$

By Proposition 2.3, $\overline{de}\Lambda_{s,t}$ determines an $\mathcal{F}_t$-measurable set-valued function $I_{s,t}(F) : \Omega \to \mathbf{K}(\mathfrak{X})$, such that the family of all L^p-integrable selections of $I_{s,t}(F)$ is

$$S^p_{I_{s,t}(F)}(\mathcal{F}_t) = \overline{de}\Lambda_{s,t}.$$

Particularly, $I_{0,t}(F)$ will be denoted by $I_t(F)$ for brevity. Therefore $\{I_t(F) : t \in [0,T]\}$ is an $\mathcal{F}_t$-adapted set-valued stochastic process.

Definition 2.1. For a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, the set-valued random variable $I_{s,t}(F)$ defined as the above is called the set-valued integral of $\{F_t, \mathcal{F}_t : t \in [0,T]\}$ with respect to the Lebesgue measure on the interval $[s,t]$. We denote it by $\int_s^t F_u du := I_{s,t}(F)$.

2.2 Stochastic integral w.r.t Brownian motion in M-type 2 Banach space

Let $\{B_t, \mathcal{F}_t : t \in [0,T]\}$ be a real valued $\mathcal{F}_t$-Brownian motion with $B_0(\omega) = 0$ a.s., where we call $\{B_t, \mathcal{F}_t : t \in [0,T]\}$ an $\mathcal{F}_t$-Brownian motion if it is an $\mathcal{F}_t$-adapted continuous martingale and for any $0 \le t \le u \le T$, $E[(B_u - B_t)^2] = u - t$ (see [12]).

Definition 2.2. ([3]) A Banach space $(\mathfrak{X}, \|\cdot\|)$ is called M-type 2 if and only if there exists a constant $C_{\mathfrak{X}} > 0$ such that for any $\mathfrak{X}$-valued martingale $\{M_k\}$, it holds that

$$\sup_k E[\|M_k\|^2] \le C_{\mathfrak{X}} \sum_k E[\|M_k - M_{k-1}\|^2]. \tag{2.2}$$

The crucial inequality (2.2) guarantee the availability to define the integration.

Now, we rewrite briefly about the stochastic integral studied in [19].

Let $\mathcal{L}^2_{step}(\mathfrak{X})$ be the subspace of those $f \in \mathcal{L}^2(\mathfrak{X})$ for which there exists a partition $0 = t_0 < t_1 < ... < t_n = T$, such that $f_t = f_{t_k}$ for $t \in [t_k, t_{k+1}), 0 \le k \le n-1, n \in \mathbb{N}$. For $f \in \mathcal{L}^2_{step}(\mathfrak{X})$, define an $\mathfrak{X}$-valued martingale by $I_T(f) := \sum_{k=0}^{n-1} f_{t_k}(B_{t_{k+1}} - B_{t_k})$.

$\mathcal{L}^2_{step}(\mathfrak{X})$ is dense in $\mathcal{L}^2(\mathfrak{X})$ (see [19]), the integrand can be extend to $\mathcal{L}^2(\mathfrak{X})$. The extension is the definition of the stochastic integral and has the following properties.

Proposition 2.5. *For $f \in \mathcal{L}^2(\mathfrak{X})$, we have*

(i) $E[I_t(f)] = 0$, $I_t(f) \in L^2(\Omega, \mathcal{F}, P; \mathfrak{X})$ *and* $\{I_t(f) : t \in [0,T]\}$ *is a measurable $\mathcal{F}_t$-martingale,*

(ii)

$$E[\|I_t(f)\|^2] \leq C_{\mathfrak{X}} E\Big[\int_0^t \|f_s\|^2 ds\Big] \text{ for all } t \in [0,T], \text{ and}$$

(iii) *There exists a t-continuous version of* $I_t(f) = \int_0^t f_s dB_s, t \in [0,T]$.

From now on, we will always assume that $\int_0^t f_s(\omega) dB_s(\omega)$ means a t-continuous version of the integral.

2.3 Set-valued stochastic differential equation

Assume $\mathfrak{X}$ is a separable M-type 2 Banach space. Let the functions
$a(\cdot,\cdot) : [0,T] \times \mathbf{K}(\mathfrak{X}) \to \mathbf{K}(\mathfrak{X})$ be $\big(\mathcal{B}([0,T]) \otimes \sigma(\mathcal{C})\big)/\sigma(\mathcal{C})$-measurable,
$c(\cdot,\cdot) : [0,T] \times \mathbf{K}(\mathfrak{X}) \to \mathfrak{X}$ be $\big(\mathcal{B}([0,T]) \otimes \sigma(\mathcal{C})\big)/\mathcal{B}(\mathfrak{X})$-measurable, and
$b(\cdot,\cdot) : [0,T] \times \mathbf{K}(\mathfrak{X}) \to \mathfrak{X}$ be $\big(\mathcal{B}([0,T]) \otimes \sigma(\mathcal{C})\big)/\mathcal{B}(\mathfrak{X})$-measurable.
Assume the above functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ also satisfy the following conditions :

$$(2.3) \qquad H\big(\{0\}, a(t,X)\big) + \|c(t,X)\| + \|b(t,X)\| \leq C\big(1 + H(\{0\}, X)\big); X \in \mathbf{K}(\mathfrak{X}), t \in [0,T]$$

for some constant C and

(2.4)

$$H\big(a(t,X), a(t,Y)\big) + \|c(t,X) - c(t,Y)\| + \|b(t,X) - b(t,Y)\| \leq DH(X,Y); X, Y \in \mathbf{K}(\mathfrak{X}), t \in [0,T]$$

for some constant D.

Let X_0 be an L^2-integrably bounded set-valued random variable. Then by Proposition 2.4, it is reasonable to define the set-valued stochastic differential equation as follows:

Definition 2.3.

$$(2.5) \qquad X_t = cl\Big\{X_0 + \int_0^t a(s, X_s) ds + \int_0^t c(s, X_s) ds + \int_0^t b(s, X_s) dB_s\Big\}, \text{ for } t \in [0,T] \text{ a.s.}$$

An $\mathcal{F}_t$-adapted, H-continuous in t almost surely and measurable set-valued process $\{X_t : t \in [0,T]\}$ is called a *strong solution* if it satisfies the equation (2.5).

3 Main results

Theorem 3.1. *Let $p \geq 1$. For a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, then for $0 \leq s \leq t \leq T$, $S^p_{I_{s,t}(F)}(\mathcal{F}_t)$ is nonempty and bounded in $L^p(\Omega, \mathcal{F}_t, P; \mathfrak{X})$ and $I_{s,t}(F)$ L^p-integrably bounded.*

When $\mathcal{F}$ is separable with respect to the probability measure P, we know both $S^p\big(F(\cdot)\big)$ and $S^p_{I_t(F)}(\mathcal{F}_t)$ are separable.

Theorem 3.2. *Assume $\mathcal{F}$ is separable with respect to the probability measure P. Then for a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, there exists a sequence $\{f^n : n = 1,2,...\} \subset S^p(F(\cdot))$ such that*

$$F(t,\omega) = cl\{f_t^n(\omega) : n = 1,2,...\} \text{ for a.e. } (t,\omega),$$

and for $0 \le s \le t \le T$

$$I_{s,t}(F)(\omega) = cl\Big\{ \int_s^t f_u^n(\omega)du : n \in \mathbb{N}\Big\} \text{ a.s,}$$

where cl denotes the closure in $\mathfrak{X}$.

Lemma 3.1. *Assume $\mathcal{F}$ is separable with respect to P. For a set valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, there exists a $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable version $\{\widetilde{I_{s,t}}(F) : t \in [0,T]\}$ of $\{I_{s,t}(F) : t \in [0,T]\}$ such that $I_{s,t}(F)(\omega) = \widetilde{I_{s,t}}(F)(\omega)$ a.s. and $\widetilde{I_{s,t}}(F)(\omega) \in \mathbf{K}_b(\mathfrak{X})$ for all $0 \le s \le t \le T$ and almost sure ω.*

From now on, if $\mathcal{F}$ is separable, we will always assume that the set-valued integral of $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$ means the $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable version $\{\widetilde{I_{s,t}}(F) : t \in [0,T]\}$. For convenience, we still denote $\widetilde{I_{s,t}}(F)(\omega)$ by $I_{s,t}(F)(\omega)$.

Theorem 3.3. *Assume $\mathcal{F}$ is separable with respect to P. For a set-valued stochastic processes $\{F_t, \mathcal{F}_t : t \in [0,T]\}, \{G_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, set*

$$\phi(t,\omega) := H\Big(\int_0^t F_s(\omega)ds, \int_0^t G_s(\omega)ds\Big) : [0,T] \times \Omega \to \mathbb{R}.$$

Then $\phi(\cdot,\cdot)$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable.

By Theorem 3.2 and Lemma 3.1, we obtain that

Theorem 3.4. *Assume $\mathcal{F}$ is separable with respect to P. Then for a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, then the following formula*

$$I_t(F)(\omega) = cl\Big\{I_s(F)(\omega) + I_{s,t}(F)(\omega)\Big\}$$

holds for $0 \le s < t \le T$ and almost sure ω, where cl stands for the closure in $\mathfrak{X}$.

Lemma 3.2. *Assume $\mathcal{F}$ is separable with respect to P. Then for a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, the set-valued integral $\{I_t(F) : t \in [0,T]\}$ is H-continuous in t a.s.*

Lemma 3.3. *Assume $\mathcal{F}$ is separable with respect to P. For a set-valued stochastic processes $\{F_t\}_{t\in[0,T]}, \{G_t\}_{t\in[0,T]} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, and for all t and almost sure ω, we have*

$$H^p\Big(\int_0^t F_s(\omega)ds, \int_0^t G_s(\omega)ds\Big) \le t^{p-1}\int_0^t H^p\big(F_s(\omega), G_s(\omega)\big)ds$$

Theorem 3.5. *Assume $\mathcal{F}$ is separable with respect to P. For set-valued stochastic processes $\{F_t\}_{t\in[0,T]}, \{G_t\}_{t\in[0,T]} \in \mathcal{L}^p(\mathbf{K}(\mathfrak{X}))$, then for $1 \le r \le p$, all t and almost sure ω, it follows that*

$$H^r\Big(\int_0^t F_s(\omega)ds, \int_0^t G_s(\omega)ds\Big) \le t^{r-1}\int_0^t H^r(F_s(\omega), G_s(\omega))ds,$$

and then

$$E\Big[H^r\Big(\int_0^t F_s ds, \int_0^t G_s ds\Big)\Big] \le t^{r-1}E\Big[\int_0^t H^r(F_s, G_s)ds\Big].$$

Theorem 3.6. *Assume $\mathcal{F}$ is separable with respect to P. Let $T > 0$, and let $a(\cdot,\cdot) : [0,T] \times \mathbf{K}(\mathfrak{X}) \to \mathbf{K}(\mathfrak{X})$, $b(\cdot,\cdot) : [0,T] \times \mathbf{K}(\mathfrak{X}) \to \mathfrak{X}$ be measurable functions satisfying conditions* (2.3) *and* (2.4). *Then for any given L^2-integrably bounded initial value X_0, there exists a strong solution to* (2.5). *The strong solution is unique in the sense of $P\Big(H(X_t, \hat{X}_t) = 0 \text{ for all } t \in [0,T]\Big) = 1$.*

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