数理解析研究所講究録
第 1636 巻 2009 年 55-62

Optimal Choice of the Best Available Applicant in the Full-information Models with Uncertain Selection

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Abstract

The problem we consider in this paper is a full-information best-choice problem in which n applicants appear sequentially, but each applicant has a right to refuse an offer with a known probability q independently of other applicants and all else. The objective is to choose the best available applicant. We obtain the optimal rule and the success probability explicitly for finite n. A remarkable feature of this problem is that, as $n \to \infty$, the optimal success probability becomes insensitive to q and approaches 0.580164. The planar Poisson process model provides more insight into this phenomenon.

key words: secretary problem, optimal stopping, planar Poisson process.

1 Introduction

A known number, n, of applicants appear sequentially in random order, and one of them must be chosen. An applicant that has been rejected cannot be recalled later. In the classical best-choice problem, often referred to as the secretary problem, the objective is to maximize the probability of choosing the very best applicant. At each time, we observe only the relative rank of the current applicant with respect to his/her predecessors. It is well known that the optimal rule lets approximately $e^{-1}n$ applicants go by and then selects the first relatively best applicant if any. The optimal probability of choosing the overall best tends to $e^{-1} \approx 0.368$ as $n \to \infty$.

In contrast to the above *no-information* version of the problem, the *full-information* analogue is the problem in which the observations are the true values of n applicants $X_1, X_2, \ldots, X_n$, assumed to be i.i.d. random variables from a known continuous distribution, taken without loss of generality to be the uniform distribution on the interval $[0, 1]$. There exists a single increasing sequence of thresholds $\{b_k, 1 \leq k\}$ with b_k defined as a solution $x \in (0, 1)$ to the equation

$$\sum_{i=1}^{k} \frac{1}{i}\left(x^{-i}-1\right)=1, \tag{1}$$

such that the optimal stopping rule is to select the *jth* applicant if this applicant is a relative maximum, i.e., $X_j = \max(X_1, \ldots, X_j)$ and $X_j \geq b_{n-j}$. Let $c \approx 0.80435$ be a root of the equation

$$\sum_{j=1}^{\infty} \frac{c^j}{j!j} = 1. \tag{2}$$

Then, as $n \to \infty$, $n(1-b_n) \to c$, and the optimal probability of choosing the largest of $X_1, X_2, \ldots, X_n$ tends to

$$e^{-c} + (e^c - c - 1)\int_1^{\infty} \frac{e^{-cx}}{x} dx, \tag{3}$$

which is evaluated as 0.580164. These best-choice problems were studied by Gilbert and Mosteller (1966) together with many other best-choice problems. See Samuels (1982, 2004) and Berezovsky and Gnedin (1984) for the limiting form (3). We refer to the above full-information problem as the GM problem henceforth.

Though many authors have so far implicitly assumed that the applicant is always *available*, that is, she accepts an offer of selection (employment) with certainty, Smith (1975) introduced the possibility of the applicant refusing an offer. He considered a best-choice problem in the no-information setting and then Petruccelli (1982) considered the corresponding full-information problems in a greater generality. However, the common objective of their problems is to choose the best overall, thus implying that the trial is automatically unsuccessful if the overall best is unavailable. Another more appropriate objective may be to choose the best available applicant, i.e., the best among all the available applicants. Tamaki (1991) considered the problems along this line in the no-information setting.

The problem we treat in this paper is considered as the full-information version of Tamaki (1991). We simply assume that each applicant is available with a known fixed probability $p (0 < p \leq 1)$ and unavailable with the remaining probability $q = 1 - p$, independent of the value of the applicant and also of one another. The decision of when to make an offer is based on both the values of the applicants and their availabilities observed so far and the objective is to maximize the probability of choosing the best available applicant. Two models are distinguished according to when the availability can be ascertained.

Model 1 The availability of the applicant can be ascertained just after her arrival. Thus we make an offer only to an available applicant because nothing is lost in doing so.

Model 2 The availability of the applicant can be ascertained only when an offer is given. If the offer is accepted the selection process terminates, but if rejected the next applicant is to be observed if any.

In both models, we give an offer to the *nth* applicant if the final stage is reached. When $q = 0$, these two models are the same and reduced to the GM problem. In this paper, we only consider Model 1.

Model 1 is considered in Section 2. A remarkable feature of this model is that asymptotically the optimal success probability is insensitive to q and approaches 0.580164 as given by (3), though it in effect depends on q for a finite n. Though this feature can be derived via the limiting argument of the finite problem, the planar Poisson process model in Section 3 provides more insight into this phenomenon. This model is known to facilitate the derivation of the asymptotic results for some full-information problems.

2 Model 1

Let X_j and I_j denote respectively the value and the availability of the *jth* applicant, $1 \le j \le n$. It is assumed that $(X_1, X_2, \ldots, X_n)$ is a sequence of n independent random variables each uniformly distributed on [0, 1] and $(I_1, I_2, \ldots, I_n)$ is also a sequence of n independent random variables each taking a value of 1 or 0 with probability p and $q = 1 - p$ respectively according to whether the applicant is available or not, that is,

$$P\{I_j = 1\} = p = 1 - P\{I_j = 0\}.$$

The two sequences $(X_1, X_2, \ldots, X_n)$ and $(I_1, I_2, \ldots, I_n)$ are also assumed to be independemt of each other.

Let J_i be the index set of the available applicants observed up to time i, i.e., $J_i = \{1 \le j \le i : I_j = 1\}, 1 \le i \le n$ and define the largest value of the first i observations as $L_i = \max_{j \in J_i} X_j$ for $J_i \ne \emptyset$ (For $J_i = \emptyset$, define $L_i = 0$ for convenience). Call X_i (or *ith* applicant) a *candidate* if X_i is available and the largest of the first i observations, i.e., $I_i = 1$ and $X_i = L_i$. The objective is to maximize the probability of *success*, i.e., choosing the applicant having the largest value L_n for $J_n \ne \emptyset$. In Model 1, an optimal rule at any decision point will not depend on the number of available applicants passed by, but only on the largest value observed so far and on the number of remaining observations. Obviously an optimal rule only chooses (makes an offer to) a candidate. Hence, if we denote by $v_k(x)$ the maximal probability of success, provided that there are k applicants yet to observe and the largest value seen so far is x, the principle of optimality gives, for $1 \le k \le n-1$,

$$v_k(x) = (q+px)v_{k-1}(x) + p\int_x^1 \max\{(q+pt)^{k-1},\, v_{k-1}(t)\}dt \tag{4}$$

where $(q+pt)^{k-1}$ is the probability that the $(n-k+1)th$ applicant is best available, i.e., $X_{n-k+1} = L_n$, given that she is a new candidate of value t (this is just the probability that all the subsequent X_j's are smaller than t or unavailable). The boundary condition is given by $v_0(x) \equiv 0$.

The main results of Model 1 can be summarized as follows.

Theorem 1 (a) *Optimal stopping rule: Let*

$$m = m(q) = \min\{k \ge 1 : b_k \ge q\},$$

where b_k is defined by (1). Then, for a given q, there exists a non-decreasing sequence of the thresholds $\{s_k(q), 1 \le k\}$ with $s_k(q)$ defined as

$$s_k(q) = \begin{cases} \frac{b_k - q}{1-q}, & \text{if } k \ge m \\ 0, & \text{if } 1 \le k < m \end{cases} \tag{5}$$

or $s_k(q) = \max\{0, (b_k - q)/(1-q)\}$ as a function of q, such that the optimal rule is to choose the first candidate X_j that exceeds the threshold $s_{n-j}(q)$.
(b) Optimal probability of success: Let $P_n(q)$ denote the optimal probability of success as a function of n and q. Then

$$\begin{aligned} P_n(q) &= \frac{1}{n}\left[1 + \sum_{i=\min(n,m)}^{n-1} \sum_{r=1}^{i} \frac{1}{n-r} b_i^{n-r}\right] \\ &\quad + q^n \sum_{r=1}^{\min(n,m)} \frac{1}{r}\left[\binom{\min(n,m)-r}{n-r} q^{-r} - \binom{\min(n,m)}{n}\right]. \end{aligned} \tag{6}$$

(c) Asymptotics: Let $c \approx 0.80435$ be a root of Equation (2). Then, for $0 \le q < 1$,

$$\lim_{n\to\infty} P_n(q) = e^{-c} + (e^c - c - 1)\int_1^\infty \frac{e^{-cx}}{x}dx \approx 0.580164, \tag{7}$$

showing the insensitivity to q of the asymptotic success probability.

Proof. (a) and (b) are omitted.
(c) Write $b_n = 1 - c_n/n$. Then $c_n \to c$. To derive (7), write (6) in the form

$$P_n(q) = \frac{1}{n}\left\{1 + \sum_{i=\min(n,m)}^{n-1} \sum_{r=1}^{i} \frac{1}{n-r}\left(1 - \frac{c_i}{i}\right)^{n-r}\right\} + A_n(q)$$

where $A_n(q)$ is the second term of (6). As $n \to \infty$, $A_n(q) \to 0$, so $P_n(q)$ converges to

$$\int_0^1 \int_0^u \frac{1}{1-v}(e^{-c})^{\frac{1-v}{u}} dvdu$$

Make the change of variables; $s = 1/u$ and $t = (1-v)/u$. Since this transformation has Jacobian $J(s,t) = \partial(u,v)/\partial(s,t) = 1/s^3$, this double integral reduces to

$$\begin{aligned}
& \int_0^1 \left(\int_1^{t+1} \frac{ds}{s^2}\right) \frac{1}{t} e^{-ct} dt + \int_1^\infty \left(\int_t^{t+1} \frac{ds}{s^2}\right) \frac{1}{t} e^{-ct} dt \\
= & \int_0^1 \frac{e^{-ct}}{1+t} dt + \int_1^\infty \left(\frac{1}{t^2} - \frac{1}{t} + \frac{1}{1+t}\right) e^{-ct} dt \\
= & \int_0^\infty \frac{e^{-ct}}{1+t} dt + \int_1^\infty \frac{e^{-ct}}{t^2} dt - \int_1^\infty \frac{e^{-ct}}{t} dt \\
= & e^{-c} + (e^c - c - 1) \int_1^\infty \frac{e^{-cx}}{x} dx,
\end{aligned}$$

which yields (7).

The neat expression (6), inspired by Sakaguchi (1973), is crucial to derive (7) as the limit of the finite problems . An interesting phenomenon is that no matter what the value of q is, we have asymptotically the same success probability 0.580164 by allowing the stopping rule to depend on q. To give a more perspective to this phenomenon which prevails in a greater generality, we reconsider the problem as a planar Poisson process model in the next section.

3 The planar Poisson process

Samuels (2004, see Sections 9 and 10) used the planar Poisson process (PPP) model with rate 1 on the semi-infinite strip $[0,1] \times [0,\infty)$ as an appropriate setting in which we can define the infinite version of the full-information problem as the limit of the corresponding finite problems. His model inverts the problem (i.e., turns the problem upside down), making the 'best' become the 'smallest'. The process is scanned from left to right by shifting a vertical detector and the scanning can be stopped each time a point in the PPP, referred to as an *atom* henceforth, is detected. Thus, in the GM problem, the objective is to stop on the lowest atom in the PPP. For a more detailed explanation, see Gnedin (1996) who used the PPP on the semi-infinite strip $[0,1] \times (-\infty, 0]$. Here we use Samuels framework.

Suppose that each atom is classified as either a type I or a type II atom. Suppose further that each atom is classified as a type I atom with probability

p and a type II atom with probability q independently of all other atoms. Then it follows that, as a simple consequence from the property of the PPP, both the type I atoms and the type II atoms constitute the PPPs with respective rates p and q and these two PPPs are independent. This is why we use the PPP with rate p on the semi-infinite strip $[0,1]\times[0,\infty)$ as the desired setting for Model 1. This is also consistent with the fact that, if the X_i' are iid uniform on [0, 1] and M is a binomial random variable with parameters (n,p) independent of X_i, then $n\{1-\max(X_1,X_2,\ldots,X_M)\}$ converges in distribution to exponential (p).

However, for more generality, we consider in this section the PPP best-choice problem with (positive) parameters (a,λ) defined as the problem of maximizing the probability of stopping on the lowest atom in the PPP with rate λ on the semi-infinite strip $[0,a]\times[0,\infty)$. This problem corresponds to Model 1 when $(a,\lambda)=(1,p)$ and to the GM problem when $(a,\lambda)=(1,1)$. The typical properties of the PPP are:

- The number of atoms in each bounded domain of the PPP with rate λ has a Poisson distribution with mean equal to $\lambda\times$ (the area of the domain).
- The numbers of atoms in disjoint domains are independent.

See Gnedin (2004, Section 2.1) for further properties of the PPP.

The goal in this section is to show that, though the optimal rule of the PPP best-choice problem with parameters (a,λ) depends on the parameters, its optimal success probability is insensitive to them (Theorem 2). Our argument goes quite parallel to that developed by Samuels (2004, see Sections 10.1 and 10.2) to find the optimal rule and the success probability of the GM problem. Let $u(t,y)$ denote the probability of success if we choose the point (t,y) in the PPP, i.e., we stop at time t with a relatively best atom having value y. Then, if we denote by $P_{ois}(k;\mu)$ the Poisson probability of k events, for a given mean μ, we have

$$u(t,y)=P_{ois}(0;\lambda y(a-t)), \tag{8}$$

because $u(t,y)$ is just the probability that there is no atom in the box domain $[t,a]\times[0,y]$ whose area is $y(a-t)$. On the other hand, let $v(t,y)$ denote the probability of success if we do not choose the point (t,y), but instead choose the point related to the next relatively best atom, if any, then

$$v(t,y)=\sum_{j=1}^{\infty}\frac{1}{j}P_{ois}(j;\lambda y(a-t)),$$

because, if there are j atoms in the box $[t,a]\times[0,y]$, the leftmost one has probability $1/j$ of being best (lowest). Solving for the locus of point (t,y) at which $u(t,y)=v(t,y)$ yields $\lambda y(a-t)=c$ for $c\approx 0.80435$ given as a root of (2). Moreover, since $u(t,y)\geq v(t,y)$ implies that $u(t',y')\geq v(t',y')$

for $t' > t$, $y' < y$, so the problem is monotone, we can conclude that the optimal rule stops with the first relatively best atom, if any, that lies below the threshold curve

$$y = \frac{c}{\lambda(a-t)}. \tag{9}$$

We now have the following result.

Theorem 2 *The optimal rule of the PPP best-choice problem with parameters (a, λ) is specified by the threshold curve (9) and its optimal success probability is given by*

$$P_{a,\lambda} = e^{-c} + (e^c - c - 1)\int_1^\infty \frac{e^{-cx}}{x}dx \approx 0.580164,$$

which is insensitive to (a, λ).

Proof. Let

T=the arrival time of the first (leftmost) point that lies below the threshold curve $y = c/\lambda(a-t)$.

S=the time when the value of the best (lowest) above-threshold arrival is now equal to the threshold.

Then T and S are independent and have the following respective densities :

$$\begin{aligned} f_T(t) &= -\frac{d}{dt}P\{T > t\} = -\frac{d}{dt}\exp\left(-\lambda\int_0^t \frac{c}{\lambda(a-r)}dr\right) \\ &= \frac{c(a-t)^{c-1}}{a^c}, \quad 0 < t < a \end{aligned} \tag{10}$$

$$\begin{aligned} f_S(s) &= -\frac{d}{ds}P\{S > s\} = -\frac{d}{ds}\exp\left(-\lambda\left[\int_0^s \left(\frac{c}{\lambda(a-s)} - \frac{c}{\lambda(a-r)}\right)dr\right]\right) \\ &= -\frac{d}{ds}\left[\left(\frac{a}{a-s}\right)^c e^{-\frac{cs}{a-s}}\right] = \frac{csa^c}{(a-s)^{c+2}}e^{-\frac{cs}{a-s}}, \quad 0 < s < a. \end{aligned} \tag{11}$$

The so called 'forward-looking argument' by Samuels(2004, see his Eq(10)) then yields

$$\begin{aligned} P_{a,\lambda} &= \int_0^a\int_0^t u\left(s, \frac{c}{\lambda(a-s)}\right) f_S(s)f_T(t)dsdt \\ &\quad + \int_0^a\int_0^s \left[\frac{\lambda(a-t)}{c}\int_0^{c/\lambda(a-t)} u(t,y)dy\right] f_T(t)f_S(s)dtds, \end{aligned}$$

which, combined with $u(t, y) = e^{-\lambda y(a-t)}$ from (8), can be simplified to

$$e^{-c}P\{S \le T\} + \frac{1}{c}(1-e^{-c})P\{S > T\}$$

$$= e^{-c} + \left(\frac{1}{c} - \frac{1}{c}e^{-c} - e^{-c}\right) P\{S > T\}. \tag{12}$$

By the way, we have from (10) and (11)

$$\begin{aligned} P\{S > T\} &= \int_0^a P\{S > t\} f_T(t) dt \\ &= c \int_0^a \frac{1}{a-t} e^{-\frac{ct}{a-t}} dt \\ &= ce^c \int_1^\infty \frac{e^{-cx}}{x} dx, \end{aligned} \tag{13}$$

where the last equality follows by making the change of variable $x = a/(a-t)$. Substituting (13) into (12) gives the desired result.

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