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A SURVEY OF CORNER SINGULARITY FOR COMPRESSIBLE VISCOUS NAVIER-STOKES FLOWS

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ABSTRACT. The theory for the corner singularity of solutions to elliptic problems on domains with corners has been well established over last fifty years by several mathematicians: for instance, Dauge, Grisvard, Kellogg, Kondrat'ev, Maźya among the references given in this note. The coefficient of the corner singularity in the singular and regular decomposition of solution at a corner of boundary is called the stress intensity factor in mechanics because it is named from measuring the unboundedness of the flux variable. Recently the theory has been investigated to the compressible viscous Stokes or Navier-Stokes equations on polygonal domains, particularly focusing on the singular behaviors of solutions near concave corners. The compressible viscous Navier-Stokes equations are of mixed type that is neither elliptic nor hyperbolic. A main mathematical and physical observation in this direction is that the corner singularity is propagated along the streamline emanating from corners and may generate certain interior layers or discontinuities in the considered domains because of the hyperbolic character in density of the continuity equation.

In this note we will describe and survey the corner singularities of solutions to compressible viscous Navier-Stokes equations on bounded domains with corners, based on the results obtained by Kweon and Kellogg so far.

1. Introduction

Since the compressibility is a pivotal aspect of high speed flows, the density is an important variable to be resolved and can have certain jump or discontinuity propagations of corner or edge singularities into the region provided that it is governed by the continuity equation. Because, if the continuity equation (derived from the mass conservation) is integrated along the streamline emanating from corners or edges, then the density function is an integral of the divergence of the velocity and is not free from propagation of corner singularities into the region (from the hyperbolic character of the continuity equation) because the velocity has the ones. So the corner singularities in compressible viscous Navier-Stokes flows preserve not only the corner singularity generated from the diffusion terms but also the

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transporting phenomena of corner singularities. Definitely they have more interesting properties to be resolved than the corner singularities of elliptic equations. Also, from numerous literatures of fluid dynamics one can observe such singular behaviors, for instance, see [2].

In particular, in fluid mechanics, flow separations, internal recirculations and discontinuities are technologically and mathematically important issues to be resolved [3, 31, 34, 37]. Such phenomena are often generated by sharp corners and edges of body contours of flow regions like driven cavities [3, 34]. They also can be observed either in the front portions of aircrafts or near the junctions of body and wing [2, 3].

Throughout this note the variables $\mathbf{u}$, ρ and p denote the velocity vector, density and pressure functions in the fluid flows, respectively; $\rho = \rho(p)$ is a strictly increasing smooth function of pressure; μ, ν denote the coefficient of viscosity with $\mu > 0$ and $\mu + \nu > 0$; $\mathbf{f}$ and g are given functions; Ω is a bounded polygon in the plane $\mathbb{R}^2$ with boundary Γ; $Q := \Omega \times (0, \mathrm{T})$, $\Sigma := \Gamma \times (0, \mathrm{T})$ where T is a positive number.

We here describe the geometry of Ω as follows. Let the polygon Ω have only one concave vertex P, placed at the origin. Let r denote the distance of a point in Ω to P. With the vertex P we associate two numbers ω_1 and ω_2, with $\omega_1 < \omega_2 < \omega_1 + 2\pi$, which give the directions of the two sides to the polygon at the vertex P. The interior angle of Ω at P is $\omega = \omega_2 - \omega_1$ and we set $\alpha = \pi/\omega$. We assume that $0 < \omega < 2\pi$, thus excluding slit domains. Let χ be a smooth cutoff function which is 1 near the vertex P and which vanishes outside a small neighborhood of P. The singular function of the Laplacian at the vertex P is defined by the formula [4, 9, 17]

$$\phi(x, y) = \chi\, r^{\alpha} \sin[\alpha(\theta - \omega_1)]. \tag{1.1}$$

A picture for such polygon and polyhedral cylinder is given in Fig. 1.1.

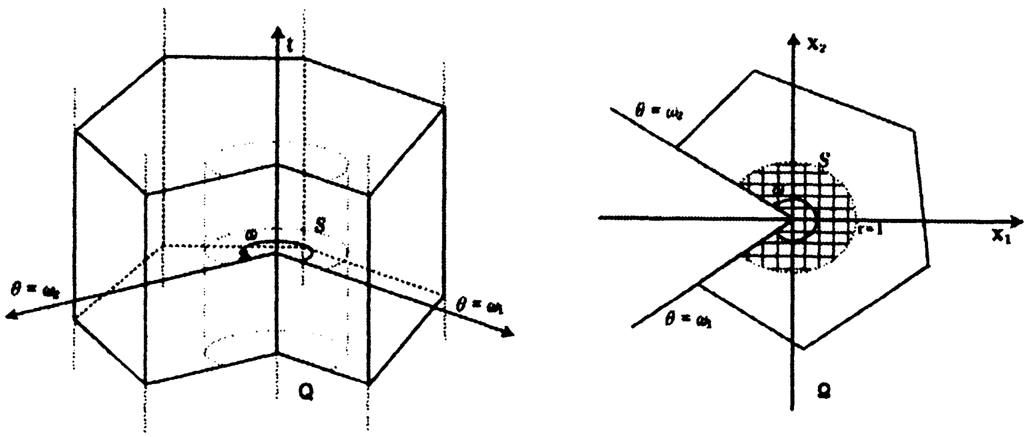

FIGURE 1. A polyhedral cylinder Q and a polygon Ω.

2. The stationary case

It is well-known in [9, 11, 13, 15, 16] that the solution $u \in H_0^1(\Omega)$ of the Dirichlet problem for the Laplace equation $-\Delta u = f$ on a bounded domain Ω with only one concave corner placed at the origin can be written as follows: There is a continuous linear functional Λ on H^{s-2} such that if $f \in H^{s-2}$ is assumed for $s > 1+\alpha$, then

$$u = \Lambda(f)\chi r^{\alpha} \sin[\alpha(\theta - \omega_1)] + w, \quad w \in H^s(\Omega), \tag{2.1}$$

satisfying the following regularity estimate

$$\|w\|_s + |\Lambda(f)| \leq C\|f\|_{s-2} \tag{2.2}$$

for a constant C. In [18, 19] Kweon and Kellogg extended the above results to the following stationary barotropic compressible viscous Navier-Stokes system with inflow boundary condition for pressure:

$$\begin{aligned} -\mu\Delta\mathbf{u} - \nu\nabla \mathrm{div}\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad &\text{in } \Omega, \\ \rho \mathrm{div}\mathbf{u} + \rho'(p)\mathbf{u}\cdot\nabla p = g \quad &\text{in } \Omega, \\ \mathbf{u} = 0 \quad &\text{on } \Gamma, \end{aligned} \tag{2.3}$$

and later generalized to the whole compressible Navier-Stokes system containing the energy equation [20]. The first attempt of the corner singularity expansion to compressible viscous Navier-Stokes equations is from the paper [18], in which a simple compressible Stokes system was considered and the essential main idea is motivated. So we will sketch its idea here. The simple compressible Stokes system that is obtained by a linearization of the system (2.3) around an ambient flows is given by

$$\begin{aligned} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad &\text{in } \Omega, \\ \mathrm{div}\mathbf{u} + \mathbf{U}\cdot\nabla p = g \quad &\text{in } \Omega, \\ \mathbf{u} = 0 \quad &\text{on } \Gamma, \\ p = 0 \quad &\text{on } \Gamma_{in}, \end{aligned} \tag{2.4}$$

where $\mathbf{U} = [1, 0]$ and $\Gamma_{in} = \{(x,y) \in \Gamma : \mathbf{U}\cdot n < 0\}$.

We here state the corner singularity result for the solution $[\mathbf{u}, p]$ of (2.4) on the polygon Ω described above. The proof is shown in [18, 19].

Theorem 2.1. *If $[\mathbf{f}, g] \in \mathbf{L}^2 \times L^2$, then there is a unique solution $[\mathbf{u}, p] \in \mathbf{H}_0^1 \times L^2$ of (2.4), satisfying the inequality $\|\mathbf{u}\|_1 + \|p\|_0 \leq C(\|\mathbf{f}\|_0 + \|g\|_0)$ where $C = C(\Omega)$.*

On the other hand, let $\omega > \pi$ and $2 \leq q < \frac{1}{1-\alpha}$. Assume that $[\mathbf{f}, g] \in \mathbf{L}^q \times H^{1,q}$. Suppose μ is sufficiently large. Then the velocity solution of (2.4) can be split as follows:

$$\mathbf{u} = \mathbf{w} + \mathbf{C}\phi, \quad \mathbf{w} \in \mathbf{H}^{2,q}, \tag{2.5}$$

where C is the stress intensity factor which can be explicitly expressed in terms of data $[\mathbf{f}, g]$. Furthermore, the triple $[\mathbf{w}, p, C]$ satisfies the a priori error estimate

$$\|\mathbf{w}\|_{2,q} + \|p\|_{1,q} + |\mathbf{C}| \leq C(\|\mathbf{f}\|_{0,q} + \|g\|_{1,q}) \tag{2.6}$$

where $C = C(\Omega, \mu)$.

Here we briefly summarize the main ideas and procedures used in obtaining Theorem 2.1. We consider two solution operators. First we define $B : \mathrm{L}^2 \mapsto \mathrm{L}^2$ by $q = BG$ where q is the solution of

$$q_x = G \quad \text{in } \Omega, \quad q = 0 \quad \text{on } \Gamma_{in}. \tag{2.7}$$

The solution q of (2.7) is given by integrating the equation (2.7) in the x direction, more explicitly,

$$q(x, y) = \int_{\delta(y)}^{x} G(s, y) \mathrm{d}s.$$

Second we let $A : \mathbf{H}^{-1} \mapsto \mathbf{H}_0^1$ be defined by $A\mathbf{F} = \mathbf{u}$ where $\mathbf{u}$ is the solution of

$$-\Delta \mathbf{u} = \mathbf{F} \quad \text{in } \Omega, \quad \mathbf{u} = 0 \quad \text{on } \Gamma. \tag{2.8}$$

Using the operator B, the solution of the continuity equation is given by $p = B(g - \mathrm{div}\mathbf{u})$. So, if $\mathbf{F} = \mu^{-1}(\mathbf{f} - \nabla p)$, then

$$\mathbf{F} = \mu^{-1}(\mathbf{f} - \nabla Bg + B\mathrm{div}\mathbf{u}). \tag{2.9}$$

If (2.9) is applied to (2.8), one has a generalized convection-diffusion equation that involves the operator B:

$$\begin{aligned} -\Delta \mathbf{u} - \mu^{-1} \nabla B(\mathrm{div}\mathbf{u}) &= \mathbf{f}_1 \ \text{ in } \Omega, \\ \mathbf{u} &= 0 \ \text{ on } \Gamma, \end{aligned} \tag{2.10}$$

where $\mathbf{f}_1 := \mu^{-1}(\mathbf{f} - \nabla Bg)$. The problem (2.10) is a reformulation of the problem (2.4).

We apply the basic theory of the corner singularity for the Laplacian to the equation (2.10). We split the velocity by

$$\mathbf{u} = \mathbf{C}\phi + \mathbf{w}.$$

For the justification of the decomposition a crucial step is to show that the function $\mathbf{C}$ is well-defined and expressed in terms of the known data. By Theorem 2.1 one has $\mathbf{C} = \Lambda(\mathbf{f}_2)$ where Λ is defined in (2.1) and

$$\mathbf{f}_2 := \mathbf{f}_1 + \mu^{-1} \nabla B(\mathrm{div}\mathbf{u}). \tag{2.11}$$

In order to show that $\mathbf{C}$ is well-defined we must show that $\mathbf{f}_2 \in \mathbf{L}^q$. In doing this a main difficult step is to prove that

$$\mathbf{C} \cdot \nabla B \nabla \phi \in \mathbf{L}^q,$$

which is shown in [18, Lemma 2.2] for the L^2 space and in [19, Lemma 2.5] for the L^q space.

On the other hand, from $-\Delta\mathbf{w} = \mathbf{f}_2 + \mathbf{C}\Delta\phi$ with the function $\mathbf{f}_2$ of (2.11) and in view of Theorem 2.1, one must have

$$\Lambda(\mathbf{f}_2 + \mathbf{C}\Delta\phi) = 0. \tag{2.12}$$

This vector equation gives two linear equations whose solution is the stress intensity factor $\mathbf{C}$. The solvability of the linear system is shown in Lemma 3.1. With this $\mathbf{C}$, an increased regularity for the remainder $[\mathbf{w}, p]$ is established.

For a finite element analysis one may need a weak formulation which can give a computable form. The weak form for the regular part $\mathbf{w}$ of (2.5) can be formulated as follows. Since the pressure function in the continuity equation of (2.4) can be integrated along the horizontal line and using the inflow boundary condition we consider the following integration formula: for given G,

$$(BG)(x,y) := \int_{\delta_-(y)}^{x} G(s,y)ds, \tag{2.13}$$

where $(\delta_-(y), y)$ is the graph of the inflow boundary Γ_{in}, so the pressure solution p is given by

$$p = B(g - \mathrm{div}\mathbf{u}). \tag{2.14}$$

Using (2.14) and letting $\mathbf{f}_1 = \mathbf{f} - \nabla Bg$, system (2.4) becomes a single vector equation

$$\begin{aligned} -\mu\Delta\mathbf{u} - \nabla B(\mathrm{div}\mathbf{u}) &= \mathbf{f}_1 \qquad \text{in } \Omega, \\ \mathbf{u} &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{2.15}$$

Inserting the decomposition $\mathbf{u} = \mathbf{w} + \mathbf{C}\phi$ into (2.12), the coefficient vector $\mathbf{C}$ can be expressed by (see [9, 10])

$$\mathbf{C} = (\mu\pi)^{-1}\int_\Omega (\mathbf{f}_1 + \nabla B\mathrm{div}\mathbf{u})\phi_- \mathbf{dx} + \pi^{-1}\int_\Omega \mathbf{u}\Delta\phi_- \mathbf{dx}. \tag{2.16}$$

Replacing the function $\mathbf{u}$ of (2.15) by $\mathbf{u} = \mathbf{w} + C\phi$ and using the fact

$$\int_\Omega \phi\Delta\phi_- \mathbf{dx} = 0,$$

the coefficient vector $\mathbf{C}$ can be expressed by (see Lemma 2.2 in Section 2)

$$\begin{aligned} \mathbf{C} &= \mathcal{M}^{-1}\Big(\int_\Omega \mu\mathbf{w}\Delta\phi_- - B\mathrm{div}\mathbf{w}\nabla\phi_- \mathbf{dx} + \int_\Omega \mathbf{f}_1\phi_- \,\mathbf{dx}\Big) \qquad (2.17) \\ &:= \mathbf{C}_1(\mathbf{w}) + \mathbf{C}_2(\mathbf{f}, g), \end{aligned}$$

where

$$\begin{aligned} &\mathcal{M} = \mu\pi\mathrm{I} + \int_\Omega B\nabla\phi \otimes \nabla\phi_- \mathbf{dx} \text{ is the (2,2)-matrix,} \\ &\mathbf{C}_1(\mathbf{w}) = \mathcal{M}^{-1}\big[- (B\mathrm{div}\mathbf{w}, \nabla\phi_-) + \mu(\mathbf{w}, \Delta\phi_-)\big], \qquad (2.18) \\ &\mathbf{C}_2(\mathbf{f}, g) = \mathcal{M}^{-1}\big[(\mathbf{f}, \phi_-) + (Bg, \nabla\phi_-)\big], \end{aligned}$$

where I is the identity matrix and $\otimes$ denotes the tensor product.

The theory of the corner singularity for the compressible Stokes system was extended to the one of the edge singularity for the compressible Stokes system [21]. The result is

Theorem 2.2. *Let Ω have only one concave corner, which coincides with the sector $\mathcal{S}$ near the origin. Suppose $\nu = 0$. If $[\mathbf{f}, g] \in \mathbf{H}^{-1} \times \mathrm{L}^2$, then there is a unique solution $[\mathbf{u}, p] \in \mathbf{H}_0^1 \times \mathrm{L}^2$ of (2.4), satisfying $\|\mathbf{u}\|_1 + \|p\|_0 \leq C(\|\mathbf{f}\|_{-1} + \|g\|_0)$ where $C = C(\Omega, \mu, \kappa)$. Also, if μ is large enough and if $[\mathbf{f}, g] \in \mathbf{H}^{s-2} \times \mathrm{H}^{s-1}$ for $1 \leq s < \alpha + 1$, then $\|\mathbf{u}\|_s + \|p\|_{s-1} \leq C(\|\mathbf{f}\|_{s-2} + \|g\|_{s-1})$ for a constant $C = C(\Omega, \mu)$. Finally, if μ is large enough and if $[\mathbf{f}, g] \in \mathbf{L}^2 \times \mathrm{H}^1$, the velocity $\mathbf{u}$ can be split as follows:*

$$\mathbf{u} = (\mathcal{E} \star \mathbf{c})\phi + \mathbf{w}, \quad \mathbf{w} := \mathbf{u} - (\mathcal{E} \star \mathbf{c})\phi, \tag{2.19}$$

$$\mathcal{E}(r, z) = \frac{r}{\pi(r^2 + z^2)}, \tag{2.20}$$

$$\mathbf{c}(z) := \frac{1}{2\pi i} \int_\gamma \langle \Lambda(\lambda); (\lambda I - \partial_{zz})^{-1} \mathbf{h}^*(z) \rangle d\lambda, \tag{2.21}$$

where $\star$ is the convolution in the z variable, $\Lambda(\lambda)$ is a continuous linear functional on $\mathrm{L}^2(\Omega)$ (see (2.10)), $\mathbf{h}^*$ is a known vector function with the components h_i^* and γ is a vertical axis satisfying $\mathcal{R}e\,\lambda > 0$, $\lambda \in \gamma$. Furthermore, we have $\mathbf{c} \in \mathbf{H}^{1-\alpha}(\mathbb{R})$, $\Delta[(\mathcal{E} \star \mathbf{c})\phi] \in \mathrm{L}^2$, and the triplet $[\mathbf{u}_R, p, \mathbf{c}]$ satisfies

$$\|\mathbf{w}\|_2 + \|p\|_1 + \|\mathbf{c}\|_{\mathrm{H}^{1-\alpha}(\mathbb{R})} \leq C(\|\mathbf{f}\|_0 + \|g\|_1), \tag{2.22}$$

where $C = C(\Omega, \alpha, \mu, \kappa)$. If Ω is convex, then $\mathbf{u} = \mathbf{w}$ satisfies (2.22).

3. The time-dependent case

In this section we will introduce some known results for the corner singularity expansion for the time-dependent compressible viscous Navier-Stokes fluid flows on a bounded domain having only one concave corner. However, the only known results, as far as we know, are the ones shown in [22, 23]. In [22] a corner singularity expansion for a linearized problem is derived and in [23] the expansion for the nonlinear problem is derived. Here we state main results for the corner singularity expansion and give some motivations used and related backgrounds.

As mentioned in the stationary case the regularity issue of solutions to the compressible Navier-Stokes system on domains with singular boundaries is very important problem to be resolved. Mostly the un-stationary compressible Navier-Stokes systems have been considered either in the whole space $\mathbb{R}^n$ or in the half space of $\mathbb{R}^n$ or in the exterior domains of bounded regions or in bounded domains with smooth boundaries. In [29] a global existence of classical solution for the un-stationary system of polytropic ideal fluids was proved in $\mathbb{R}^3$ assuming the data has high regularity order and is close to a stable equilibrium. The case of the half space or of an exterior domain was studied in [30] and the one of a bounded domain with smooth boundary in $\mathbb{R}^3$ was studied in [36], showing the existence of a global in time solution. Furthermore, in [27, 28] the global in time existence of weak solutions in the sense of distributions with bounded physical energy in the spirit of Leray's weak solutions is studied and in [8] integrability up to the boundary is studied in a bounded domain with Lipschitz boundary in $\mathbb{R}^3$. Also, in [7] regularity of weak solutions is proven in the plane $\mathbb{R}^2$ under periodic boundary condition, and in [5] an optimal regularity of global strong solutions is investigated under initial data close to a stable equilibrium, and in [6] uniqueness for compressible flows with data having critical regularity is stated.

However, if one try to understand such properties of solutions in domains with singular boundaries, the issues related to the singular boundaries have to be dealt with. Besides, the issues have been involved and critical in the computational fluid mechanics [2, 31, 34, 37]. Not only stresses and pressure singularities but also many interesting physical phenomena occur around the corners; for instance, eddy, recirculation, flow separation and discontinuity, etc. Hence it will be worthwhile to give a rigorous mathematical analysis for the occurrence of the singularities.

Main issues not resolved were to give an explicit description for the singular behaviors of solution to the system in domains with singular boundaries having corners, edges, turning points, etc, and to establish an increased regularity for the smoother part. However, known results for the unsteady case are very few in the literature; we refer to [10, 11, 12] for for the Laplace problem with parameter or the heat equation and [24] for the time-dependent incompressible Navier-Stokes system. In the unsteady case the coefficient of the corner singularity is a function of time, so the space of singular functions is infinite dimensional. However the coefficients of singularities at corners decay exponentially as the distance to corners increases.

The time-dependent compressible Navier-Stokes system (see [2]) to be considered in a bounded polygon $\Omega \subset \mathrm{R}^2$ is

$$\begin{aligned} \rho\mathbf{u}_t - \mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 \quad & \text{in } Q, \\ \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \quad & \text{in } Q, \\ \mathbf{u} = 0 \quad & \text{on } \Sigma, \\ \mathbf{u}(0) = \mathbf{u}_0, \ \rho(0) = \rho_0 \quad & \text{on } \Omega, \end{aligned} \tag{3.1}$$

where $Q := \Omega \times (0, \mathrm{T})$ with a number $\mathrm{T} > 0$, $\Sigma := \partial\Omega \times (0, \mathrm{T})$ is the lateral boundary of Q; $\mathbf{u}$ and p are the velocity and pressure variables; μ and ν are the coefficients of viscosity with $\mu > 0$ and $\mu + \nu > 0$; $\rho = \rho(p)$ is a strictly increasing smooth function of pressure p; $\mathbf{u}_0$ and ρ_0 are given initial data with $\mathbf{u}_0|_{\partial\Omega} = 0$ and $\rho_0 = \rho(p_0)$. We shall take the simpler case $\nu = 0$.

We define a function $\mathcal{E}$ by

$$\mathcal{E}(r,t) := re^{-r^2/ct}\Big/\sqrt{c\pi t^3}, \quad t > 0, \tag{3.2}$$

where c is a number. We here state a main regularity result for (3.1), which is shown in [23].

Theorem 3.1. *Let* $2 < q < 1/(1-\alpha)$. *Assume* $\mathbf{u}_0 \in \mathbf{H}^{2,q} \cap \mathbf{H}_0^1$ *and* $p_0 \in \mathrm{H}^{1,q}$. *Suppose the Reynolds number* R_e *is sufficiently small. Let* $0 < \mathrm{T} \leq R_e^{-1}$ *be a number. For any positive constant* C_1, *there is a constant* C_2 *such that if* $\|\mathbf{u}_0\|_{2,q} + \|p_0\|_{1,q} \leq C_1$ *and for some constant* $\tilde{\rho} > 0$

$$\|\nabla \mathbf{u}_0\|_{1,q} + \|\nabla p_0\|_{1,q} + \|\tilde{\rho} - \rho_0\|_\infty \leq C_2,$$

then there is a unique solution $[\mathbf{u}, p] \in \mathrm{L}^q(0, \mathrm{T}; \mathbf{H}_0^{1,q}) \times \mathrm{L}^q(0, \mathrm{T}; \mathrm{H}^{1,q})$ *of system* (3.1). *For each* $n \in \mathcal{I}$ *there is a vector function* $\mathbf{c}(t)$ *on the half interval* $\mathbb{R}^+$, *with zero outside* $(0, \mathrm{T})$, *such that the velocity* $\mathbf{u}$ *can be split as follows:*

$$\mathbf{u} = (\mathcal{E} \star \mathbf{c})\, \phi + \mathbf{u}_R, \quad \mathbf{u}_R := \mathbf{u} - (\mathcal{E} \star \mathbf{c})\, \phi + \mathbf{u}_0,$$

where $\star$ *denotes convolution in time* t. *Also the function* $\mathbf{c} \in \mathbf{H}^{1/q' - \alpha/2, q}(0, \mathrm{T})$ *where* $1/q + 1/q' = 1$, *and the pair* $[\mathbf{u}_R, p] \in \mathrm{L}^q(0, \mathrm{T}; \mathbf{H}^{2,q}) \times \mathrm{L}^q(0, \mathrm{T}; \mathrm{H}^{1,q})$, *satisfies the inequality*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0 \leq t \leq \mathrm{T}} \|\mathbf{u}_R(t) - \mathbf{u}_0\|_{1,q} + \|\mathbf{u}_R - \mathbf{u}_0\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{H}^{2,q})} \\
&+ \|\mathbf{u}_R'\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{L}^q)} + \|\mathbf{c}\|_{\mathrm{H}^{1/q'-\alpha/2,q}(0,\mathrm{T})} \\
&+ \|p - p_0\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{H}^{1,q})} + \|p'\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{L}^q)} \\
&+ \|p - p_0\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}^{1,q})} \leq C_3,
\end{aligned} \tag{3.3}$$

where $C_3 = C(C_1, C_2)$. *If* $1 < q < 2/(2-\alpha)$, *so* $\mathcal{I} = \emptyset$, *then* $\mathbf{u} = \mathbf{u}_R$ *satisfies* (3.3).

Remark 3.1. *With the restriction* $\mathrm{T} \leq R_e^{-1}$ *the Jacobian determinant of the streamline function directed by the velocity vector field will not vanish. The smallness condition of the Reynolds number* R_e *implies that the considered flow can be thought of a little perturbation around a laminar constant flow. The coefficient function in the singular part of the velocity is convolution integral in time* t *as follows:*

$$(\mathcal{E} \star \mathbf{c})(t) = \int_0^t \mathcal{E}(\sigma)\mathbf{c}(t-\sigma)d\sigma,$$

so it is not zero for all time $t > 0$ *(provided that* $\mathbf{c} \neq 0$*) and decays exponentially as the distance* r *to the vertex* P *increases.*

In the lowest order level the corner singularity corresponding to the pressure does not have to be split. However, if $\mathbf{u}_s(\mathbf{x}, t) := (\mathcal{E}(\mathbf{x}, \cdot) \star \mathbf{c})(t)\, \phi(\mathbf{x})$ *is defined, the pressure singularity is defined by*

$$p_s(\mathbf{x}, t) := \int_0^t (\kappa^{-1} \mathrm{div} \mathbf{u}_s)(\mathbf{h}(\varphi, s), s)\, \mathrm{d}s,$$

where $\mathbf{h}$ *is the particle trajectory mapping and* φ *its inverse function. Hence by the formula, we observe that the corner singularity is propagated along the streamlines emanating from the corners and the derivatives of* p_s *can be infinite near the concave vertices.*

The reason of choosing the interval $(2, 1/(1-\alpha))$ *in Theorem 1.1 is as follows. If* $q \geq 2/(2-\alpha)$, *the velocity* $\mathbf{u}$ *is split. To bound the pressure* p *near the concave vertices in the* $\mathrm{L}^q(0,\mathrm{T};\mathrm{H}^{1,q})$*–norm, the index* q *is required to be strictly less than* $1/(1-\alpha)$ *with* $\alpha < 1$ *and the inequality* $q > 2$ *is required to guarantee that the density function is well-defined. Near the convex vertices* $(\alpha > 1)$, *the index* q *is chosen in the interval* $(1, \infty)$ *and so* q *can be chosen in the range* $2 < q < \infty$.

To show Theorem 1.1 we transform the problem (3.1) into the one with zero initial data and linearize the resulted one in a suitable way. We explain some main procedures of showing Theorem 3.1. First, we consider a linearized problem for the system (3.1) (see [22, 23]) but here introduce a simple linearized version, for simplicity:

$$\begin{aligned} R_e^{-1}(\mathbf{u}_t - \nu\Delta\mathbf{u}) + \mathbf{U}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } Q, \\ R_e^{-1}p_t + \mathbf{U}\cdot\nabla p + \mathrm{div}\mathbf{u} = g \quad & \text{in } Q, \\ \mathbf{u} = 0 \quad & \text{on } \Sigma, \\ \mathbf{u}(\cdot,0) = 0,\ p(\cdot,0) = 0 \quad & \text{on } \Omega, \end{aligned} \tag{3.4}$$

where R_e is the Reynolds number for the given ambient flow $\mathbf{U}$, ν is the viscous number, the functions $\mathbf{U}$, $\mathbf{f}$ and g are given.

From the system (3.4) we see that the first equation is parabolic(or elliptic in the steady case) in the velocity variable and the second one is hyperbolic in the pressure one. So it is of mixed type. To solve this problem one has to use methods(schemes) considering these two characters. Therefore we consider two solution operators: One is the solution operator $A : \mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{-1}) \mapsto \mathrm{L}^2(0,\mathrm{T};\mathrm{H}_0^1)$, defined by $u = Af$, with $u_t \in \mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{-1})$, where u is the solution of

$$\begin{aligned} u_t - \Delta u = f \quad & \text{in } Q, \\ u = 0 \quad & \text{on } \Sigma, \\ u(0) = 0 \quad & \text{on } \Omega, \end{aligned} \tag{3.5}$$

where $Q = \Omega \times (0,\mathrm{T})$ and $\Sigma = \Gamma \times [0,\mathrm{T}]$. The other one is the operator $B : \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2) \mapsto \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2)$, defined by $q = B\mathrm{G}$, where q is the solution of

$$q_t + \mathbf{U}\cdot\nabla q = \mathrm{G} \ \text{ in } Q, \quad q(\cdot,0) = 0 \ \text{ on } D, \tag{3.6}$$

where $\mathbf{U}$ is a given vector field.

Since the corner singularity expansion to the problem (3.5) can be obtained by taking the inverse Laplace transform to the corner singularity expansion for the Laplace problem with parameter, we consider

$$\begin{aligned} -\Delta v + \lambda v = g \text{ in } \Omega, \\ v = 0 \text{ in } \Gamma, \end{aligned} \tag{3.7}$$

where λ is not a real negative number to stay away from the eigenvalues of the Laplace operator Δ. The corner singularity result (see [10, 11, 12]) of the solution v of (3.7) can be stated as follows:

Lemma 3.1. *Let $1 < q < 2/(1-\alpha)$. Let $g \in \mathrm{L}^q$ or $g \in \mathrm{H}^{-1,q}$. Let $v = A^\lambda g$ be the solution of (3.2). Then A^λ is a bounded operator from $\mathrm{H}^{-1,q}$ to $\mathrm{H}^{1,q}$, satisfying $\|A^\lambda g\|_{1,q} + |\lambda|\|v\|_{-1,q} \le C\|g\|_{-1,q}$. (i) If $q < 2/(2-\alpha)$ then $v \in \mathrm{H}^{2,q}$ and satisfies*

$$\|v\|_{2,q} + |\lambda|^{1/2}\|v\|_{1,q} + |\lambda|\|v\|_{0,q} \le C\|g\|_{0,q}.$$

(ii) Let $2/(2-\alpha) < q < 1/(1-\alpha)$. There exist a continuous linear form $\Lambda(\lambda)$, which is denoted by $\Lambda^\lambda := \Lambda(\lambda)$, on L^q and a function $\psi(\lambda) \notin \mathrm{H}^{2,q}$. Then the solution u of (3.2) may be split as follows:

$$v = \Lambda^\lambda[g]\,\psi(\lambda) + v_R, \quad v_R := v - \Lambda^\lambda[g]\,\psi(\lambda). \tag{3.8}$$

Furthermore, there is a constant C such that

$$\|v_R\|_{2,q} + |\lambda|^{1/2}\|v_R\|_{1,q} + |\lambda|\|v_R\|_{0,q} \le C\|g\|_{0,q}, \tag{3.9}$$

$$|\Lambda^\lambda[g]| \le C(1+|\lambda|)^{\alpha/2-1/q'}\|g\|_{0,q}. \tag{3.10}$$

The singular function ψ can be given by

$$\begin{aligned} \psi(\lambda) &= e^{-r\sqrt{\lambda}}\,\phi, && \text{if } \alpha > 2/q' - 1, \\ \psi(\lambda) &= (1 + r\sqrt{\lambda})e^{-r\sqrt{\lambda}}\,\phi, && \text{if } \alpha \le 2/q' - 1. \end{aligned} \tag{3.11}$$

In order to cast the corner singularity result of the Laplace problem with parameter into the one of the Heat equation (3.5) we apply the inverse Laplace transform to the expansion given in Lemma 3.1 and can formulate the following theorem (see [10, 11, 12]):

Theorem 3.2. *Let $1 < q < 2/(1-\alpha)$ and $\mathbb{R}_+ = (0, \infty)$. If $f \in \mathrm{L}^q(\mathbb{R}_+; \mathrm{H}^{-1,q})$ then there is a unique solution u of (3.5), satisfying the inequality*

$$\begin{aligned} &\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{L}^q)} + \|u\|_{\mathrm{L}^q(\mathbb{R}_+;\mathrm{H}^{1,q})} + \|u'\|_{\mathrm{L}^q(\mathbb{R}_+;\mathrm{H}^{-1,q})} \\ &\le C\|f\|_{\mathrm{L}^q(\mathbb{R}_+;\mathrm{H}^{-1,q})}. \end{aligned} \tag{3.12}$$

On the other hand, let $2/(2-\alpha) < q < 1/(1-\alpha)$. If $f \in \mathrm{L}^q(\mathbb{R}_+; \mathrm{L}^q)$, then the solution u is split as follows:

$$\begin{aligned} u(t) &= (\mathcal{E} \star \Phi(f))(t)\,\phi + u_R(t), \\ \Phi(f)(t) &= \frac{1}{2\pi i}\int_\gamma \Big\langle \Lambda(\lambda); (\lambda I - \partial_t)^{-1} \Big\rangle d\lambda f(t), \\ u_R(t) &= u(t) - (\mathcal{E} \star \Phi(f))(t)\,\phi, \end{aligned} \tag{3.13}$$

satisfying $\Phi(f) \in \mathrm{H}^{1/q'-\alpha/2,q}(\mathbb{R}_+)$ and $\Lambda(\lambda)$ is defined in Lemma 2.1. Here the curve γ is a vertical axis satisfying $\mathcal{R}e\,\lambda < 0$, $\lambda \in \gamma$. Furthermore the regular part u_R and the function $\Phi(f)$ satisfy

$$\begin{aligned} &\|u_R\|_{\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{H}^{1,q})} + \|u_R\|_{\mathrm{L}^q(\mathbb{R}_+;\mathrm{H}^{2,q})} + \|u_R'\|_{\mathrm{L}^q(\mathbb{R}_+;\mathrm{L}^q)} \\ &+\|\Phi(f)\|_{\mathrm{H}^{1/q'-\alpha/2,q}(\mathbb{R}_+)} \le C\|f\|_{\mathrm{L}^q(\mathbb{R}_+;\mathrm{L}^q)}. \end{aligned} \tag{3.14}$$

If $q < 2/(2-\alpha)$, the solution $u = u_R$ satisfies the inequality (3.14).

Next we will discuss about roles of the transport equation (3.6). As noticed from the continuity equation in (3.4), to understand the transport equation (3.6) well is very important because it describes physical phenomena carrying fluids in one place over some other places, for example, in this contents, carrying fluids having a corner singularity into the region. Hence the role of the vector field $\mathbf{U}$ appearing in the equation is critical since it gives direction to the considered fluid flows and also it is related to the corner singularity expansion because it comes from a linearization of the velocity vector field. Hence one needs to specify a Banach space for the vector field $\mathbf{U}$. We set, for $q > 2$,

$$\begin{aligned} \mathbf{W} : &= \mathrm{L}^q(0,\mathrm{T};\mathbf{H}_0^{2,q}) \cap \mathrm{L}^\infty(0,\mathrm{T};\mathbf{H}^{1,q}) \cap \mathrm{H}^{1,q}(0,\mathrm{T};\mathbf{L}^q), \\ \mathcal{Z} : &= \mathbf{H}^{1/q'-\alpha/2,q}(0,\mathrm{T}). \end{aligned}$$

Let $\|\cdot\|_{\mathbf{W}}$ and $\|\cdot\|_{\mathcal{Z}}$ be the norms induced by the spaces $\mathbf{W}$ and $\mathcal{Z}$, respectively. For handling nonlinearity later it is assumed that the vector $\mathbf{U}$ vanishes on $\Sigma \cup \partial\Omega$, and has the corner singularity decomposition of the form

$$\mathbf{U} = (\mathcal{E} \star \mathbf{d})\,\phi + \mathbf{U}_R, \tag{3.15}$$

where $\mathbf{U}_R \in \mathbf{W}$, and $\mathbf{d} \in \mathcal{Z}$ is a given function of only t variable, with zero outside $(0, \mathrm{T})$, and $\mathcal{E}$ is given in (3.2).

By a particle trajectory mapping by the vector field $[1, \mathbf{U}]$, for each fixed $\bar{\mathbf{x}}$ we consider a curve $(\mathbf{h}(\bar{\mathbf{x}}, t), t)$ where $\mathbf{h}$ is the solution of

$$\mathbf{h}_t(\bar{\mathbf{x}}, t) = \mathbf{U}(\mathbf{h}(\bar{\mathbf{x}}, t), t), \quad \mathbf{h}(\bar{\mathbf{x}}, 0) = \bar{\mathbf{x}}. \tag{3.16}$$

Since $\mathbf{U} = 0$ on $\Sigma \cup \partial\Omega$, the curve $(\mathbf{h}(\bar{\mathbf{x}}, t), t)$ emanates from the points in $\Omega \times \{0\}$. From the theory of the ordinary differential equations, if the vector function $\mathbf{U}(\mathbf{x}, t)$ is Lipschitz continuous in $\mathbf{x}$, the solution $\mathbf{h}$ to (3.16) exists, is unique, and is continuous differentiable in t and continuous in $\bar{\mathbf{x}}$. For fixed t the equation $\mathbf{x} = \mathbf{h}(\bar{\mathbf{x}}, t)$ has a well-defined solution $\bar{\mathbf{x}}$, where we write $\bar{\mathbf{x}} = \varphi(\mathbf{x}, t) = (\varphi^1(\mathbf{x}, t), \varphi^2(\mathbf{x}, t))$ for a vector function φ. Under a suitable condition on $\mathbf{U}$ the Jacobian determinants $J_{\mathbf{h}}$, J_φ of $\mathbf{h}$ and φ, respectively, will not be zero on the interval $[0, \mathrm{T}]$ for a number $\mathrm{T} > 0$.

Next we give a Lemma (see [23]), which gives some criteria for the particle trajectory mapping $\mathbf{h}(\bar{\mathbf{x}}, t)$ under the condition that the vector $\mathbf{U}$ has the decomposition (3.15).

Lemma 3.2. *If $1 < q < 2/(2-\alpha)$, let $\mathbf{U} \in \mathbf{W}$. If $2 < q < 1/(1-\alpha)$, let $\mathbf{U}$ have the decomposition given in* (3.15). *Then the vector functions $\mathbf{h}$ and φ are well defined and continuously differentiable on the cylinder $Q = \Omega \times (0, \mathrm{T})$. The Jacobian determinants $J_{\mathbf{h}}$, J_φ do not vanish on the interval $[0, \mathrm{T}]$ and are estimated by a constant that depends only on Ω and the quantity $\gamma_0 := \|\mathbf{U}_R\|_{\mathbf{W}} + \|\mathbf{d}\|_{\mathcal{Z}}$.*

By Lemma 3.2 a criteria for the trajectory mapping $\mathbf{h}(\bar{\mathbf{x}}, t)$ is given. Based on this we give a solution formula for the transport equation (3.6). In fact it can be obtained by integrating along the trajectories generated by the vector $[1, \mathbf{U}]$. Using the trajectory mapping $\mathbf{h}(\bar{\mathbf{x}}, t)$ and its inverse $\varphi(\mathbf{x}, t)$, the solution of (3.6) is given by

$$(BG)(\mathbf{x}, t) = \int_0^t G(\mathbf{h}(\varphi, s), s)\, \mathrm{d}s. \tag{3.17}$$

Note that regularities of the function BG are essential in establishing the corner singularity expansion of solution of (3.4) and its regularity for the remainder (for details, see [23]): Let $1 < q < \infty$. For $0 \le s \le 1$ the mapping

$$B : \mathrm{L}^q(0, \mathrm{T}; \mathrm{H}^{s,q}) \to \mathrm{L}^q(0, \mathrm{T}; \mathrm{H}^{s,q})$$

is a bounded operator, satisfying $\|BG\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{H}^{s,q})} \le C\|G\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{H}^{s,q})}$ where $C = C(D, \gamma_0)$.

We next cite a result for the corner singularity expansion result of (3.4), which can be derived by the property of the transport operator B and the corner singularity result of the Heat equation (see [23] for details):

Theorem 3.3. *Let $0 < \mathrm{T} \le R_e^{-1}$ and $1 < q < 2/(1-\alpha)$. Let $\mathbf{f} \in \mathrm{L}^q(0, \mathrm{T}; \mathbf{H}^{-1,q})$ and $g \in \mathrm{L}^q(0, \mathrm{T}; \mathrm{L}^q)$. Suppose that the Reynolds number R_e is sufficiently small. There is a unique solution $[\mathbf{u}, p]$ of (3.4). Additionally, if we assume that $\mathbf{f} \in \mathrm{L}^q(0, \mathrm{T}; \mathbf{L}^q)$ and $g \in \mathrm{L}^q(0, \mathrm{T}; \mathrm{H}^{1,q})$ for $2 < q < 1/(1-\alpha)$, then the velocity solution $\mathbf{u}$ can be split as follows:*

$$\mathbf{u} = (\mathcal{E} \star \mathbf{c})\,\phi + \mathbf{u}_R, \quad \mathbf{u}_R := \mathbf{u} - (\mathcal{E} \star \mathbf{c})\,\phi,$$

where $\mathbf{c}(t)$ is a vector function of the time variable t. Also the function $\mathbf{c} \in \mathbf{H}^{1/q'-\alpha/2,q}(0, \mathrm{T})$ and there is a constant $C_4 = C(\|\mathbf{U}_R\|_{\mathbf{W}}, \|\mathbf{d}\|_Z)$ such that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0 \le t \le \mathrm{T}} \|\mathbf{u}_R(t)\|_{1,q} + \|\mathbf{u}_R\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{H}^{2,q})} \\
&+ \|\mathbf{u}_R'\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{L}^q)} + \|\mathbf{c}\|_{\mathrm{H}^{1/q'-\alpha/2,q}(0,\mathrm{T})} \\
&+ \|p\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{H}^{1,q})} + \|p'\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{L}^q)} + \|p\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}^{1,q})} \\
&\le C_4 R_e\big(\|\mathbf{f}\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{L}^q)} + \|Bg\|_{\mathrm{L}^q(0,\mathrm{T};\mathrm{H}^{1,q})}\big).
\end{aligned} \tag{3.18}$$

If $1 < q < 2/(2-\alpha)$, so $\mathcal{I} = \emptyset$, then $\mathbf{u} = \mathbf{u}_R$ satisfies (3.18).

Finally we review the procedure of showing Theorem 3.1 for the corner singularity expansion of (3.1). First a suitable linearized version of (3.1) is needed, and second the known results for the Heat equation (3.5) and the transport equation (3.6) are used (see Theorem 3.2 and the properties of the operator B). Third, a similar result like Theorem 3.3 for the considered linear problem need to be shown and finally the Schauder fixed point theory can be applied to get the solution of (3.1), having the corner singularity expansion. All these details are given in the paper [23]. □

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