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Social Decision Considering Subjective Feelings

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Abstract. As a method of obtaining a fair social decision, we propose the fuzzy social decision procedure in which each member can choose freely appraisal criteria for his preferences and, taking the ambiguity of human judgement into consideration, the comprehensive evaluation for each alternative is made by a fuzzy integral.

Key words : social decision, voting system, evaluation, fuzzy integral

1 Introduction

The main subject in the social decision theory is as follows : How should we put together opinions of all members in order to obtain a fair social decision? As a method of compiling opinions various voting systems are used. The most popular voting system is the majority rule in which each voter writes the most favorable alternative and an alternative obtained the maximum number of votes wins. Since this system is simple in both voting and counting votes, it is well-used, but it is an excessive simplification of information to take only the best alternative into consideration. In actuality the voter who hesitates between the first and the second alternatives, has dissatisfaction with the ignoration of the second alternative.

As a voting system considering all preference orders of each voter, the Condorcet procedure and the Borda procedure have been proposed. The Condorcet procedure decides the social preference order for all alternatives by collecting the social preference for each pair of alternatives which is decided by the majority rule by all voters. But it is well-known that the Condorcet procedure may result in the “voting paradox”, that is, the cyclic preference order occurs and then the social preference order can't be decided.

On the other hand, in the Borda procedure points are allocated to each alternative according to the preference order of each voter, that is, in the case of m alternatives, an alternative with the k-th preference order is allocated $(m-k+1)$ points. The social preference order for alternatives is decided in order of the sum of allocated points. For the Borda procedure the following characteristics are well-known.

(1) Each voter must decide the preference order for all alternatives, but the difference between neighbouring orders is always one. Then the Borda procedure can't represent strictly the degree of strength of preference.

(2) When the set of alternatives varies, it may occur that the social preference order among old alternatives varies.

(3) A desirable social decision procedure must have the property of duality, that is, when all preference orders of all voters are reversed, the social preference order must not be the same as before. But the Borda procedure doesn't have this property.

(4) Under the Borda procedure, it may occur that in order to make the specific alternative the social selection a certain voter varies his own preference orders and succeeds.

In general both the Condorcet and the Borda procedures consider not the strength itself but the order only of preference and then can't represent true preferences strictly. Moreover under these procedures the appraisal criteria are not taken up positively. Discussions of the various voting systems are shown in D.Black [1].

We propose the fuzzy social decision (FSD) procedure with the following characteristics:

(1) We take up appraisal criteria positively and let each member select them freely.

(2) Each member puts subjective weights among his own appraisal criteria.

(3) Taking the ambiguity of appraisal into consideration, the fuzzy integral is used in a comprehensive evaluation of an alternative by each member.

(4) The social decision is made by not the preference order but the sum of strengths of preference.

For making a social decision, traditional procedures require each member to express his own preference order for all alternatives, but the FSD procedure goes into the details of the process of individual preference, considers the subjectivity and the fuzziness in evaluations and makes a social decision by strengths of preferences. Then the FSD procedure is expected to solve many weak points of traditional procedures. Comparing the FSD procedure with the majority rule, it is sure that the FSD procedure imposes more burdens on both members and counters. But since it has respect for the subjective preference by each member as much as possible, it is worth adopting as occasion calls.

2 The FSD procedure

A society being composed of many members wants to decide the social preference order for many alternatives. Each member has allotted points according to his social position and allocates them among alternatives. To compile individual preferences of all members we propose the FSD procedure which details are as follows:

Step 1: Each member enumerates appraisal criteria for deciding his preferences. He can choose freely the number and contents of the criteria.

Step 2: As the AHP (Analytic Hierarchy Process : T.L.Saaty [3]) method, each member repeats paired comparisons among his appraisal criteria. As a result, his unconscious subjective weights on criteria are clarified. Moreover the consistency among paired comparisons is verified.

Step 3: Subjective weights on criteria are transformed into a fuzzy measure. In the case of a λ-fuzzy measure, in order to decide the value of parameter λ, in addition the member is asked his weight of a certain set of alternatives.

Step 4: Each member marks all alternatives under each of his own appraisal criteria. In this occasion maximum marks must be all the same.

Step 5: Considering the fuzziness of human judgement, the comprehensive evaluation of each alternative is made according to a fuzzy integral of marks with respect to the fuzzy measure mentioned in step 3.

Step 6: Each member allocates his allotted points to all alternatives in proportion to comprehensive evaluations.

Step 7: The score of each alternative is decided by summing up points allocated by all members.

Step 8: The social preference order follows the order of scores.

Tasks of each member are as follows:
Enumerating criteria (step1), Paired comparisons (step2),
Answering the question to decide the parameter λ (step3),
Marking an alternative under each criterion (step4)
Other tasks are carried out by the election committee.

Here we show the definition of a fuzzy measure (L.A.Zadeh[5]). We consider a finite set $X = \{x_1, \cdots, x_n\}$ and let A, B be subsets of X.

Definition 1. If a set function $g(\cdot)$ on the family of subsets of X satisfys the following conditions, it is called a fuzzy measure on X.

(i) $g(\phi) = 0\ ,\ g(X) = 1$
where ϕ is a null set.

(ii) $A \subseteq B \Rightarrow g(A) \leq g(B)$ (monotonicity) .

Definition 2. For a constant $\lambda(> -1)$, if a fuzzy measure $g_\lambda(\cdot)$ on X satisfys the following condition, it is called a λ-fuzzy measure.

(iii) For any subsets A and B of X $(A \cap B \neq \phi)$,

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B). \tag{1}$$

Remark 1. The condition $\lambda > -1$ is necessary to satisfy the monotonicity.

Next we show the definition of the Choquet integral (G.Choquet[2], D.Schmeidler[4]). Let $h(\cdot)$ be a real-valued function on X satisfying

$$h(x_1) \geq h(x_2) \geq \cdots \geq h(x_n). \tag{2}$$

Moreover we put

$$H_i = \{x_1, \cdots, x_i\} \qquad (i = 1, \cdots, n). \tag{3}$$

Definition 3. The right side of the following equation (4) is called the Choquet integral of a function $h(\cdot)$ on X with respect to the fuzzy measure $g(\cdot)$.

$$(C) \int h\, dg = \sum_{i=1}^{n} [h(x_i) - h(x_{i+1})] g(H_i) \tag{4}$$

where $h(x_{n+1}) = 0$.

Remark 2. The Choquet integral (4) represents an area of the part under the function $h(\cdot)$, weighted by the fuzzy measure $g(\cdot)$.

3 Numerical Example

We consider an election problem in a society consisted of ten members (voters). We suppose that there are four alternatives A, B, C and D. We explain in detail step 1 ~ step 6 in the case of member 3 as follows:

Step 1: We suppose that member 3 had enumerated three appraisal criteria x_1(pledge), x_2(public opinion) and x_3(political party).

Step 2: Letting member 3 carry out paired comparisons among criteria x_1, x_2 and x_3, we had obtained the result in Table 1. The weights of criteria can be obtained by the geometric average (G.A.) method (for example, the G.A. of pledge is $\sqrt[3]{1 \times 3 \times 7} = 2.759$). The weights in Table 1 are the standardizations of G.A.s.

Table 1. Paired comparisons and weights by member 3.

	x_1	x_2	x_3	G.A.	weight
x_1	1	3	7	2.759	0.649
x_2	1/3	1	5	1.186	0.279
x_3	1/7	1/5	1	0.306	0.072

In general, when a paired comparison matrix is given by $A = [a_{ij} | i, j = 1, \cdots, n]$ and its weight vector is given by $w = (w_1, \cdots, w_n)$, the consistency of paired comparisons is well-known to be given by the consistency index

$$C.I. = \frac{1}{n-1}(\lambda_{\max} - n) \tag{5}$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix A. In the case of the G.A. method, $\lambda_{\max}$ is estimated by

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{w_i}\sum_{j=1}^{n} a_{ij}w_j \, .$$

It is well-known that the consistency holds if $C.I. \leq 0.1 \sim 0.15$. In our numerical example, $C.I. \fallingdotseq 0.031$ and therefore the consistency of the paired comparisons in Table 1 holds.

Step 3: The weight vector of appraisal criteria is given by

$$w = c(0.649, 0.279, 0.072)$$

where c is a positive constant. Using the weight vector, we constitute a λ-fuzzy measure $g_\lambda(\cdot)$ on the family of subsets of $X = \{x_1, x_2, x_3\}$. We put

$$g_\lambda(\{x_1\}) = 0.649c, \qquad g_\lambda(\{x_2\}) = 0.279c, \qquad g_\lambda(\{x_3\}) = 0.072c \, . \tag{6}$$

By the equation (1), we obtain

$$g_\lambda(\{x_1, x_2\}) = 0.928c + 0.181\lambda c^2 \tag{7}$$

$$g_\lambda(\{x_2, x_3\}) = 0.351c + 0.020\lambda c^2 \tag{8}$$

$$g_\lambda(\{x_1, x_3\}) = 0.721c + 0.047\lambda c^2 \tag{9}$$

$$\begin{aligned} g_\lambda(\{x_1, x_2, x_3\}) &= (0.928c + 0.181\lambda c^2) + 0.072c \\ &\quad + \lambda(0.928c + 0.181\lambda c^2) \times 0.072c = 1 \end{aligned} \tag{10}$$

where the last equality of the equation (10) is derived from $g_\lambda(X) = 1$ in Definition 1.

But from relations (6)~(10) only, two constants c and λ can't be decided. Then we ask member 3 his subjective weight of double appraisals $\{x_1, x_2\} = \{pledge,\ public\ opinion\}$. It is assumed that member 3 has answered the weight was 0.95. Then

$$g_\lambda(\{x_1, x_2\}) = 0.95c\,. \tag{11}$$

From the simultaneous equations (7), (10) and (11), we obtain

$$c = 0.971\ ,\ \lambda = 0.125\,. \tag{12}$$

Substituting (12) into (6)~(9), we can decide the λ-fuzzy measure as follows:

$$g_\lambda(\phi) = 0\ ,\ g_\lambda(X) = 1 \tag{13}$$

$$g_\lambda(\{x_1\}) = 0.63\ ,\ g_\lambda(\{x_2\}) = 0.27\ ,\ g_\lambda(\{x_3\}) = 0.07 \tag{14}$$

$$g_\lambda(\{x_1, x_2\}) = 0.92\ ,\ g_\lambda(\{x_2, x_3\}) = 0.34\ ,\ g_\lambda(\{x_1, x_3\}) = 0.71 \tag{15}$$

Step 4: We have gotten member 3 to mark all alternatives under each of criteria on a maximum scale of 10 points and obtained the results in Table 2.

Table2. Appraisals of alternatives by member 3.

	x_1	x_2	x_3
A	4	2	10
B	5	3	4
C	0	0	0
D	1	4	8

Step 5: By the definition (4) of the Choquet integral, comprehensive evaluations of alternatives by member 3 are given by

$$\begin{aligned}
\mu_A &= 2 \times 1 + (4-2) \times 0.71 + (10-4) \times 0.07 = 3.84 \\
\mu_B &= 3 \times 1 + (4-3) \times 0.71 + (5-4) \times 0.63 = 4.34 \\
\mu_C &= 0 \times 1 + (0-0) \times 0.92 + (0-0) \times 0.63 = 0 \\
\mu_D &= 1 \times 1 + (4-1) \times 0.34 + (8-4) \times 0.07 = 2.3
\end{aligned} \tag{16}$$

Step 6: It is assumed that allotted points of member 3 are 10 points. Since each member allocates his points in proportion to comprehensive evaluation, member 3 allocates 3.6, 4.2, 0, 2.2 points to alternatives A, B. C, D respectively.

We deal with the case of other members by the same manner as above mentioned. We abbreviate their details. Of course, the number and contents of criteria are selected freely by each member. In order to compare the FSD procedure with the Borda procedure, we suppose that the allotted points of all members are the same 10 points. Let's assume that the allocation of points by all members are given in Table 3. From Table 3 the social preference order by the FSD procedure is D≻A≻B≻C where "$a \succ b$" means that alternative a is preferred rather than b.

Table3. The allocations of points by the FSD procedure.

	1	2	3	4	5	6	7	8	9	10	total	order
A	3.0	0.8	3.6	2.2	0.0	4.0	2.5	5.7	2.0	4.0	27.8	2
B	0.8	2.4	4.2	3.0	0.0	3.0	2.0	1.5	3.7	3.0	23.6	3
C	3.8	3.2	0.0	0.2	0.0	2.7	5.5	0.0	2.5	2.0	19.9	4
D	2.4	3.6	2.2	4.6	10.0	0.3	0.0	2.8	1.8	1.0	28.7	1

On the other hand, appling the Borda procedure under the preference orders shown in Table3, the allocations of points by all members are as Table 4. Then from Table 4 the social preference order by the Borda procedure is A≻B≻D≻C.

Table4. The allocations of points by the Borda procedure.

	1	2	3	4	5	6	7	8	9	10	total	order
A	3	1	3	2	3	4	3	4	2	4	29	1
B	1	2	4	3	2	3	2	2	4	3	26	2
C	4	3	1	1	1	2	4	1	3	2	22	4
D	2	4	2	4	4	1	1	3	1	1	23	3

Comparing the FSD procedure with Borda procedure, the results differ extremely in the order of alternative D. It seems that the allocation of points by member 5 has a great influence on the order of alternative D. In the case of the FSD procedure a member can express his own preference freely. Since member 5 prefers alternative D strongly, he allocates all of his allotted points (10 points) to alternative D. On the other hand, in the case of the Borda procedure a member is imposed to set his preference orders among alternatives, moreover the difference bteween neighbouring orders is always one. Then the Borda procedure can't represent strictly that member 5 prefers alternative D very strongly. In this meaning the FSD procedure can express the reality of preference better than Borda procedure.

4 Discussion

We enumerate characteristics of the FSD procedure as follows:

(1) In the FSD procedure a member judges his own preferences consciously under evident criteria which are chosen freely by the member.

(2) The FSD procedure weights criteria by accumulating paired comparisons. Moreover its consistency is verified.

(3) Taking ambiguities in human judgements into consideration, the FSD procedure uses a fuzzy integral in comprehensive evaluation of each alternative.

(4) Each member allocates his allotted points among alternatives in proportion to comprehensive evaluations. Then the strength of preference can be reflected directly.

(5) Though the majority rule considers the first alternative only in the preference order of each member, the FSD procedure considers other alternatives too.

(6) Though the appraisal by the Borda procedure is relative, one by the FSD procedure is absolute. Then the " voting paradox " doesn't occur and even if the set of alternatives varies, the preference orders among old alternatives are unchangeable.

(7) In a multi-valued society, discussions among members may change thier appraisal criteria and their weights. The FSD procedure can reflect the change since criteria are shown evidently in it.

(8) In the case of many alternatives it is difficult to mark relative preference order by looking around all alternatives. But in the FSD procedure since a member appraises each alternative under each criterion absolutely, the task is easier.

(9) But the FSD procedure compels each member to carry out considerable tasks.

On the whole, though the FSD procedure demands somewhat time and effort, it has high use value since it can reflect opinions of all members as much as possible.

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