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A Continuous review inventory model with stochastic price procured in the spot market

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Abstract Not only the amount of product demand but also the prices of the product have a strong impact on a manufacturer's revenue. In this paper we consider a continuous-time inventory model where the spot price of the product stochastically fluctuates according to a Brownian motion. Should information of the spot price be available, the manufacturer wishes to buy the product from the spot market if profitable. The purpose of this paper is to find an optimal procurement policy so as to minimize the total expected discounted costs over an infinite planning horizon. We extend Sulem (1986) model into the one in which the market price of the product follows geometric Brownian motions. Then we obtain the optimal cost as the solution of Quasi-variational inequality, and show that there exists an optimal procurement policy as an (s, S) policy. We shall clarify the dependence of such optimal (s, S) policy on the spot price at the procurement epoch. These values of the (s, S) policy can be used and revised in the succeeding ordering cycles. Finally, some numerical examples are provided to investigate analytical properties of the expected cost function as well as of the optimal policy.

1. Introduction

Many manufactures that use the spot market to procure in supply chains are facing a fluctuation of market prices. In this paper we consider a continuous-time inventory model in which the spot price of the product stochastically fluctuates according to a Brownian motion. The inventory level can be monitored on a continuous time basis. Our objective is to determine the procurement policy as an (s, S) policy to reduce the risk of the spot price. When the inventory level drops down to the reorder point, a pair of order quantity and reorder point for the next cycle is determined after observing the spot price.

Price uncertainty has been taken into account by several researchers in the context of an inventory policy. Goel and Gutierrez [7] considered the value of incorporating information about spot and futures market prices in procurement decision making. Guo et al. [8] characterized analytical properties of the optimal policy for a firm facing random demand.

On the other hand, there are many articles [1], [2], [3], [5], [6], [10] related to an (s, S) policy under the continuous time review. Sulem [10] analyzed the optimal ordering policy applying impulse control to an inventory system with a stochastic demand followed by a diffusion process. Furthermore, Benkherouf [3] extend the Sulems model to the case of general storage and shortage penalty cost function. We also extend the Sulem model into the case in which the market price of the product follows geometric Brownian motions, but demand is deterministic.

The reminder of this paper is organized as follows. In section 2 we present the model formulation. It is shown in section 3 that the optimal cost can be derived as the solution of

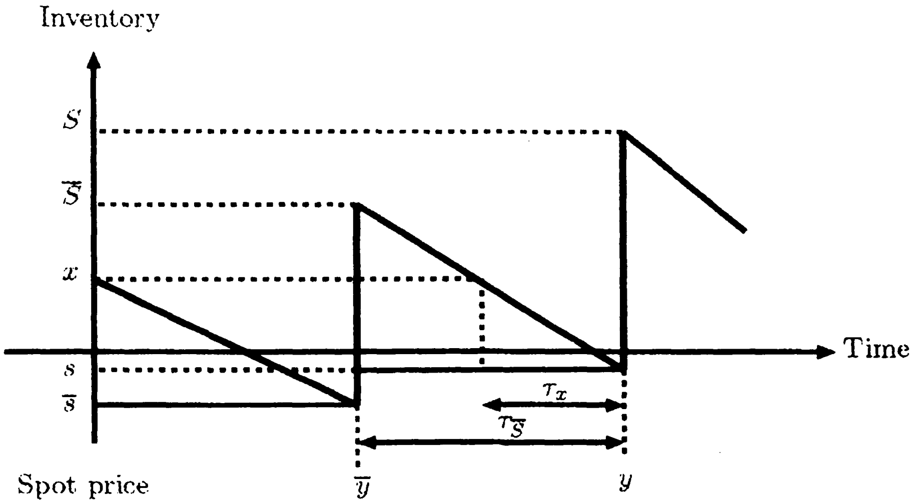

Figure 1: Inventory flow

Quasi-variational inequality, and we show in section 4 that there exists an optimal procurement policy which is the type of an (s, S) policy. In section 5 we discuss the case of the specific type of spot price, and clarifies the impact of the spot price on the value function. Section 6 concludes the paper.

2. Notations and Assumptions

The analysis is based on the following assumptions:

(i) Time is continuous and inventory is continuously reviewed.

(ii) Demand is g units per unit time in one cycle. Unsatisfied demand is backlogged.

(iii) A critical-level (s, S, y) policy is in place, which means that the inventory level x is drops to an reorder point s, then the inventory level increases up to S. And then the next s and S are determined, based on the observation of the spot price $y(t)$ at time t. Since s and S are changing at the beginning of each cycle, we suppose that $\overline{s}$ and $\overline{S}$ are reorder point and order-up-to level for the last cycle, respectively. Therefore, $\overline{S}$ represents the initial inventory level at the beginning of the next cycle, (see Figure 1).

(iv) The set up cost is K and the unit cost is equal to the spot price $y(\cdot)$. The shortage cost p and holding cost q are given by the function f:

$$f(x) = \begin{cases} -px & \text{for } x < 0, \\ qx & \text{for } x \geq 0. \end{cases} \tag{1}$$

(v) The spot price at time t, $y(t)$, follows a geometric Brownian motion, that is,

$$dy(t) = y(t)(\mu dt + \sigma dw(t)). \tag{2}$$

where $w(t)$ follows a standard Brownian motion.

A procurement policy consists of a sequence $V = \{(\theta_i, \xi_i), i = 1, 2, \cdots\}$ of i-th ordering time θ_i and order quantity ξ_i. Let $u(x, y)$ be the optimal total expected discounted cost over an infinite planning horizon when an initial inventory level is given by x and the spot price by y. Then $u(x,y)$ can be written as

$$u(x,y) = \inf_V \left(E_y \left[\int_0^\infty f(x(t))e^{-\alpha t}dt + \sum_{i\geq 1}(K + y(\theta_i)\xi_i)e^{-\alpha\theta_i} \right] \middle| x(0) = x, y(0) = y \right) \tag{3}$$

where the inventory level $x(t)$ is given by

$$dx(t) = -gdt + \sum_{i\geq 1} \xi_i \delta(t - \theta_i). \tag{4}$$

with $x(0) = x$ and $\alpha > 0$ is interest rate. In equation (4), $\delta(\cdot)$ denotes the Dirac function, that is,

$$\delta(z) = \begin{cases} 1 & \text{for } z = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Note that $u(x,y)$ is continuous and everywhere differentiable in x, and also it is twice differentiable in y.

Our objective is to find the optimal procurement policy $(s,\ S)$ at which the minimum value function u be attained.

3. QVI Problem and Optimal Procurement Policy

In this section, we deal with equation (3) as a Quasi-Variational Inequality (QVI) problem (Bensoussan and Lions [4]).

First, if the procurement is not made at least during a small time interval $(t, t+\epsilon)$, then we have following inequation;

$$u(x,y) \leq \int_t^{t+\epsilon} f(x(s))e^{-\alpha(s-t)}ds + u(x(t+\epsilon), y(t+\epsilon))e^{-\alpha\epsilon}. \tag{6}$$

We can expand the right hand side of equation (6) up to first order in ϵ and then we have

$$\epsilon f(x) + u(x,y) - g\epsilon\frac{\partial u}{\partial x} + \mu y\epsilon\frac{\partial u}{\partial y} + \frac{1}{2}\epsilon\sigma^2 y^2\frac{\partial^2 u}{\partial y^2} - \alpha\epsilon u(x,y) + o(\epsilon^2). \tag{7}$$

Hence, making ϵ tend to 0, equation (6) can be deduced to

$$\frac{1}{2}\sigma^2 y^2\frac{\partial^2 u}{\partial y^2} + \mu y\frac{\partial u}{\partial y} - g\frac{\partial u}{\partial x} - \alpha u \geq -f(x). \tag{8}$$

On the other hand, if the procurement is made at time t, the inventory level jumps from x to an $x+\xi$. We assume that the order quantity is delivered immediately, so the spot price before the procurement is equal to the price after procurement. Thus, we obtain

$$u(x,y) \leq K + \inf_{\xi\geq 0}(y(t)\xi + u(x+\xi, y(t))). \tag{9}$$

Therefore, the equation (3) is given by a solution of the QVI problem:

$$\min(Au + f, Mu - u) = 0 \tag{10}$$

where

$$Au(x,y) := \frac{1}{2}\sigma^2 y^2\frac{\partial^2 u}{\partial y^2} + \mu y\frac{\partial u}{\partial y} - g\frac{\partial u}{\partial x} - \alpha u, \tag{11}$$

$$Mu(x,y) := K + \inf_{\xi\geq 0}(y(t)\xi + u(x+\xi, y(t))). \tag{12}$$

4. Solution of QVI Problem

In this section, we solve the QVI problem (10) quoted by Sulem [10] in part. We divide the inventory space into two regions; for no procurement,

$$G = \{x \in \mathcal{R} : u(x, y) < Mu(x, y)\} = \{x \in \mathcal{R} : x > s\} \tag{13}$$

then, we have

$$Au = f. \tag{14}$$

And its complement is given by

$$\overline{G} = \{x \in \mathcal{R} : u(x, y) = Mu(x, y)\} = \{x \in \mathcal{R} : x \leq s\} \tag{15}$$

and for $x \in \overline{G}$, we have

$$\begin{aligned} u(x, y) &= K + \inf_{\xi \geq 0}(y(t)\xi + u(x + \xi, y(t))) && (16) \\ &= K + y(t)(S - x) + u(S, y(t)). && (17) \end{aligned}$$

Due to the deterministic demand, we take the inventory level as the elapsed time from the beginning of the cycle. Thus, we describe the spot price y_x depending on the inventory level x.

Since u is continuously differentiable in x, in inventory space $\overline{G}$, we can get the boundary conditions on u.

(i) Continuity of the derivative of u at the boundary point s:

$$\lim_{x \downarrow s} \frac{\partial u(x, y)}{\partial x} = -y_s. \tag{18}$$

(ii) The infimum in equation (16) is attained at $\overline{S}$:

$$\lim_{x \uparrow \overline{S}} \frac{\partial u(x, y)}{\partial x} = -\overline{y}, \tag{19}$$

where $\overline{y}$ is the spot price at the beginning of the cycle.

(iii) u is continuous at s:

$$u(S, y) = u(s, y) - K - y_s(S - s). \tag{20}$$

(iv) The growth condition of u:

$$\lim_{x \to \infty} \frac{u(x, y)}{f(x)} < +\infty. \tag{21}$$

To obtain the value function u, we solve the partial differential equation (14) with the initial and boundary conditions (18)-(21). First, we set

$$u(x, y) = w(\tau, z)e^{kz + l\tau}, \quad z = \log y, \quad \tau = \frac{x - s}{g}. \tag{22}$$

This results in the equation

$$\left(l-\mu k+\alpha+\frac{1}{2}k(1-k)\sigma^2\right)w+\frac{\partial w}{\partial \tau} \\ +\left(\left(\frac{1}{2}-k\right)\sigma^2-\mu\right)\frac{\partial w}{\partial z}-\frac{\sigma^2}{2}\frac{\partial^2 w}{\partial z^2}=f(g\tau+s)e^{-(kz+l\tau)} \quad (23)$$

where we choose k and l satisfying

$$k=-\frac{1}{\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right), \quad l=-\frac{1}{2\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)^2-\alpha. \quad (24)$$

Then, we can get the non-homogeneous heat equation

$$\frac{\partial w}{\partial \tau}-\frac{\sigma^2}{2}\frac{\partial^2 w}{\partial z^2}=f(g\tau+s)\exp\left\{\frac{1}{\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)z+\left(\frac{1}{2\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)^2+\alpha\right)\tau\right\} \quad (25)$$

with

$$\begin{aligned} w(0,z) &= e^{\frac{1}{\sigma^2}\left(\mu-\frac{\sigma^2}{2}\right)z}\{v(\tau_S,z)+K+e^z(S-s)\}\equiv m(z), && (26)\\ \tau_S &= \frac{S-s}{g}. && (27)\end{aligned}$$

The solution to the diffusion equation problem is given by

$$w(\tau,z)=w_1(\tau,z)+w_2(\tau,z), \quad (28)$$

where $w_1(\tau,z)$ and $w_2(\tau,z)$ are solutions of following problems:

$$\begin{aligned} \frac{\partial w_1}{\partial \tau} &= \frac{\sigma^2}{2}\frac{\partial^2 w_1}{\partial z^2}, \quad w_1(0,z)=m(z), && (29)\\ \frac{\partial w_2}{\partial \tau} &= \frac{\sigma^2}{2}\frac{\partial^2 w_2}{\partial z^2}+h(\tau,z), \quad w_2(0,z)=0. && (30)\end{aligned}$$

Here, we set the right hand side of equation (25) as $h(\tau,z)$. The solutions w_1, w_2 of each problem (29) and (30) are, respectively, given by

$$\begin{aligned} w_1(\tau,z) &= \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty} w_1(0,\xi)\exp\left\{-\frac{(z-\xi)^2}{2\sigma^2\tau}\right\}d\xi \\ &= \frac{1}{\sqrt{2\pi}}e^{n_-(\eta,z)}\int_{-\infty}^{\infty} v\left(\tau_S,\tau\left(\mu-\frac{\sigma^2}{2}\right)+\lambda\sigma\sqrt{\tau}+z\right)e^{-\frac{\lambda^2}{2}}d\lambda \\ &\quad +Ke^{n_-(\eta,z)}+(S-s)e^{n_+(\eta,z)}, && (31)\end{aligned}$$

and

$$\begin{aligned} w_2(\tau,z) &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\tau}\frac{1}{\sqrt{\tau-\delta}}\left(\int_{-\infty}^{\infty}h(\delta,\xi)\exp\left(-\frac{(z-\xi)^2}{2\sigma^2(\tau-\delta)}\right)d\xi\right)d\delta \\ &= e^{n_-(\tau,z)}\int_0^{\tau}f(g\delta+s)e^{\alpha\delta}d\delta, && (32)\end{aligned}$$

where

$$n_{\pm}(\tau, z) = \frac{1}{2\sigma^2}\left(\mu \pm \frac{\sigma^2}{2}\right)\left(\tau\left(\mu \pm \frac{\sigma^2}{2}\right) + 2z\right). \tag{33}$$

Therefore, from equations (22), (28), (31) and (32), $u(x,y)$ can be rewritten as follows;

$$u(x,y) = e^{-\alpha\tau_x}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(S, y_x e^{\tau_x\left(\mu-\frac{\sigma^2}{2}\right)+\lambda\sigma\sqrt{\tau_x}})e^{-\frac{\lambda^2}{2}}d\lambda \right.$$
$$\left. +K + (S-s)y_x e^{\mu\tau_x} + \int_0^{\tau_x} f(g\delta + s)e^{\alpha\delta}d\delta\right]. \tag{34}$$

In the last term of equation (34), it can be rewritten as

$$\begin{aligned} D(x) &\equiv \int_0^{\tau_x} f(g\delta+s)e^{\alpha\delta}d\delta \\ &= \begin{cases} \frac{p}{\alpha}\left(s-\frac{g}{\alpha}\right) + \frac{q}{\alpha}\left(x - \frac{g}{\alpha}\right)e^{\alpha\tau_x} + \frac{g}{\alpha^2}(p+q)e^{-\frac{\alpha}{g}s} & \text{for } x \geq 0, \\ -\frac{p}{\alpha}\left\{\left(x-\frac{g}{\alpha}\right)e^{\alpha\tau_x} - s + \frac{g}{\alpha}\right\} & \text{for } x < 0, \end{cases} \end{aligned} \tag{35}$$

and $D(s) = 0$. Therefore, we have

$$\begin{aligned} u(x,y) &= e^{-\alpha\tau_x}[E[u(S, y_x e^{\tau_x(\mu-\frac{\sigma^2}{2})+\sigma\sqrt{\tau_x}X})] + K + (S-s)y_x e^{\mu\tau_x} + D(x)] \\ &= e^{-\alpha\tau_x}[u(S,y) + K + (S-s)y_x e^{\mu\tau_x} + D(x)] \end{aligned} \tag{36}$$

where X is a standard normal random variable.

Lemma 1. *The optimal cost function $u(x,y)$ is given by*

$$u(x,y) = \frac{g}{\alpha}\left(\frac{q}{g}\overline{S} + \overline{y}\right)e^{\alpha(\tau_{\overline{S}}-\tau_x)} + \left\{y_x e^{(\mu-\alpha)\tau_x} - \left(1-\frac{\mu}{\alpha}\right)\overline{y}e^{\mu\tau_{\overline{S}}-\alpha\tau_x}\right\}(S-s) + L(x) \tag{37}$$

where

$$L(x) = \begin{cases} \frac{q}{\alpha}\left\{\left(x-\frac{g}{\alpha}\right) - \left(\overline{S}-\frac{g}{\alpha}\right)e^{\alpha(\tau_{\overline{S}}-\tau_x)}\right\} & \text{for } x \geq 0, \\ -\frac{p}{\alpha}\left(x-\frac{g}{\alpha}\right) - \frac{q}{\alpha}\left(\overline{S}-\frac{g}{\alpha}\right)e^{\alpha(\tau_{\overline{S}}-\tau_x)} - \frac{g}{\alpha^2}(p+q)e^{-\frac{\alpha}{g}x} & \text{for } x < 0. \end{cases} \tag{38}$$

Note that $L(x) < 0$ for all x.

Remark. Note that equation (37) can be reduced to the deterministic-demand case of Sulem's model when we assume $\mu = \sigma = 0$, $y = \overline{y}$, $S = \overline{S}$. In this case, the optimal cost $\tilde{u}(x)$ can be reduced to be

$$\tilde{u}(x) = \left\{\frac{(q+\alpha y)g}{\alpha^2}e^{\frac{\alpha S}{g}} - \frac{qg}{\alpha^2}\right\}e^{-\frac{\alpha x}{g}} + \frac{r}{\alpha}x + \frac{rg}{\alpha^2}\left(e^{-\frac{\alpha x}{g}} - 1\right) \tag{39}$$

where

$$r = \begin{cases} -p & \text{for } x < 0, \\ q & \text{for } x \geq 0. \end{cases} \tag{40}$$

Lemma 2. *If there exists an optimal order point s exists for any y, s is strictly negative.*

Next we show the properties of u and the existence of optimal policy based on Benkherouf [3]. Let us denote $H(x,y) = u(x,y) + y_x x$. Then, equation (14) can be rewritten as

$$g\frac{\partial H}{\partial x} + \alpha H = gy_x + \alpha y_x x + f(x) + \mu y_x e^{(\mu-\alpha)\tau_x}(S-s). \tag{41}$$

Lemma 3. *There exists a pair (s, S) such that s and S satisfy equations (18)-(21).*

Theorem 1. *If $(\mu-\alpha)y_x + p > 0$ for $x < 0$, then an optimal policy (s, S) is solution of the QVI problem, and the value of (s, S) is given by the solution of following simultaneous equation:*

$$\frac{g}{\alpha}\left(\frac{p}{g}s - y_s\right) + \frac{q}{\alpha}\left(S - \frac{g}{\alpha}\right) + \frac{g}{\alpha}\left(\overline{y} + \frac{q}{\alpha}\right)e^{\alpha(\tau_{\overline{S}}-\tau_S)} + K$$
$$+\left\{y_s e^{(\mu-\alpha)\tau_S} + \left(1-\frac{\mu}{\alpha}\right)\left(y_s - \overline{y}e^{\mu\tau_{\overline{S}}-\alpha\tau_S}\right)\right\}(S-s) = 0 \tag{42}$$

$$\frac{g}{\alpha}\left(\frac{q}{g}\overline{S} + \overline{y}\right)e^{\alpha\tau_{\overline{S}}} - \frac{q}{\alpha}\left(S - \frac{g}{\alpha}\right) - \frac{g}{\alpha}\left(\overline{y} + \frac{q}{\alpha}\right)e^{\alpha(\tau_{\overline{S}}-\tau_S)} - D(\overline{S}) - K$$
$$-\left\{y_s e^{(\mu-\alpha)\tau_S} + \overline{y}e^{\mu\tau_{\overline{S}}}\left(1-\frac{\mu}{\alpha}\right)\left(1 - e^{-\alpha\tau_S}\right)\right\}(S-s) = 0 \tag{43}$$

Lemma 5. *The value (s, S) satisfying equations (42) and (43) is a unique solution of the QVI problem (10).*

Theorem 2. *If $(\mu-\alpha)y_x + p \leq 0$ for $x < 0$, there is no solution to the QVI problem (10).*

5. The Type of a Specific Spot Price

Since equations (42) and (43) depend on the spot price y_s at the end of cycle, we assume that the price is equal to the expectation of spot price at the end of cycle in order to estimate the total cost at the beginning of cycle. Thus, we assign

$$y_x = \overline{y}e^{\mu(\tau_{\overline{S}}-\tau_x)} \tag{44}$$

to equation (37).

Then, we have

$$u(x) = \frac{g}{\alpha}\left(\frac{q}{g}\overline{S} + \overline{y}\right)e^{\alpha(\tau_{\overline{S}}-\tau_x)} - \frac{\mu}{\alpha}\overline{y}e^{\mu\tau_{\overline{S}}-\alpha\tau_x}(S-s) + L(x). \tag{45}$$

Lemma 6. *$u(x)$ is increasing in $\overline{y}$ for $\mu \leq \frac{1+\alpha\tau_{\overline{S}}\tau_S}{\tau_{\overline{S}}(1+\tau_S)}$, and is decreasing in $\overline{y}$ for $\mu > \frac{1+\alpha\tau_{\overline{S}}\tau_S}{\tau_{\overline{S}}(1+\tau_S)}$.*

Lemma 7. *$u(x)$ is increasing in μ.*

6. Concluding Remarks and Further Research

In this paper, we showed the existence of an optimal policy for the inventory model that permits the manufactures to procure the products from the spot market. We obtained the optimal cost function as the solution of Quasi-variational inequality, and showed that there exists an optimal procurement policy which is described by the form of an (s, S) policy. For future research we may extend the inventory model with procurement from spot market into the model where demand also follows diffusion process and incorporate the supply contracts like option or futures into the model.

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