数理解析研究所講究録
第 1637 巻 2009 年 57-65

An Application Model of a Nonlinear Difference Equation whose the all Eigenvalues are 1

桜美林大学・リベラルアーツ学群　鈴木麻美 (Mami Suzuki)
Department of Mathematics and Science, College of Liberal Arts,
J. F. Oberlin University.

1 Introduction

In the previous work, we consider the following second order nonlinear difference equations,

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)). \end{cases} \tag{1.1}$$

Here $X(x,y)$, $Y(x,y)$ are holomorphic functions and expanded in a neighborhood of $(0,0)$ in the following form

$$\begin{cases} X(x,y) = x + y + \sum_{i+j\geqq 2} c_{ij}x^i y^j = x + X_1(x,y), \\ Y(x,y) = y + \sum_{i+j\geqq 2} d_{ij}x^i y^j = y + Y_1(x,y), \end{cases} \tag{1.2}$$

where $X_1(x,y) \not\equiv 0$ or $Y_1(x,y) \not\equiv 0$.

Hereafter we consider t to be a complex variable. We define domain $D_1(\kappa_0, R_0)$ by

$$D_1(\kappa_0, R_0) = \{t \, : \, |t| > R_0, |\arg[t]| < \kappa_0\}, \tag{1.3}$$

where κ_0 is any constant such that $0 < \kappa_0 \leqq \frac{\pi}{4}$, and R_0 is sufficiently large number which may depend on X and Y. Further we define domain $D^*(\kappa, \delta)$ by

$$D^*(\kappa, \delta) = \{x \, ; \, |\arg[x]| < \kappa, \, 0 < |x| < \delta\}, \tag{1.4}$$

where δ is a small constant, and κ is a constant such that $\kappa = 2\kappa_0$, i.e., $0 < \kappa \leqq \frac{\pi}{2}$.

Here we defined $g_0^{\pm}$ as follows for the coefficients of $X(x,y)$ and $Y(x,y)$

$$g_0^+(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4}, \tag{1.5}$$

$$g_0^-(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) - \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4}, \tag{1.6}$$

respectively, and

$$A_2 = g_0^+(c_{20}, d_{11}, d_{30}) + c_{20}, \ \ A_1 = g_0^-(c_{20}, d_{11}, d_{30}) + c_{20},$$

We have proved the following Theorem 1 in previous works in [7].

Theorem 1. *Suppose $X(x,y)$ and $Y(x,y)$ are expanded in the following forms (1.2),*

$$\begin{cases} X(x,y) = x + y + \sum_{i+j\geqq 2} c_{ij}x^i y^j = x + X_1(x,y), \\ Y(x,y) = y + \sum_{i+j\geqq 2} d_{ij}x^i y^j = y + Y_1(x,y). \end{cases} \tag{1.2}$$

(1) Suppose $d_{20} = 0$, and we assume the following conditions,

$$A_2 n \neq c_{20} - d_{11} - g_0^+(c_{20}, d_{11}, d_{30}) \tag{1.7}$$
$$A_1 n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}) \tag{1.8}$$

for all $n \in \mathbb{N}$, $(n \geqq 4)$, then we have formal solutions $x(t)$ of the difference system (1.1) the following forms

$$-\frac{1}{A_2 t}\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j}\left(\frac{\log t}{t}\right)^k\right)^{-1}, \ -\frac{1}{A_1 t}\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j}\left(\frac{\log t}{t}\right)^k\right)^{-1}, \tag{1.9}$$

where $\hat{q}_{jk}$ are constants defined by X and Y.

(2) Further suppose $R_1 = \max(R_0, 2/(|A_2|\delta)))$ and we assume $A_2 < 0$, $A_1, A_2 \in \mathbb{R}$, there are two solutions $x_1(t)$ and $x_2(t)$ of (1.1) such that

(i) $x_s(t)$ are holomorphic in $D_1(\kappa_0, R_1)$, and $x_s(t) \in D^*(\kappa, \delta)$ for $t \in D_1(\kappa_0, R_1)$, $s = 1, 2$,

(ii) $x_s(t)$ are expressible in the following form

$$x_1(t) = -\frac{1}{A_1 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}, \ x_2(t) = -\frac{1}{A_2 t}\left(1 + b_2\left(t, \frac{\log t}{t}\right)\right)^{-1}. \tag{1.10}$$

Here $b_1(t, \log t/t)$, $b_2(t, \log t/t)$ are asymptotically expanded in $D_1(\kappa_0, R_1)$ such that

$$b_1\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geqq 1} \hat{q}_{jk(1)} t^{-j}\left(\frac{\log t}{t}\right)^k,$$

$$b_2\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geqq 1} \hat{q}_{jk(2)} t^{-j}\left(\frac{\log t}{t}\right)^k,$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$.

In this proof, we use results of T. Kimura [2] and the following Theorem C in [6], though we need some modifications of Kimura's results.

Theorem C. *Suppose $X(x,y)$ and $Y(x,y)$ are defined in (1.2). We assume $d_{20}=0$ and the following conditions,*

$$A_2 n \neq c_{20} - d_{11} - g_0^+(c_{20}, d_{11}, d_{30}), \tag{1.7}$$
$$A_1 n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}), \tag{1.8}$$

for all $n \in \mathbb{N}$, $(n \geqq 4)$.

(1) We have two formal solutions

$$\Psi^+(x) = \sum_{n \geqq 2}^{\infty} a_n^+ x^n, \ \Psi^-(x) = \sum_{n \geqq 2}^{\infty} a_n^- x^n$$

of

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \tag{1.11}$$

where a_n^+, a_n^- are given by X and Y.

(2) Further we assume $A_1, A_2 \in \mathbb{R}$, $A_2 < 0$. For any κ $(0 < \kappa \leqq \frac{\pi}{2})$ and small $\delta > 0$, there is a constant δ, and two solutions $\Psi^+(x)$, $\Psi^-(x)$ of

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \tag{1.11}$$

which are holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that

$$\Psi^+(x) \sim \sum_{n=2}^{\infty} a_n^+ x^n, \quad \text{and} \quad \Psi^-(x) \sim \sum_{n=2}^{\infty} a_n^- x^n. \tag{1.12}$$

as $x \to 0$ through $D^*(\kappa, \delta)$.

When we assume $d_{20} \neq 0$, then there are no analytic solution of (1.11).

2 An Application

Next we will consider the following population model (P)

$$u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)}, \tag{P}$$

where $\alpha = 1 + r$, β are constants. This model is proposed by Prof. D. Dendrios [1]. Here r is the net (births minus death) endogenous population (stock) growth rate. The second term, in the right side hand, is a function depicting net immigration at $t+1$, which should be considered as a "momentum" to grow from t to $t+1$. We assume that α and β are constants such that $\alpha > 0$, i.e., $r > -1$ and $\beta > 0$ in (P).

Let

$$u(t+2) = u_1(t+2) + u_2(t+2),$$

where $u_1(t+2) = \alpha u(t+1)$, $u_2(t+2) = \beta\frac{u(t+1)-\alpha u(t)}{\alpha u(t)}$. Here $u_1(t+2)$ is a term for endogenous population growth rate from $t+1$ to $t+2$, and $u_2(t+2)$ is a term for net in-migration rate. We have

$$u_1(t+2) = \alpha u(t+1) = \alpha\{u_1(t+1) + u_2(t+1)\},$$

$$u_2(t+2) = \beta\frac{u(t+1) - \alpha u(t)}{\alpha u(t)} = \beta\frac{u_2(t+1)}{u_1(t+1)},$$

where $u_1(t+1)$ is the endogenous population growth rate from t to $t+1$, and $u_2(t+1)$ due to net in-migration rate. We may write (P) as :

$$u(t+2) - \alpha u(t+1) = \frac{c}{u(t)}\{u(t+1) - \alpha u(t)\}, \quad c = \frac{\beta}{\alpha}.$$

When $\alpha \neq 1$, (P) admits the unique equilibrium value $c = \frac{\beta}{\alpha}$, and we can have general analytic solutions such that $u(t+n) \to c$, as $n \to \infty$ ($n \in \mathbb{N}$), making use of theorems of previous studies [7] and [8].

When $\alpha = 1$, we note that, any value can be equilibrium point of (P). Suppose the equation (P) has a solution $u(t)$ such that $u(t+n) \to u_0 > 0$, as $n \to \infty$, in the case $\alpha = 1$. ¿From [6], we have the following three cases.

1) $u(t_0+n) \downarrow u_0 \geq c$ as $n \to \infty$,

2) $u(t_0+n) \uparrow u_0 > c$, as $n \to \infty$,

or

3) there is a n_0, such that $u(t_0+n_0) \leq 0$, (extermination) .

However we have not been able to prove the existence of a solution of (P) in [6]. Now we will show a solution of (P). Here we have the following Proposition 6, analogous to Theorem C.

Proposition 6. *Suppose $X(x,y)$ and $Y(x,y)$ are defined $d_{20} = 0$, and we assume the following condition,*

$$A_1 n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}), \tag{2.13}$$

for all $n \in \mathbb{N}$, ($n \geqq 4$).

(1) We have a formal solution $\Psi^-(x) = \sum_{n\geq 2}^{\infty} a_n^- x^n$ of

$$\Psi(X(x,\Psi(x))) = Y(x,\Psi(x)), \tag{1.11}$$

where a_n^- are given by X and Y.

(2) We assume $A_1, A_2 \in \mathbb{R}$, $(A_1 \leqq A_2)$ and $A_1 < 0$. For any κ $(0 < \kappa \leqq \frac{\pi}{2})$ and small $\delta > 0$, there is a constant δ, and a solution $\Psi^-(x)$ of (1.11) which is holomorphic and can be expanded asymptotically in $D^*(\kappa,\delta)$ such that

$$\Psi^-(x) \sim \sum_{n=2}^{\infty} a_n^- x^n,$$

as $x \to 0$ through $D^*(\kappa,\delta)$.

When we assume $d_{20} \neq 0$, then there are no analytic solution of (1.11).

From Proposition 6, we have the following lemma 7, analogous to Theorem 1.

lemma 7. *Suppose $X(x,y)$ and $Y(x,y)$ are expanded in the following forms*

$$\begin{cases} X(x,y) = x + y + \sum_{i+j\geqq 2} c_{ij}x^iy^j = x + X_1(x,y), \\ Y(x,y) = y + \sum_{i+j\geqq 2} d_{ij}x^iy^j = y + Y_1(x,y). \end{cases} \tag{1.2}$$

(1) Suppose $d_{20} = 0$ and we assume the following conditions,

$$A_1 n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}) \tag{1.8}$$

for all $n \in \mathbb{N}$, $(n \geqq 4)$. Then the difference system

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases} \tag{1.1}$$

has a formal solution $x(t)$ of the following form

$$-\frac{1}{A_1 t}\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j}\left(\frac{\log t}{t}\right)^k\right)^{-1}, \tag{1.9}$$

$\hat{q}_{jk}$: constants defined by X and Y.

(2) Further suppose $R_1 = \max(R_0, 2/(|A_1|\delta)))$, and we assume $A_1 < 0$, there is a solution $x_1(t)$ of (1.1) such that
(i) $x_1(t)$ is holomorphic and $x_1(t) \in D^*(\kappa,\delta)$ for $t \in D_1(\kappa_0, R_1)$,
(ii) $x_1(t)$ are expressible in the following form

$$x_1(t) = -\frac{1}{A_1 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}. \tag{1.10}$$

Here $b_1(t, \log t/t)$, is asymptotically expanded in $D_1(\kappa_0, R_1)$ such that

$$b_1\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geqq 1} \hat{q}_{jk(1)} t^{-j} \left(\frac{\log t}{t}\right)^k,$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$.

3 The Population Model (P)

In the equation (P),

$$u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)}, \tag{P}$$

we put $u(t) = v(t) + \frac{\beta}{\alpha}$. We have

$$v(t+2) = (1+\alpha)v(t+1) - v(t) + F(v(t), v(t+1)),$$

where $F(v(t), v(t+1))$ is defined by

$$\begin{aligned} F(v(t), v(t+1)) = & -\frac{\alpha}{\beta} v(t)v(t+1) + \frac{\alpha^2}{\beta} v(t)^2 \\ & + (v(t+1) - \alpha v(t) + \frac{\beta}{\alpha} - \beta) \sum_{i \geqq 2} (-\frac{\alpha}{\beta})^i v(t)^i. \end{aligned} \tag{3.1}$$

Next put $v(t+1) = \xi(t)$, $v(t) = \eta(t)$, we have

$$\begin{pmatrix} \xi(t+1) \\ \eta(t+1) \end{pmatrix} = \begin{pmatrix} \alpha+1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} + \begin{pmatrix} F(\eta(t), \xi(t)) \\ 0 \end{pmatrix}.$$

When $\alpha = 1$, we have eigenvalues λ and μ of $M = \begin{pmatrix} \alpha+1 & -1 \\ 1 & 0 \end{pmatrix}$ such that $\lambda = \mu = 1$. Further put $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} \xi \\ \eta \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}$, we have the following difference equation

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + P^{-1} \begin{pmatrix} F(x(t)+y(t), x(t)+2y(t)) \\ 0 \end{pmatrix}. \tag{3.2}$$

Since $P^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$ we can write the equations (3.2) as follows,

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases} \tag{1.3}$$

such that

$$\begin{cases} X(x,y) = x + y - F(x+y, x+2y) = x + \Big(y + \sum_{i+j\geq 2} c_{ij}x^i y^j\Big) \\ \qquad\quad = x + X_1(x,y), \\ Y(x,y) = y + F(x+y, x+2y) = y + \Big(\sum_{i+j\geq 2} d_{ij}x^i y^j\Big) \\ \qquad\quad = y + Y_1(x,y), \end{cases} \tag{1.2'}$$

where $d_{ij} = -c_{ij}$.
¿From the definition of the function F by (3.1), when $\alpha = 1$, we have

$$F(x+y, x+2y) = -\frac{1}{\beta}(xy + y^2) + \frac{1}{\beta^2}(x^2y + 2xy^2 + y^3) + \sum_{i+j\geqq 4, j\geqq 1} \gamma_{ij}x^i y^j, \tag{3.3}$$

where γ_{ij} are constants which consist of β. ¿From (3.3), we have the coefficients of X and Y in (1.2') as follows,

$$c_{20} = d_{20} = 0, \;\; c_{n0} = d_{n0} = 0, (n \geqq 3),$$

$$d_{11} = -\frac{1}{\beta} < 0, \;\; d_{02} = -\frac{1}{\beta} < 0, \;\; d_{21} = \frac{1}{\beta^2},$$

and have

$$A_1 = g_0^-(c_{20}, d_{11}, d_{30}) + c_{20} = \frac{-(0+\frac{1}{\beta}) - \sqrt{(0+\frac{1}{\beta})^2 + 0}}{4} + 0 = -\frac{1}{2\beta} < 0,$$

$$A_2 = g_0^+(c_{20}, d_{11}, d_{30}) + c_{20} = \frac{-(0+\frac{1}{\beta}) + \sqrt{(0+\frac{1}{\beta})^2 + 0}}{4} + 0 = 0, \tag{3.4}$$

Here we can not have the condition $A_1 \leqq A_2 < 0$, however we have $A_1 < A_2 = 0$. Thus we put $a_2 = g_0^-(c_{20}, d_{11}, d_{30})$, $A_1 = a_2 + c_{20} < 0$. Since

$$c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}) = \frac{3}{2\beta} > 0,$$

we have

$$A_1 n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}),$$

for all $n \in \mathbb{N}$.

By Proposition 6, the functional equation

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \tag{1.11}$$

which X and Y are defined by (1.2') has a formal solution $\Psi^-(x) = \sum_{n\geqq 2}^{\infty} a_n^- x^n$, where a_n^- are defined by X and Y.

Further, for any κ, $0 < \kappa \leqq \frac{\pi}{2}$, there are a $\delta > 0$ and a solutions $\Psi^-(x)$ of (1.11), which are holomorphic and can be expanded asymptotically such that

$$\Psi^-(x) \sim \sum_{n=2}^{\infty} a_n^- x^n,$$

as $x \to 0$ through

$$D^*(\kappa, \delta) = \{x\,;\, |\arg[x]| < \kappa,\ 0 < |x| < \delta\}. \tag{1.4}$$

¿From Lemma 7, we have a formal solution $x(t)$ of (3.2) such that

$$-\frac{1}{A_1 t}\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1} = \frac{2\beta}{t}\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1}, \tag{3.5}$$

where $\hat{q}_{jk}$ are constants defined by X and Y in (1.2').

Further suppose $R_1 = \max(R_0, 2/(|A_1|\delta)))$, since $A_1 = -\frac{1}{2\beta} < A_2 = 0$, there is a solution $x(t)$ of (3.2) such that

(i) $x(t)$ are holomorphic and $x(t) \in D^*(\kappa, \delta)$ for $t \in D_1(\kappa_0, R_1)$,
(ii) $x(t)$ is expressible in the following form

$$x(t) = -\frac{1}{A_1 t}\left(1 + b\Big(t, \frac{\log t}{t}\Big)\right)^{-1} = \frac{2\beta}{t}\left(1 + b\Big(t, \frac{\log t}{t}\Big)\right)^{-1}. \tag{3.6}$$

Here $b(t, \log t/t)$ are asymptotically expanded in $D_1(\kappa_0, R_1)$ such that

$$b\Big(t, \frac{\log t}{t}\Big) \sim \sum_{j+k\geqq 1} \hat{q}_{jk(1)} t^{-j} \left(\frac{\log t}{t}\right)^k,$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$.

By the definition (1.7), we have $y(t) = \Psi(x(t))$. Since

$$\begin{pmatrix} u(t+1) - \frac{\beta}{\alpha} \\ u(t) - \frac{\beta}{\alpha} \end{pmatrix} = \begin{pmatrix} v(t+1) \\ v(t) \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

we have a solution $u(t)$ of the (P), such that

$$u(t) = x(t) + y(t) + \frac{\beta}{\alpha} = x(t) + \Psi(x(t)) + \frac{\beta}{\alpha} \sim x(t) + \sum_{n\geqq 2}^{\infty} a_n^- x(t)^n + \frac{\beta}{\alpha}$$

where $x(t)$ is given in the equation (3.6) as $t \to \infty$ through $D_1(\kappa_0, R_1)$.

Here we have proved existence a solution of the Population Model (P).

References

[1] D. Dendrinos, Private communication.

[2] T. Kimura, *On the Iteration of Analytic Functions*, Funkcialaj Ekvacioj, **14**, (1971), 197-238.

[3] M. Suzuki, *Difference Equation for a Population Model* , Discrete Dynamics in Nature and Society, **5**, (2000), 9-18.

[4] M. Suzuki, *Holomorphic solutions of a functional equation and their applications to nonlinear second order difference equations* , Aequationes mathematicae, **74**, (2007), 7-25.

[5] M. Suzuki, *Analytic General Solutions of Nonlinear Difference Equations* , Annali di Matematica Pure ed Applicata, 187, (2008), 353-368.

[6] M. Suzuki, *Holomorphic solutions of some functional equation which has a relation with nonlinear difference systems* , Sūrikaisekikenkyūsho Kōkyūroku No. 1547 (2007), 78–86.

[7] M. Suzuki, *Analytic sol solutions of a nonlinear two variables Difference system*, Sūrikaisekikenkyūsho Kōkyūroku No. 1582 (2008), 53–82.