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On Floating Body Problem

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This paper consists of two parts. In Sections 1, 2 and 3, I give my recent work [8] of 2-dimensional floating bodies. In Section 4, I give two important formulas of 3-dimensional floating bodies. These formulas are already known, maybe, but I do not know where the statements and the proofs are given. In Section 5, we apply the result of Section 4 to Ulam's problem.

1 Ulam's Floating Body Problem of Two Dimension

S. M. Ulam posed a problem: *If a body of uniform density floats in water in equilibrium in every direction, must it be a sphere?* See [3] or [9] for detail. The problem is still open. However, in two dimensional case of the problem, Auerbach [1] gives a counter-example.

Theorem 1. ([1]) *There is a non-circular figure $D \subset \mathbb{R}^2$ of density $\rho = 1/2$ which floats in equilibrium in every direction.*

Before we state our result, we define some terminology of two-dimensional floating bodies. Consider a figure $D \subset \mathbb{R}^2$ whose perimeter ∂D is a simple closed curve, and take a number $0 < \rho < 1$. For a given angle $0 \leqq \theta < 2\pi$, there is a directed line L_θ of slope angle θ which divides the area of D in the ratio $\rho : 1 - \rho$. In this paper, we assume the following three conditions:

(C1) ∂D is of class C^1.

(C2) L_θ meets ∂D at exactly two points, say, P and Q.

(C3) Neither the tangent at P nor at Q is not parallel to the line PQ.

We call ρ the *density* of D, and the segment PQ the *water line* of slope angle θ. We denote by D_u and D_a the divided figures of area ratio $\rho : 1 - \rho$. We call D_u and D_a the *underwater* and *abovewater* parts of D, respectively. We denote by G_u and G_a the centroids of D_u and D_a, respectively. We say that D *floats in equilibrium* in direction $\mathbf{e}_2(\theta) = (-\sin\theta, \cos\theta)$ if the line G_uG_a is parallel to $\mathbf{e}_2(\theta)$.

If the figure D of density ρ floats in equilibrium in every direction, we call $D \subset \mathbb{R}^2$ an *Auerbach figure* of an *Auerbach density* ρ. It is known that, if $D \subset \mathbb{R}^2$ is an Auerbach figure, then the water surface divides ∂D in constant ratio, say, $\sigma : 1 - \sigma$. See (ii) of Corollary 7. We call σ the *perimetral density* of the Auerbach figure D.

If D is an Auerbach figure of density $\rho = 1/2$, then the water lines L_θ and $L_{\theta+\pi}$ are the same but opposite directed lines. Thus it is of perimetral density $\sigma = 1/2$. In the proof of Theorem 1, the condition $\rho = 1/2$ is essential. It is difficult to make an Auerbach figures of density $\rho \neq 1/2$. So a question arises: *Is there a non-circular Auerbach figure of density $\rho \neq 1/2$?*

Recently, Wegner [10] gave an positive answer to this question. Wegner's examples exhibit more interesting fact. That is, for given integer $p \geqq 3$, one of his examples has $(p-2)$ different Auerbach densities. So one Auerbach figure can have many perimetral densities.

On the other hand, Bracho, Montejano and Oliberos [2] gave a following result.

Theorem 2. ([2]) *If there is an Auerbach figure $D \subset \mathbb{R}^2$ of perimetral density $\sigma = 1/3$ or $1/4$, then it is a circle.*

The purpose of the first part of this paper is to prove the following theorem.

Theorem 3. (1) *If an Auerbach figure $D \subset \mathbb{R}^2$ has three perimetral densities σ_1, σ_2 and σ_3, and if $\sigma_1+\sigma_2+\sigma_3=1$, then it is a circle. (These σ_i's are not necessarily different.)*

(2) *If an Auerbach figure $D \subset \mathbb{R}^2$ has four perimetral densities σ_1, σ_2, σ_3 and σ_4, and if $\sigma_1+\sigma_2+\sigma_3+\sigma_4=1$, then it is a circle. (These σ_i's are not necessarily different.)*

The above theorem is a generalization of Theorem 2. Certainly, putting $\sigma_1=\sigma_2=\sigma_3=1/3$ gives the 1/3 case of Theorem 2, and putting $\sigma_1=\sigma_2=\sigma_3=\sigma_4=1/4$ gives the 1/4 case of Theorem 2.

2 Auerbach Figures

In this section, we give a short survey of Auerbach figures.

Theorem 4. ([1], [10]) *If a figure $D \subset \mathbb{R}^2$ is Auerbach, then the water line is of constant length.*

We give a proof of the above theorem in Section 5.

Theorem 5. *If a figure $D \subset \mathbb{R}^2$ is Auerbach, and if PQ is the water line of slope angle θ, then there is a 2π-periodic function f of class C^2 such that the position vectors of P and Q are given by*

$$\mathbf{p}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + \big(f'(\theta) - l\big)\mathbf{e}_1(\theta), \qquad \mathbf{q}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + \big(f'(\theta) + l\big)\mathbf{e}_1(\theta), \tag{1}$$

where $\mathbf{e}_1(\theta) = (\cos\theta, \sin\theta)$, $\mathbf{e}_2(\theta) = (-\sin\theta, \cos\theta)$, and l is half the length of PQ.

Proof. Assume that D is an Auerbach figure. Then by Theorem 4, the waterline is of constant length. Since $\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta)\}$ is a basis of $\mathbb{R}^2$, we can represent the position vectors of the points P and Q as follows:

$$\mathbf{p}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + g(\theta)\mathbf{e}_1(\theta), \quad \mathbf{q}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + \big(g(\theta) + 2l\big)\mathbf{e}_1(\theta). \tag{2}$$

Suppose that the chord P^*Q^* of C is the water line of slope angle $\theta+h$. Then the position vector of the intersection H of the chords PQ and P^*Q^* are given by

$$\overrightarrow{OH} = -f(\theta)\mathbf{e}_2(\theta) + \lambda\mathbf{e}_1(\theta) = -f(\theta+h)\mathbf{e}_2(\theta+h) + \mu\mathbf{e}_1(\theta+h). \tag{3}$$

By taking the inner product of (3) and $\mathbf{e}_2(\theta+h)$, we have that $f(\theta+h) = \lambda\sin h + f(\theta)\cos h$. Thus we obtain that

$$f'(\theta) = \frac{f(\theta+h) - f(\theta)}{h} + o(1) = \lambda\frac{\sin h}{h} - f(\theta)\frac{1-\cos h}{h} + o(1) = \lambda + o(1). \tag{4}$$

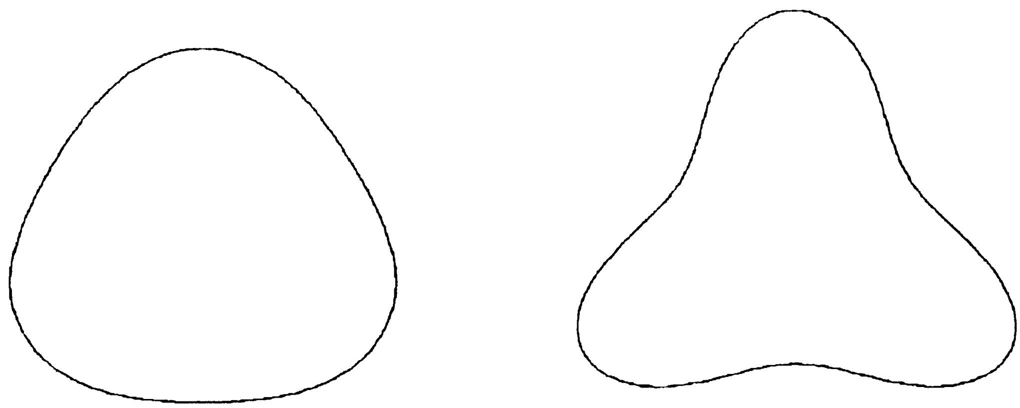
We can evaluate the areas of the sectors HPP^* and HQQ^* by

$$\begin{aligned} &\frac{1}{2}HP^2\,h + o(h) = \frac{1}{2}\,|\; g(\theta) - f'(\theta)\;|^2\,h + o(h) \quad \text{and} \\ &\frac{1}{2}HQ^2\,h + o(h) = \frac{1}{2}\,|\; g(\theta) - f'(\theta) + 2l\;|^2\,h + o(h), \end{aligned} \tag{5}$$

respectively. Since these two areas are equal, we obtain that $g(\theta) = f'(\theta) - l$. Hence we have proved (1). By taking the inner product of (1) and $\mathbf{e}_1(\theta)$, we have that $f'(\theta) = \mathbf{p}(\theta)\cdot\mathbf{e}_1(\theta) + l$. Thus the function $f(\theta)$ is of class C^2. □

The following result is a "proof" of Theorem 1.

Example 6. Put $f(\theta) = -k\cos 3\theta$ in Equation (1). Then the curve is rotational symmetric with respect to the angle $2\pi/3$. So it surrounds an Auerbach figure of density 1/2. See Theorem 11. The figures of $k/l = 0.03$ and $k/l = 0.1$ are drawn as follows:



The following result gives geometric properties of Auerbach figures.

Corollary 7. *If a figure $D \subset \mathbb{R}^2$ is Auerbach, and if PQ is the water line of slope angle θ, then:*
(i) *The vectors $\mathbf{p}'(\theta)$ and $\mathbf{q}'(\theta)$ are symmetric with respect to the line PQ.*
(ii) *The arc PQ of ∂D is of constant length.*

Proof. By differentiating (1), we have that

$$\mathbf{p}'(\theta) = s(\theta)\mathbf{e}_1(\theta) - l\mathbf{e}_2(\theta), \qquad \mathbf{q}'(\theta) = s(\theta)\mathbf{e}_1(\theta) + l\mathbf{e}_2(\theta), \tag{6}$$

where $s(\theta) = f(\theta) + f''(\theta)$. Since the line PQ is parallel to the vector $\mathbf{e}_1(\theta)$, we have proved (i).
(ii) By (6), we have that $|\mathbf{p}'(\theta)| = |\mathbf{q}'(\theta)| = \sqrt{s(\theta)^2 + l^2}$. This implies that the points P and Q move at the same speed along ∂D. Thus we have proved (ii). □

Remark. By integrating (6), we have that

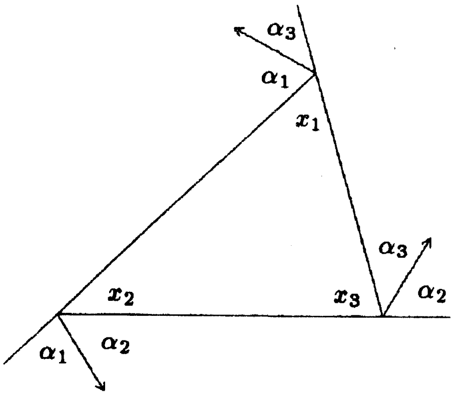
$$\mathbf{p}(\theta) = \mathbf{c} + \int_0^\theta s(\phi)\mathbf{e}_1(\phi)\, d\phi - l\mathbf{e}_1(\theta), \quad \mathbf{q}(\theta) = \mathbf{c} + \int_0^\theta s(\phi)\mathbf{e}_1(\phi)\, d\phi + l\mathbf{e}_1(\theta), \tag{7}$$

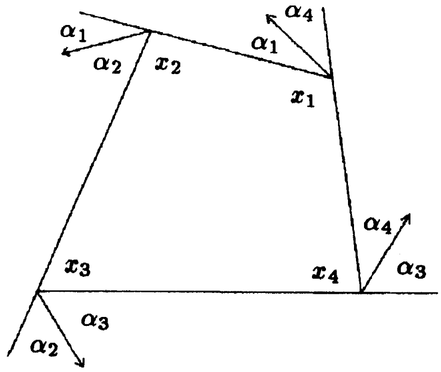
where $\mathbf{c}$ is a constant vector. These formulas are same as those given in Section 2 of [10].

3 Proof of Theorem 3

Proof of Theorem 3. (i) Let P_1, P_2 and P_3 be three points of ∂D such that for each $i = 1, 2, 3$, the line P_iP_{i+1} can be a water surface of perimetral density σ_i. (The indices are taken cyclic in modulo 3.) For each $i = 1, 2, 3$, we denote by $\mathbf{p}_i(\theta)$ the position vector of P_i, by x_i the angle $\angle P_{i-1}P_iP_{i+1}$ and by α_i the angle between $\mathbf{p}_i'(\theta)$ and P_iP_{i+1}. By (i) of Corollary 7, the angle between $P_{i-1}P_i$ and $\mathbf{p}_i'(\theta)$ is equal to α_i. So we obtain that $x_1 + \alpha_3 + \alpha_1 = \pi$, $x_2 + \alpha_1 + \alpha_2 = \pi$ and $x_3 + \alpha_2 + \alpha_3 = \pi$. Since $x_1 + x_2 + x_3 = \pi$, we have that $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. See the figure below left. So we obtain that $\alpha_1 = x_3$. By the converse of Alternate Segment Theorem, $\mathbf{p}_1'(\theta)$ tangents to the circumcircle of the triangle $P_1P_2P_3$. Thus P_1 varies on the circumcircle. Hence D is a circle.

(ii) Let P_1, P_2, P_3 and P_4 be four points of ∂D such that for each $i = 1, 2, 3, 4$, the line P_iP_{i+1} can be a water surface of perimetral density σ_i. (The indices are taken cyclic in modulo 4.) By the same notation and argument used in (i) of this theorem, we otain that $x_1 + \alpha_4 + \alpha_1 = \pi$, $x_2 + \alpha_1 + \alpha_2 = \pi$, $x_3 + \alpha_2 + \alpha_3 = \pi$ and $x_4 + \alpha_3 + \alpha_4 = \pi$. See the figure below right. Since $x_1 + x_2 + x_3 + x_4 = 2\pi$, we have that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi$. So we obtain that $x_1 + x_3 = \pi$. By the converse of Inscribed Quadrangle Theorem, the quadrangle $P_1P_2P_3P_4$ inscribes to a circle. Thus P_3P_1 is of constant length, and therefore, it can be a water line of perimetral density $\sigma_3 + \sigma_4$. Hence, by (i) of this theorem, D is a circle. □





4 On formulas for 3-dimensional floating bodies

Suppose that a solid $\mathcal{F} \subset \mathbb{R}^3$ has a volume V and a uniform density ρ $(0 < \rho < 1)$ and that it floats on water so that a unit vector $\boldsymbol{n} \in \mathbb{R}^3$ is the vertical and upward direction. The *water surface* is a plane which is orthogonal to $\boldsymbol{n}$ and cuts $\mathcal{F}$ in ratio $1-\rho : \rho$. By Archimedes Principle, the volume of the undersurface part of $\mathcal{F}$ is equal to ρV. In the paper, we assume that the boundary of $\mathcal{F}$ is sufficiently differentiable and do not tangent to the water surface. From now, we consider that $\mathcal{F}$ is fixed and $\boldsymbol{n}$ varies, that is, we consider that $\mathcal{F}$ is not inclined but the water surface is inclined. We call $\boldsymbol{n}$ the *vertical vector* and orthogonal vectors to $\boldsymbol{n}$ *horizontal vectors*.

We can solve the stability problem of floating bodies by analyzing the relation between the positions of its center of gravity and its center of buoyancy. The *center of buoyancy* is the center of gravity of the underwater part of the floating body. The center of gravity G does not depend on $\boldsymbol{n}$, but the center of buoyancy B is a function of $\boldsymbol{n}$.

The gravity acts vertically downward on G and the buoyancy acts vertically upward on B. If G and B do not lie on same vertical line, then the floating body rotates. We say that $\mathcal{F}$ is in *equilibrium state* with respect to the direction $\boldsymbol{n}$ if G and B lie on the same vertical line, that is, $(\boldsymbol{B}-\boldsymbol{G}) \cdot \boldsymbol{u} = 0$ for every horizontal vector $\boldsymbol{u}$.

Suppose that the floating body $\mathcal{F}$ is in equilibrium state with respect to $\boldsymbol{n}_0$. Then fix a horizontal vector $\boldsymbol{t}$ and consider the vector $\boldsymbol{u}_0$ which is made by the vector product $\boldsymbol{u}_0 = \boldsymbol{n}_0 \times \boldsymbol{t}$. Rotate the vectors $\boldsymbol{u}_0$ and $\boldsymbol{n}_0$ counterclockwise with respect to $\boldsymbol{t}$, that is,

$$\boldsymbol{u} = \boldsymbol{u}_0 \cos\theta + \boldsymbol{n}_0 \sin\theta, \qquad \boldsymbol{n} = -\boldsymbol{u}_0 \sin\theta + \boldsymbol{n}_0 \cos\theta. \tag{8}$$

It means that the water line inclines counterclockwise by the angle θ, that is, the floating body inclines clockwise by the angle θ. We call $\boldsymbol{t}$ the *vector of rotation axis*.

Assume that the floating body inclines by a sufficiently small angle θ. If the function $F(\theta) = (\boldsymbol{B}-\boldsymbol{G})\cdot\boldsymbol{u}$ is monotone increasing, then the gravity and buoyancy act as the floating body returns to the original state. If the function $F(\theta)$ is monotone decreasing, then the gravity and buoyancy act as the floating body inclines more. So we define as follows:

Definition. We say that a floating body $\mathcal{F} \subset \mathbb{R}^3$ in equilibrium is *stable* (*unstable*) with respect to a rotation axis $\boldsymbol{t}$ if $F(\theta)$ is monotone increasing (decreasing). We say that $\mathcal{F}$ is *stable* if it is stable with respect to every rotation axis $\boldsymbol{t}$.

We denote by E the cross section of $\mathcal{F}$ cut by the water surface. We make a coordinate plane whose origin is the center of gravity of E and x- and z-axis have the same direction as $\boldsymbol{u}$ and $\boldsymbol{t}$, respectively. Then we can regard E as the figure in the xz-plane. Consider the following quantity:

$$I(\boldsymbol{n}, \boldsymbol{t}) = \iint_E x^2 \, dxdz. \tag{9}$$

We call it the *moment of inertia* with respect to $\boldsymbol{n}$ and $\boldsymbol{t}$. From now, we fix vectors $\boldsymbol{n}_0$ and $\boldsymbol{t}$, and consider the moment of inertia as a function of θ, that is, we denote $I(\theta) = I(\boldsymbol{n}, \boldsymbol{t})$.

Suppose that a floating body $\mathcal{F}$ is in equilibrium with respect to the direction $\boldsymbol{n}_0$. Then the following formula holds:

$$F(\theta) = \int_0^\theta \left(\frac{I(\phi)}{\rho V} - \overline{GB}_0\right) \cos(\theta - \phi)\, d\phi, \tag{F1}$$

where B_0 is the center of buoyancy of $\mathcal{F}$ with respect to the direction $\boldsymbol{n}_0$. Remark that $\overline{GB}_0$ is a distance between G and B_0, and so, a constant number. We can approximate (F1) as follows:

$$F(\theta) = \left(\frac{I(0)}{\rho V} - \overline{GB}_0\right)\theta + O(\theta^2). \tag{10}$$

By using (10), we deduce the following corollary:

Corollary 8. *If $I(0) > \rho V \overline{GB}_0$, then $\mathcal{F}$ is stable with respect to the axis $\boldsymbol{t}$. If $I(0) < \rho V \overline{GB}_0$, then $\mathcal{F}$ is unstable with respect to the axis $\boldsymbol{t}$.*

Set $U(\theta) = (\boldsymbol{G} - \boldsymbol{B}) \cdot \boldsymbol{n}$. We call it the *potential function* of the floating body $\mathcal{F}$. Suppose that a floating body $\mathcal{F}$ is in equilibrium with respect to the direction $\boldsymbol{n}_0$. Then the following formula holds:

$$U(\theta) = \int_0^\theta \left(\frac{I(\phi)}{\rho V} - \overline{GB}_0\right) \sin(\theta - \phi)\, d\phi \tag{F2}$$

By (F1) and (F2), we find that

$$U'(\theta) = F(\theta), \qquad F'(\theta) = -U(\theta) + \frac{I(\theta)}{\rho V} - \overline{GB}_0. \tag{11}$$

By the first formula of (11), we obtain the following corollary:

Corollary 9. *The floating body $\mathcal{F}$ is stable with respect to the axis $\boldsymbol{t}$ if and only if the function $U(\theta)$ takes a local minimum at $\theta = 0$. The floating body $\mathcal{F}$ is unstable with respect to the axis $\boldsymbol{t}$ if and only if the function $U(\theta)$ takes a local maximum at $\theta = 0$.*

Proof of (F1) and (F2). Rotate $\boldsymbol{u}$ and $\boldsymbol{n}$ by a small angle ε with respect to the axis $\boldsymbol{t}$. Denote them by $\boldsymbol{u}_\varepsilon$ and $\boldsymbol{n}_\varepsilon$, that is,

$$\boldsymbol{u}_\varepsilon = \boldsymbol{u}\cos\varepsilon + \boldsymbol{n}\sin\varepsilon, \qquad \boldsymbol{n}_\varepsilon = -\boldsymbol{u}\sin\varepsilon + \boldsymbol{n}\cos\varepsilon. \tag{12}$$

We denote by $\mathcal{W}$ and $\mathcal{W}_\varepsilon$ the under surface parts of $\mathcal{F}$ with respect to the vertical vectors $\boldsymbol{n}$ and $\boldsymbol{n}_\varepsilon$, respectively, and by B and B_ε the centers of buoyancy of them. By the definition, B and B_ε are the center of gravity of $\mathcal{W}$ and $\mathcal{W}_\varepsilon$, respectively. Then put

$$\mathcal{W}_3 = \mathcal{W} \cap \mathcal{W}_\varepsilon, \qquad \mathcal{W}_1 = \mathcal{W} \setminus \mathcal{W}_3 \text{ and } \mathcal{W}_2 = \mathcal{W}_\varepsilon \setminus \mathcal{W}_3, \tag{13}$$

and denote by B_3, B_1 and B_2 the centers of gravity of them, respectively. Since the volumes of $\mathcal{W}$ and $\mathcal{W}_\varepsilon$ are equal to ρV, the volumes of $\mathcal{W}_1$ and $\mathcal{W}_2$ are equal, putting V_1. Then by the property of center of gravity, we obtain that

$$\boldsymbol{B}_\varepsilon = \left(1 - \frac{V_1}{\rho V}\right)\boldsymbol{B}_3 + \frac{V_1}{\rho V}\boldsymbol{B}_2, \qquad \boldsymbol{B} = \left(1 - \frac{V_1}{\rho V}\right)\boldsymbol{B}_3 + \frac{V_1}{\rho V}\boldsymbol{B}_1. \tag{14}$$

By the above equality, we obtain that

$$\boldsymbol{B}_\varepsilon - \boldsymbol{B} = \frac{V_1}{\rho V}(\boldsymbol{B}_2 - \boldsymbol{B}_1)\ . \tag{15}$$

We cut the cross section E into the following two parts:

$$E_2 = E \cap \mathcal{W}_2, \qquad E_1 = E \cap \mathcal{W}_1. \tag{16}$$

We take the center of gravity of E as the origin, and take x, y and z-axis so that they have the same directions as $\boldsymbol{u}$, $\boldsymbol{n}$ and $\boldsymbol{t}$, respectively. We denote by $x = \delta$ the border line of E_2 and E_1, and set

$$\begin{aligned}\tilde{E}_2 =&\{(r\cos\theta + \delta, r\sin\theta, z) \mid (r+\delta, z) \in E_2,\ 0 \leqq \theta \leqq \varepsilon\},\\ \tilde{E}_1 =&\{(-r\cos\theta + \delta, -r\sin\theta, z) \mid (-r+\delta, z) \in E_1,\ 0 \leqq \theta \leqq \varepsilon\}.\end{aligned} \tag{17}$$

Then we obtain that

$$\begin{aligned}V_1 &= \iiint_{\mathcal{W}_2} dxdydz = \iiint_{\tilde{E}_2} r\ drd\theta dz + O(\varepsilon^2)\\ &= \varepsilon \iint_{(r+\delta,z)\in E_2} r\ drdz + O(\varepsilon^2)\\ &= \varepsilon \iint_{E_2} (x-\delta)\ dxdz + O(\varepsilon^2).\end{aligned} \tag{18}$$

Similarly, we obtain that

$$V_1 = \iiint_{\mathcal{W}_2} dxdydz = \varepsilon \iint_{E_1} (\delta - x)\ dxdz + O(\varepsilon^2). \tag{19}$$

By (18) and (19), we obtain that

$$\varepsilon\Big(\iint_E x\ dxdz - \delta \iint_E\ dxdz\Big) = \varepsilon \iint_{E_2} (x-\delta)\ dxdz - \varepsilon \iint_{E_1} (\delta - x)\ dxdz = O(\varepsilon^2). \tag{20}$$

Since we take the center of gravity of E as the origin, we obtain that $\iint_E x\ dxdz = 0$. Putting it into (20), we obtain that $\delta = O(\varepsilon)$. Since $\boldsymbol{u}$ is the unit vector of direction x-axis, we obtain that

$$\begin{aligned}V_1 \boldsymbol{B}_2 \cdot \boldsymbol{u} &= \iiint_{\mathcal{W}_2} x\ dxdydz = \iiint_{\tilde{E}_2} r(r\cos\theta + \delta)\ drd\theta dz + O(\varepsilon^2)\\ &= \iint_{(r+\delta,z)\in E_2} r(r\sin\varepsilon + \delta\varepsilon)\ drdz + O(\varepsilon^2)\\ &= \varepsilon \iint_{E_2} x(x-\delta)\ dxdz + O(\varepsilon^2) = \varepsilon \iint_{E_2} x^2\ dxdz + O(\varepsilon^2).\end{aligned} \tag{21}$$

Similarly, we obtain that

$$V_1 \boldsymbol{B}_1 \cdot \boldsymbol{u} = -\varepsilon \iint_{E_1} x^2\ dxdz + O(\varepsilon^2), \tag{22}$$

$$V_1 \boldsymbol{B}_2 \cdot \boldsymbol{n} = O(\varepsilon^2), \qquad V_1 \boldsymbol{B}_1 \cdot \boldsymbol{n} = O(\varepsilon^2). \tag{23}$$

By using (15), (21) and (22), we obtain that

$$(\boldsymbol{B}_\varepsilon - \boldsymbol{B}) \cdot \boldsymbol{u} = \frac{\varepsilon}{\rho V} \iint_E x^2\ dxdz + O(\varepsilon^2) = \frac{\varepsilon I(\theta)}{\rho V} + O(\varepsilon^2). \tag{24}$$

Similarly, we obtain that

$$(\boldsymbol{B}_\varepsilon - \boldsymbol{B}) \cdot \boldsymbol{n} = O(\varepsilon^2). \tag{25}$$

By dividing (24) and (25) by ε, and taking $\varepsilon \to 0$, we obtain that

$$\boldsymbol{B}'(\theta) \cdot \boldsymbol{u} = \frac{I(\theta)}{\rho V}, \qquad \boldsymbol{B}'(\theta) \cdot \boldsymbol{n} = 0. \tag{26}$$

By putting (8) into (26), we obtain that

$$\boldsymbol{B}'(\theta)\cdot\boldsymbol{u}_0=\frac{I(\theta)}{\rho V}\cos\theta,\qquad \boldsymbol{B}'(\theta)\cdot\boldsymbol{n}_0=\frac{I(\theta)}{\rho V}\sin\theta. \tag{27}$$

By replacing θ of (27) by ϕ, and by integrating it on the interval $0\leqq\phi\leqq\theta$, we obtain that

$$(\boldsymbol{B}-\boldsymbol{B}_0)\cdot\boldsymbol{u}_0=\frac{1}{\rho V}\int_0^\theta I(\phi)\cos\phi\,d\phi,\qquad (\boldsymbol{B}-\boldsymbol{B}_0)\cdot\boldsymbol{n}_0=\frac{1}{\rho V}\int_0^\theta I(\phi)\sin\phi\,d\phi. \tag{28}$$

By (8) and (28), we obtain that

$$\begin{aligned}(\boldsymbol{B}-\boldsymbol{B}_0)\cdot\boldsymbol{u}&=\frac{1}{\rho V}\int_0^\theta I(\phi)(\cos\theta\cos\phi+\sin\theta\sin\phi)\,d\phi\\&=\frac{1}{\rho V}\int_0^\theta I(\phi)\cos(\theta-\phi)\,d\phi.\end{aligned} \tag{29}$$

Similarly, we obtain that

$$\begin{aligned}(\boldsymbol{B}-\boldsymbol{B}_0)\cdot\boldsymbol{n}&=-\frac{1}{\rho V}\int_0^\theta I(\phi)(\sin\theta\cos\phi-\cos\theta\sin\phi)\,d\phi\\&=-\frac{1}{\rho V}\int_0^\theta I(\phi)\sin(\theta-\phi)\,d\phi.\end{aligned} \tag{30}$$

Hence we obtain that

$$\begin{aligned}F(\theta)&=(-(\boldsymbol{G}-\boldsymbol{B}_0)+(\boldsymbol{B}-\boldsymbol{B}_0))\cdot\boldsymbol{u}\\&=-(\boldsymbol{G}-\boldsymbol{B}_0)\cdot(\boldsymbol{u}_0\cos\theta+\boldsymbol{n}_0\sin\theta)+(\boldsymbol{B}-\boldsymbol{B}_0)\cdot\boldsymbol{u}\\&=-\overline{GB}_0\sin\theta+\frac{1}{\rho V}\int_0^\theta I(\phi)\cos(\theta-\phi)\,d\phi\\&=\int_0^\theta\Big(\frac{I(\phi)}{\rho V}-\overline{GB}_0\Big)\cos(\theta-\phi)\,d\phi.\end{aligned} \tag{31}$$

Similarly, we obtain that

$$U(\theta)=\int_0^\theta\Big(\frac{I(\phi)}{\rho V}-\overline{GB}_0\Big)\sin(\theta-\phi)\,d\phi. \tag{32}$$

Hence we have proved (F1) and (F2). □

5 Application to Ulam's problem

Firstly, we consider three dimensional case. Remark that a floating body $\mathcal{F}\in\mathbb{R}^3$ is in equilibrium in every direction if and only if $F(\theta)=0$ for every rotation axis $\boldsymbol{t}$. If $F(\theta)=0$, then by (11), we have that $I(\boldsymbol{n},\boldsymbol{t})=I(\theta)$ is constant. Hence we have proved the following theorem:

Theorem 10. *If a floating body $\mathcal{F}$ is in equilibrium in every direction, then the moment of inertia $I(\boldsymbol{n},\boldsymbol{t})$ is constant for every $\boldsymbol{n}$ and $\boldsymbol{t}$.*

Secondly, we consider two dimensional case. Consider a rod of base $D\subset\mathbb{R}^2$ and height h. Suppose that it floats so that the rotation axis $\boldsymbol{t}$ is parallel to the axis of the rod and fixed. In this case, denote by $\ell=\ell(\theta)$ the length of the water line. Then we obtain that

$$I(\theta)=\int_{-h/2}^{h/2}\Big\{\int_{-\ell/2}^{\ell/2}x^2\,dx\Big\}dz=\frac{h}{12}\ell^3. \tag{33}$$

By Theorem 10 and (33), we have proved Theorem 4.

The converse of Theorem 4 holds under a rotational symmetry condition.

Theorem 11. *If a figure $D \subset \mathbb{R}^2$ is rotational symmetric with respect to some angle, and if the water line is of constant length, then it is Auerbach.*

Proof. By (33), the moment of inertia $I(\theta)$ is constant. So by (28), we obtain that

$$\boldsymbol{B} = \boldsymbol{G} + \left(\frac{I(0)}{\rho V} - \overline{GB_0}\right)\boldsymbol{n}_0 + \frac{I(0)}{\rho V}(\boldsymbol{u}_0 \sin\theta - \boldsymbol{n}_0 \cos\theta). \tag{34}$$

So the locus of B is a circle. Since D is rotational symmetric, so is the locus of B. Thus we obtain that

$$\boldsymbol{B} = \boldsymbol{G} + \frac{I(0)}{\rho V}(\boldsymbol{u}_0 \sin\theta - \boldsymbol{n}_0 \cos\theta). \tag{35}$$

By (35), we obtain that $F(\theta) = 0$. Hence D is Auerbach. □

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